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MATHEMATICAL MODELLING OF A HELICOPTER ROTOR TRACK AND BALANCE: THEORY

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Non-uniformity of helicopter blades results in vibrations at low frequencies. These vibrations result in increased fatigue of the crew, discomfort to passengers and maintenance and reliability problems. Non-uniformities include geometric, mass, structural and aerodynamic aspects. Typical blades have special devices by which intentional non-uniformities can be introduced, in order to cancel the influence of "natural" non-uniformities of the blades. Non-uniformities also result in out-of-track behavior of the blades. In many cases the corrections are aimed at decreasing the out-of-track behavior of the blades, based on the assumption that this effort will also decrease the vibrations that are transferred from the rotor to the hub. Unfortunately, tracking does not necessarily result in optimal reduction of the vibrations. The present paper presents a general mathematical definition of track and balance, and the relation between them. This mathematical model opens the way for a thorough investigation on optimal rotor tuning.

1. INTRODUCTION

One of the most significant problems of helicopters today is their vibrations. Vibrations result in the following: (a) increased fatigue of the crew; (b) increased fatigue of mechanical parts that leads to maintenance problems that affect the vehicle's availability; (c) higher probability of malfunctions in the avionics or other "delicate" systems; (d) in many cases, a high level of vibrations that limits the operational envelope; (e) increased discomfort for passengers in civil applications.

It is common to divide the helicopter vibrations into three categories: (a) high frequency vibrations, where the frequency of vibration (f) is very high compared to the rotor rotational frequency (f_R), ($f \geqslant 20\, f_R$)—these vibrations are mainly caused by the engine or gear-box; (b) moderate frequency vibrations ($20\, f_R > f \geqslant 5\, f_R$)—the tail rotor, and to a lesser extent the (main) rotor, are the main sources of these vibrations; (c) low frequency vibrations ($5\, f_R > f$) that are caused mainly by the rotor—these vibrations have the most severe effect on human tolerance and fatigue of mechanical parts.

The present research is concerned mainly with vibrations that belong to the last category. When the low frequency vibrations that are caused by the rotor are considered, they may be divided into two different kinds: (a) vibration that is inherent to the asymmetric nature of a rotor in forward flight, and is present even in cases where all the blades are identical; (b) vibration due to non-uniformity of the blades—this non-uniformity may be the result of inaccuracies in the production procedure, or appear and grow during the operation of the helicopter (thus requiring continuous maintenance and inspection).

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In the case of the first kind of vibration, the rotor acts as a mechanical filter and applies on the fuselage only vibrations having specific discrete frequencies. These frequencies equal (mbf_R), where b is the number of blades, and m is a positive integer (1, 2, . . .). As indicated above, the first kind of vibration is inherent to a rotor in forward flight and thus always exists. Nevertheless, vibration amplitudes can be reduced by using various techniques [1].

The second kind of vibration, due to non-uniformity of the blades, has attracted only a limited amount of scientific interest, although ground crews are occupied with it on a continuous basis [2]. These vibrations, unlike the first kind, include all the rotor harmonics (mf_R). As can be seen in references [3, 4], for most practical cases, a vibration having a frequency of f_R has the most severe effect on the human body tolerance.

The non-uniformities of the blades may be the result of aerodynamic, mass or structural non-uniformities along the blades. Moreover, they also may include non-uniformity in the characteristics of the connection between the roots of the blades and the hub. In order to cope with these non-uniformities, rotor blades have various devices to introduce intentional non-uniformity among them, in order to cancel the results of the unintentional non-uniformity (denoted ‘‘natural’’ in what follows). These devices are shown schematically in Figure 1, and they include aerodynamic and mass devices: (a) pitch rod setting—by changing the length of a certain pitch rod, the pitch angle of the blade is varied, thus resulting in variations in the aerodynamic loads that act along the blade; (b) trailing edge tabs—by bending these tabs the cross-sectional camber is changed, and thus the aerodynamic loads are varied; (c) balance weights—these weights can be attached at specific predetermined locations along the blade, changing the blade mass distribution.

Non-uniformity of the blades also leads to differences between the motions of the various blades. Of special interest, in this case, is the flapping motion, since it is relatively easy to measure the flapping amplitude of each blade. In fact, instead of measuring the flapping, usually the blade tip path is measured.

Thus, all efforts to decrease non-uniformity of the blades started as efforts to ‘‘track’’

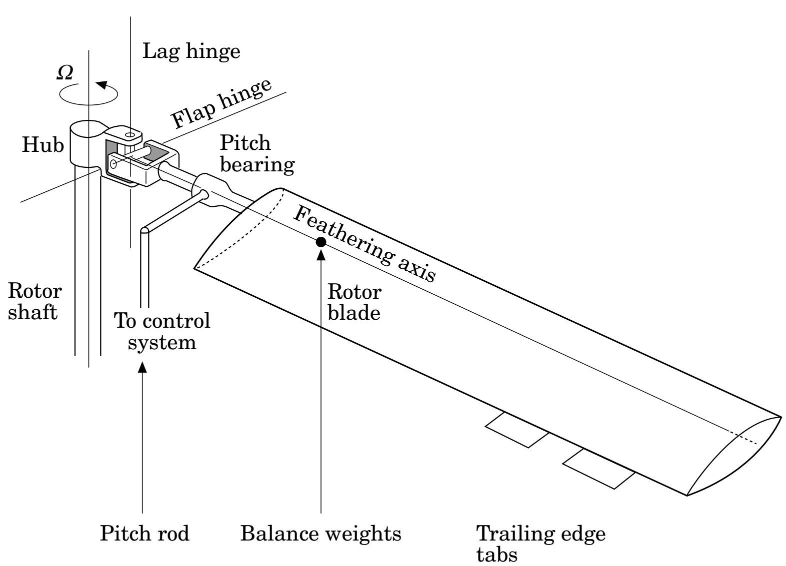

Figure 1. The track and balance elements of a typical blade.

the blades: namely, trying to reach a state where the tip paths of all the blades coincide. Various methods were developed in order to measure the ''out-of-track'' magnitude of the various blades. Nagy and Greguss [5] mentioned two simple mechanical methods that were used for many years: the brush and flag methods. These authors then described an advanced optical method based on the diffuse reflection of laser light by the running blades. Nowadays, various advanced optical methods are used to measure blade out-of-track.

Nevertheless, it should be recalled that rotor tuning is carried out in order to reduce the low frequency vibrations that are transmitted from the rotor to the helicopter fuselage. It turns out, and is well known to ground crews and helicopter maintenance experts, that tracking does not always result in an optimal reduction of the vibrations. Moreover, ground crews can always recall cases of rotors in-track that resulted in high levels of vibrations. This fact was the motivation behind the development of advanced tuning systems, that also include (besides measuring the out-of-tracking) various accelerometers that are distributed over the fuselage and measure the fuselage vibrations. The rotor tuning in this case includes track and balance, namely reduction of the rotor out-of-track, as well as the fuselage vibrations. The tuning procedure is guided by a computer code that uses the measurements of all the sensors, at various air speeds, as input data.

While various manufacturers offer advanced, fairly complicated, systems for rotor tuning, the authors are not aware of any detailed analytical study on the optimal balancing of helicopter rotors, and its relation to rotor tracking. The purpose of this paper is to present such a study. The paper is divided into two parts. In the first part, the derivation of the analytical model will be presented. Tracking and balancing will be defined mathematically. In the second part [6] the mathematical model will be validated and then used in order to study the optimal tuning of a rotor, and its relation to tracking.

2. THE SYSTEMS GEOMETRY

Three different co-ordinate systems are defined: all are right-hand Cartesian systems.

(a) The hub system of co-ordinates (x_H, y_H, z_H): this is a system that is attached to the helicopter fuselage and does not rotate with the rotor; The origin of this system, point O_H, is located at the hub center, as shown in Figure 2; the axis z_H coincides with the axis of the rotor shaft, while the x_H-axis points forward and y_H points to the left (when looking from above).

(b) The rotating system of the kth blade (x_{Rk}, y_{Rk}, z_{Rk}): this system is attached to the rotor and rotates with it, at a constant angular speed Ω. The origin of this system coincides with the origin of the hub system. The axis z_{Rk} coincides with the axis z_H. The axis x_{Rk} lies along a line that connects the hub center with the point of attachment of the kth blade (see Figure 2).

(c) The system of co-ordinates of the kth blade (x_{Bk}, y_{Bk}, z_{Bk}): this system is attached to the blade root and moves with it, relative to the rotor system. In Figure 3 this motion includes a flapping angle β_k, a lead–lag angle ζ_k and a pitch angle θ_{Dk}. In the case in which $\beta_k = \zeta_k = \theta_{Dk} = 0$, the co-ordinate directions of the rotating and blade systems coincide. The flapping angle β_k is of special interest. If elastic deformations are neglected, then this angle practically determines the blade tip path. Elastic deformations may affect the tip path if they are not negligible.

It is assumed that the helicopter is in a straight flight, at a constant speed (namely, that any angular rates or linear accelerations are negligibly small). In this case all the variables

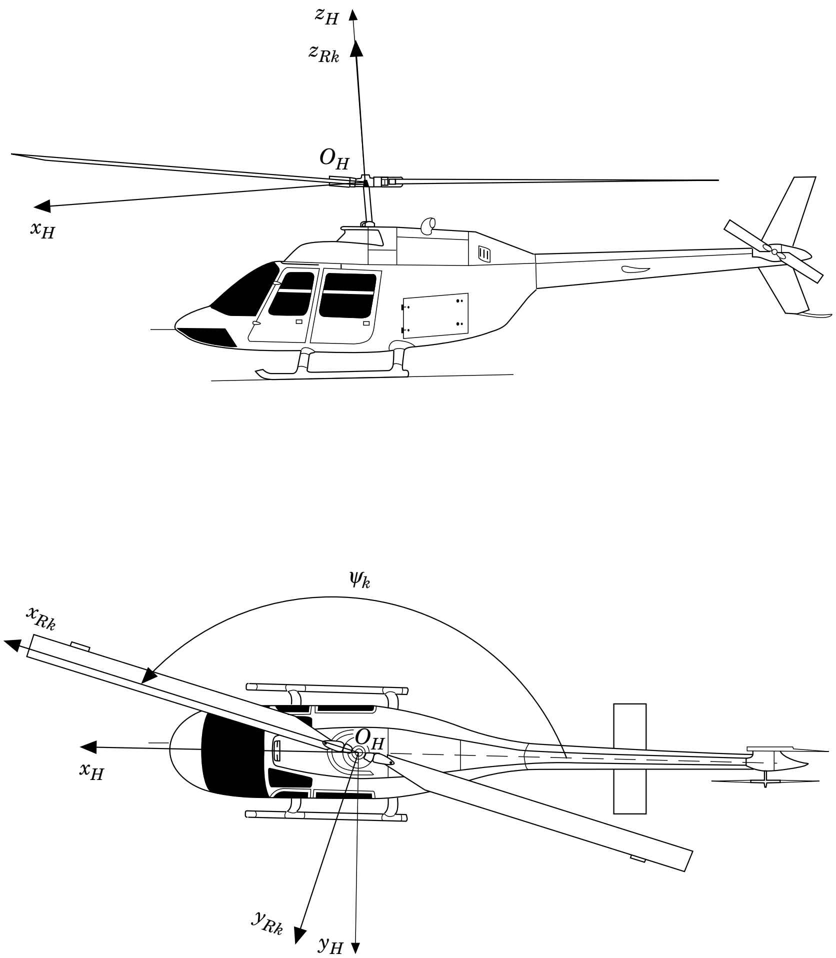

Figure 2. The hub and rotating systems of co-ordinates.

are periodic, where the basic frequency is the rotor frequency. Thus, if one considers the flapping angle of the kth blade, β_k, it can be described as an infinite Fourier series:

$$\beta_k = \beta_k^0 + \sum_{i=1}^{\infty} \beta_k^{is} \sin(i\psi_k) + \sum_{i=1}^{\infty} \beta_k^{ic} \cos(i\psi_k). \tag{2.1}$$

ψ_k is the azimuth angle of the kth blade (see Figure 2). The blade numbers range between 0 and (b − 1), and increase in the counter-clockwise direction, when looking from above. For convenience, and without losing any generality, blade number zero is chosen as the representative blade. Thus, since the blades are uniformly distributed over the disk,

$$\psi_k = \psi + 2\pi k/b, \tag{2.2}$$

where ψ is the azimuth angle of the representative blade.

The coefficients β_k^0, β_k^{is} and β_k^{ic} are functions of the blade properties, rotor angular speed, speed of flight, shaft angles and air density. In practical cases, the coefficients decrease as i is increased. Since interest here is in low frequency vibrations, the present derivation will

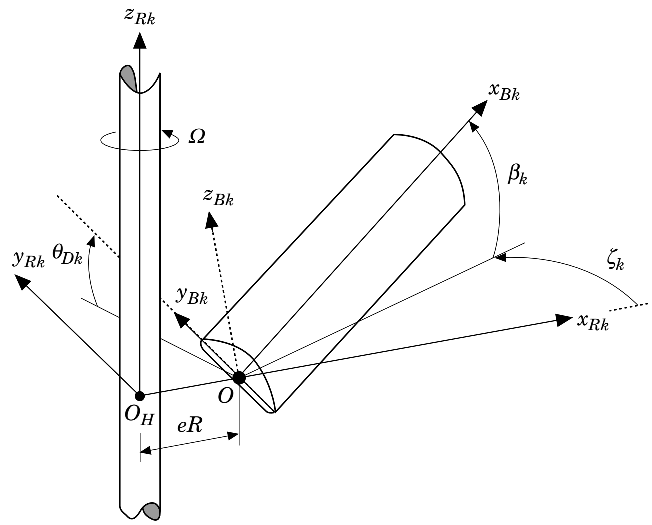

Figure 3. The kth blade motion and system of co-ordinates.

be confined to five harmonics, higher ones being neglected. If necessary, a similar derivation can be carried out for higher numbers of harmonics ($i > 5$). Thus

$$\beta_k = \beta_k^0 + \sum_{i=1}^{5} \beta_k^{is} \sin(i\psi_k) + \sum_{i=1}^{5} \beta_k^{ic} \cos(i\psi_k) \tag{2.3}$$

Equation (2.3) indicates that the blade flapping is defined by a column vector of order 11, $\boldsymbol{\beta}_k^C$ (the upper index C indicates that this is a vector of coefficients), defined as

$$\boldsymbol{\beta}_k^C = \{\beta_k^0, \beta_k^{1s}, \beta_k^{2s}, \ldots, \beta_k^{5s}, \beta_k^{1c}, \beta_k^{2c}, \ldots, \beta_k^{5c}\}, \qquad 0 \leqslant k \leqslant b-1. \tag{2.4}$$

In the present case, in which non-uniformity of the blades is considered, it is convenient to describe $\boldsymbol{\beta}_k^C$ as the sum of two vectors of order 11,

$$\boldsymbol{\beta}_k^C = \boldsymbol{\beta}_N^C + \boldsymbol{\Delta\beta}_k^C \tag{2.5}$$

where $\boldsymbol{\beta}_N^C$ is the flapping of a certain nominal (master) blade (this vector is not a function of k), while $\boldsymbol{\Delta\beta}_k^C$ represents the deviation of the flapping of the kth blade, relative to the nominal one.

3. LOADS TRANSFERRED FROM THE ROTOR TO THE FUSELAGE

Each blade transfers a load to the hub, which can be described by a force vector and a moment vector, at the hub center. If all these loads are added properly, the resultant load that is transferred from the rotor to the hub is obtained. While it is convenient to describe the loads that act on a certain blade by their components in the rotating system of co-ordinates of that blade, the loads that are transmitted to the fuselage by the entire rotor are calculated in the hub non-rotating system. Thus, the procedure of the loads calculations also includes co-ordinate transformation.

3.1. THE FORCE TRANSFERRED BY A CERTAIN BLADE TO THE HUB

The force that is transferred from the kth blade to the hub is described by its components along the axes (x_{Rk}, y_{Rk}, z_{Rk}), $(F^R_{xk}, F^R_{yk}, F^R_{zk})$. These components can be described by using a Fourier series (recall that five terms are retained in the present derivation):

$$F^R_{\alpha k} = F^{R0}_{\alpha k} + \sum_{i=1}^{5} [F^{Ris}_{\alpha k} \sin (i\psi_k) + F^{Ric}_{\alpha k} \cos (i\psi_k)], \qquad \alpha \equiv x, y, z. \tag{3.1}$$

The coefficients in equation (3.1) have two lower indices: the first one indicates the direction of the component (x_{Rk}, y_{Rk} or z_{Rk}), while the second one indicates the blade number ($0 \leqslant k \leqslant b-1$). There are also three upper indices: the first one (R) indicates that the components along the rotating system of co-ordinates are concerned and second one (i) is the harmonic number, while the third index indicates if a cosine (c) or a sine (s) component is considered. Thus, similar to the flapping angles, each of the force components can be described by a vector of order 11 that includes all the coefficients of equation (3.1).

$$\mathbf{f}^{CR}_{\alpha k} = \{F^{R0}_{\alpha k}, F^{R1s}_{\alpha k}, F^{R2s}_{\alpha k}, \ldots, F^{R5s}_{\alpha k}, F^{R1c}_{\alpha k}, F^{R2c}_{\alpha k}, \ldots, F^{R5c}_{\alpha k}\},$$

$$\alpha \equiv x, y, z, \quad k = 0, \ldots, b-1. \tag{3.2}$$

The upper index CR indicates that this is the vector of coefficients (C) in the blade rotating system (R). The lower indices indicate that the α component of the force, of the kth blade, is dealt with.

In the case of perfectly uniform (identical) blades, all the vectors $\mathbf{f}^{CR}_{\alpha k}$, of all the blades, are identical. Since in the present investigation the case of non-uniform blades is considered, it is convenient, similar to flapping, to describe the force vector as the sum of two other vectors of order 11:

$$\mathbf{f}^{CR}_{\alpha k} = \mathbf{f}^{CR}_{\alpha N} + \Delta\mathbf{f}^{CR}_{\alpha k}, \qquad \alpha \equiv x, y, z. \tag{3.3}$$

The vector $\mathbf{f}^{CR}_{\alpha N}$ is the coefficients' vector of a "nominal" blade and is identical for all the blades. $\Delta\mathbf{f}^{CR}_{\alpha k}$ represents the deviations of each blade from the nominal one. This vector may differ from one blade to another (function of k).

3.2. THE RESULTANT FORCE TRANSFERRED FROM THE ROTOR TO THE HUB

The resultant force that is transferred from the rotor to the hub is obtained by a vector summation of all the forces that are transferred to the hub by the individual blades. It is convenient to describe this force by its components in the hub system of co-ordinates, (F^H_x, F^H_y, F^H_z), thus presenting the vibrations as they are "felt" by the fuselage. From Figure 2 it is easy to see that

$$F^H_x = \sum_{k=0}^{b-1} (-F^R_{xk} \cos \psi_k + F^R_{yk} \sin \psi_k), \tag{3.4a}$$

$$F^H_y = \sum_{k=0}^{b-1} (-F^R_{xk} \sin \psi_k - F^R_{yk} \cos \psi_k), \qquad F^H_z = \sum_{k=0}^{b-1} F^R_{zk}. \tag{3.4b, c}$$

As was done in equation (3.3) it is very convenient to express the components F_x^H, F_y^H and F_z^H, as follows:

$$F_\alpha^H = F_{\alpha N}^H + \Delta F_\alpha^H, \qquad \alpha \equiv x, y, z. \tag{3.5}$$

$F_{\alpha N}^H$ is the force component that would have been transmitted to the hub if all the blades were identical to the nominal one. ΔF_α^H comprises the contributions due to deviations of the various blades, from the nominal one. Since the present investigation deals with vibrations due to non-uniformity of the blades, the derivations will concentrate on ΔF_α^H.

Now equation (3.3) is substituted into equation (3.1), and the result is substituted into equations (3.4a–c). In addition, use is made of equation (2.2), while products of sines and cosines are replaced by sums of sines and cosines, so that the final result is

$$\Delta F_\alpha^H = \Delta F_\alpha^{H0} + \sum_{i=1}^{5} [\Delta F_\alpha^{His} \sin(i\psi) + \Delta F_\alpha^{Hic} \cos(i\psi)], \qquad \alpha \equiv x, y, z. \tag{3.6}$$

Examples of detailed expressions for some of the coefficients in equation (3.6) are presented in the Appendix. It should be emphasized that harmonics higher than a fifth, which are obtained during the derivations, are neglected.

As in the case of the vector of coefficients in the rotating system, $\mathbf{f}_{\alpha k}^{CR}$, it is also convenient to define a vector of order 11, of coefficients (C) in the hub system (H), $\mathbf{\Delta f}_\alpha^{CH}$:

$$\mathbf{\Delta f}_\alpha^{CH} = \{\Delta F_\alpha^{H0}, \Delta F_\alpha^{H1s}, \Delta F_\alpha^{H2s}, \ldots, \Delta F_\alpha^{H5s}, \Delta F_\alpha^{H1c}, \ldots, \Delta F_\alpha^{H5c}\}, \qquad \alpha \equiv x, y, z. \tag{3.7}$$

3.3. THE RESULTANT MOMENT ABOUT THE HUB CENTER THAT IS TRANSFERRED FROM THE ROTOR TO THE HUB

Each blade also transfers to the hub a moment, about the hub center. This moment presents in general the contribution of two effects, as follows.

(a) The blade is attached to the hub at a certain offset, denoted eR in Figure 3 (e itself is the "non-dimensional offset"). Thus, the forces that are transferred from the blade to the hub, at the attachment point, are exerting a moment about the hub center.

(b) While articulated blades transfer (to the hub) only negligible moments at the point of attachment, hingeless blades are capable of transferring significant moments.

As in the derivations in section 3.2, the resultant moment about the hub center, which is transferred from the rotor to the hub, is obtained by a vector summation of all the contributions of the individual blades. The resultant moment is described by its components in the hub system of co-ordinates: M_x^H, M_y^H and M_z^H. As in the case of the components of the resultant force (see equation (3.5)), the moment components are also described as the sum of nominal values, together with perturbations:

$$M_\alpha^H = M_{\alpha N}^H + \Delta M_\alpha^H, \qquad \alpha \equiv x, y, z. \tag{3.8}$$

The perturbations are described by Fourier series (see also equation (3.6)):

$$\Delta M_\alpha^H = \Delta M_\alpha^{H0} + \sum_{i=1}^{5} [\Delta M_\alpha^{His} \sin(i\psi) + \Delta M_\alpha^{Hic} \cos(i\psi)], \qquad \alpha \equiv x, y, z. \tag{3.9}$$

Examples of detailed expressions for some of the coefficients in equation (3.9), under certain assumptions, are presented in the Appendix.

As for the vector $\mathbf{\Delta f}_\alpha^{CH}$, defined in equation (3.7), it is also convenient to define vectors, of order 11, of the moment coefficients in the hub system:

$$\mathbf{\Delta m}_\alpha^{CH} = \{\Delta M^{H0}, \Delta M_\alpha^{H1s}, \Delta M_\alpha^{H2s}, \ldots, \Delta M_\alpha^{H5s}, \Delta M_\alpha^{H1c}, \ldots, \Delta M_\alpha^{H5c}\}, \qquad \alpha \equiv x, y, z. \quad (3.10)$$

4. THE INFLUENCE OF NON-UNIFORMITY ON THE LOADS TRANSFERRED FROM THE ROTOR TO THE HUB AND ON THE BLADES' FLAPPING

4.1. DEFINITIONS OF NON-UNIFORMITY

As indicated above, it is convenient to define a nominal blade (which actually may not exist). It is easy to look upon the nominal blade as the blade that was defined by the rotor designer: therefore the parameters that define this blade are called the *design parameters*. In general, each blade differs from the nominal one. The differences appear in the parameters that define the blade (geometric, inertia, structural and aerodynamic parameters). The differences between the parameters of a certain blade, and the nominal ones, are called the *perturbation parameters*, abbreviated to *perturbations*. The perturbation sources are as follows: (a) inaccuracies in the production procedure; (b) the evolution of defects during the helicopter operation; (c) intentional perturbations that are introduced by the ground crew.

The first two sources are unintentional, and result in what is denoted here the "*natural*" *perturbations*. The third source represents corrections that are introduced in order to reduce the effect of the natural perturbations. These are known as *correction perturbations*. The correction perturbations may include the angles of the trailing-edge tabs, pitch rod settings, and changes of the balance weights.

It is assumed that there are N_d parameters that define the natural perturbations of each blade. The natural perturbations of the kth blade are described by a vector of order N_d, $\mathbf{d}_k$:

$$\mathbf{d}_k = \{d_{1k}, d_{2k}, \ldots, d_{N_d k}\}, \qquad 0 \leqslant k \leqslant (b-1). \quad (4.1)$$

There are N_e correction parameters that define the correction perturbations, associated with each blade. The correction parameters of the kth blade are described by a vector of order N_e, $\mathbf{e}_k$:

$$\mathbf{e}_k = \{e_{1k}, e_{2k}, \ldots, e_{N_e k}\}, \qquad 0 \leqslant k \leqslant (b-1). \quad (4.2)$$

4.2. THE INFLUENCE OF NON-UNIFORMITY ON THE FORCES TRANSFERRED FROM AN INDIVIDUAL BLADE TO THE HUB

According to the assumptions of the present derivation, the force that is transferred from the kth blade to the hub, at any moment, is defined by a vector of coefficients of order 33, $\mathbf{f}_k^{CR}$ (see also equation (3.3)):

$$\mathbf{f}_k^{CR} = \{\mathbf{f}_{xk}^{CR}, \mathbf{f}_{yk}^{CR}, \mathbf{f}_{zk}^{CR}\}. \quad (4.3a)$$

In a similar manner, a vector of perturbations (relative to the nominal blade), $\mathbf{\Delta f}_k^{CR}$, is also defined:

$$\mathbf{\Delta f}_k^{CR} = \{\mathbf{\Delta f}_{xk}^{CR}, \mathbf{\Delta f}_{yk}^{CR}, \mathbf{\Delta f}_{zk}^{CR}\}. \quad (4.3b)$$

Again, the upper indices CR indicate that coefficients (C) in the rotating (R) system are considered. It is convenient to describe $\mathbf{\Delta f}_k^{CR}$ as comprised of the sum of two other vectors of order 33:

$$\mathbf{\Delta f}_k^{CR} = \mathbf{\Delta f}_k^{CR,D} + \mathbf{\Delta f}_k^{CR,E}. \quad (4.4)$$

The additional upper index D indicates that the natural (unintentional) force perturbations are considered, which are defined by the vector $\mathbf{d}_k$. The upper index E indicates that the correction (intentional) perturbations are considered, which are defined by the vector $\mathbf{e}_k$.

It is now assumed that, as in almost all practical cases, the vectors $\mathbf{d}_k$ and $\mathbf{e}_k$ represent small perturbations. Thus, if the perturbations in the forces are expressed as Taylor series in the elements of $\mathbf{d}_k$ and $\mathbf{e}_k$, then it is possible to consider only the linear terms in the perturbations: therefore

$$\Delta\mathbf{f}_k^{CR} = \mathbf{S}^{FCR,D}\mathbf{d}_k + \mathbf{S}^{FCR,E}\mathbf{e}_k, \tag{4.5}$$

where $\mathbf{S}^{FCR,D}$ and $\mathbf{S}^{FCR,E}$ are matrices of order $33 \times N_d$ and $33 \times N_e$, respectively, defined as

$$\mathbf{S}^{FCR,D} = \mathbf{f}_k^{CR}\{\partial/\partial d_{1k}, \partial/\partial d_{2k}, \ldots, \partial/\partial d_{N_d k}\}, \tag{4.6}$$

$$\mathbf{S}^{FCR,E} = \mathbf{f}_k^{CR}\{\partial/\partial e_{1k}, \partial/\partial e_{2k}, \ldots, \partial/\partial e_{N_e k}\}, \tag{4.7}$$

where

$$\mathbf{f}^{CR}\ (\partial/\partial q) \equiv (\partial/\partial q)\mathbf{f}^{CR}. \tag{4.8}$$

The matrices $\boldsymbol{S}^{FCR,D}$ and $\mathbf{S}^{FCR,E}$ are identical for all the blades and the index k in equation (4.6) can be chosen arbitrarily. All the derivatives are taken at the nominal state.

It should be noted that equation (4.5) is based on the assumption that the perturbations in the loads that are transferred from the kth blade to the hub are functions only of the perturbations associated with that specific blade; namely, the vectors $\mathbf{d}_k$ and $\mathbf{e}_k$. In other words, it is assumed that inter-blade influences on the vibrations are very small and thus can be neglected. Yet, it is well known that inter-blade mechanical [7, 8] or aerodynamic [9–12] couplings do exist and may have non-negligible influences. The addition of these influences to the present analysis is straightforward and does not present any difficulties, as will be indicated in what follows.

4.3. THE INFLUENCE OF NON-UNIFORMITY ON THE RESULTANT LOADS TRANSFERRED FROM THE ROTOR TO THE HUB

In section 3.2, a summation of all the forces transferred from the individual blades to the hub was presented. This summation is accompanied by co-ordinate transformation, from the rotating system (R) to the hub system (H). The same procedure can be carried out for the perturbations in the forces, which were described in section 4.2.

A similar derivation can be carried out for the moments that are transferred from the rotor to the hub.

It is convenient to define a vector of order 55, $\boldsymbol{\Delta\ell}^{CH}$, that includes the coefficients (C) of the perturbations in the loads that are transferred from the rotor to the hub, described by their components in the hub (H) system (see equations (3.7) and (3.10)):

$$\boldsymbol{\Delta\ell}^{CH} = \{\Delta\mathbf{f}_x^{CH}, \Delta\mathbf{f}_y^{CH}, \Delta\mathbf{f}_z^{CH}, \Delta\mathbf{m}_x^{CH}, \Delta\mathbf{m}_y^{CH}\}. \tag{4.9}$$

$\boldsymbol{\Delta\ell}^{CH}$ expresses, in fact, the vibrational loads that are exerted on the fuselage, due to non-uniformities of the blades. The vector of components $\Delta\mathbf{m}_z^{CH}$ is not included in equation (4.9), since it mainly affects the drive train dynamics and its contribution to the vibrations is less important.

After carrying out the summation and transformation from the rotating system of co-ordinates to the hub system, one obtains that

$$\Delta \ell^{CH} = \mathbf{S}^{LCH,D}\mathbf{d} + \mathbf{S}^{LCH,E}\mathbf{e}. \tag{4.10}$$

$\mathbf{S}^{LCH,D}$ and $\mathbf{S}^{LCH,E}$ are matrices of orders $55 \times bN_d$ and $55 \times bN_e$, respectively. $\mathbf{d}$ is a vector of order bN_d that describes the natural perturbations of all the blades:

$$\mathbf{d} = \{\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_{b-1}\}. \tag{4.11}$$

$\mathbf{e}$ is a vector of order bN_e, that describes the correction perturbations of all the blades:

$$\mathbf{e} = \{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{b-1}\}. \tag{4.12}$$

The matrices $\mathbf{S}^{LCH,D}$ and $\mathbf{S}^{LCH,E}$ are called the sensitivity matrices of the perturbations in the resultant loads that are transferred from the rotor to the hub, with respect to the natural perturbations and the correction perturbations, respectively. The matrices can be calculated by using detailed models of the rotor. Such models can differ in their complexity, concerning the dynamic, structural and aerodynamic aspects. On the other hand, these matrices can be determined on the basis of flight tests. Although flight tests may be much more complicated and expensive, their results are expected to be more accurate.

4.4. THE INFLUENCE OF NON-UNIFORMITY ON THE FLAPPING OF THE BLADES

As defined by equation (2.5), the vector $\Delta\boldsymbol{\beta}_k^C$ describes the deviation in the flapping of the kth blade, relative to the nominal one. As was done for equation (4.5), it is possible to obtain the following expression:

$$\Delta\boldsymbol{\beta}_k^C = \Delta\boldsymbol{\beta}_k^{CD} + \Delta\boldsymbol{\beta}_k^{CE} = \mathbf{S}^{\beta C,D}\mathbf{d}_k + \mathbf{S}^{\beta C,E}\mathbf{e}_k. \tag{4.13}$$

$\Delta\boldsymbol{\beta}_k^{CD}$ and $\Delta\boldsymbol{\beta}_k^{CE}$ are vectors of coefficients or order 11 that define the coefficients of the natural flapping perturbations and correction flapping perturbations of the kth blade.

Again, as in equation (4.5), equation (4.13) is also based on the assumption that inter-blade coupling effects are small and can be neglected.

The matrices $\mathbf{S}_k^{\beta C,D}$ and $\mathbf{S}_k^{\beta C,E}$ are of order $11 \times N_d$ and $11 \times N_e$, respectively. These matrices include partial derivatives as follows:

$$\mathbf{S}_k^{\beta C,D} = \boldsymbol{\beta}_k^C\{\partial/\partial d_{1k}, \partial/\partial d_{2k}, \ldots, \partial/\partial d_{N_d k}\}, \tag{4.14}$$

$$\mathbf{S}_k^{\beta C,E} = \boldsymbol{\beta}_k^C\{\partial/\partial e_{1k}, \partial/\partial e_{2k}, \ldots, \partial/\partial e_{N_e k}\}. \tag{4.15}$$

Since the matrices are identical for all the blades, there is not any influence of the choice of k. All of the derivations are calculated in the nominal state.

When tracking is considered, it is convenient to define the vector $\Delta\boldsymbol{\beta}^C$ as follows:

$$\Delta\boldsymbol{\beta}^C = \{\Delta\boldsymbol{\beta}_0^C, \Delta\boldsymbol{\beta}_1^C, \ldots, \Delta\boldsymbol{\beta}_{b-1}^C\}. \tag{4.16}$$

According to equation (4.13),

$$\Delta\boldsymbol{\beta}^C = \mathbf{S}^{\beta,D}\mathbf{d} + \mathbf{S}^{\beta,E}\mathbf{e}. \tag{4.17}$$

Vectors **d** and **e** were defined by equations (4.11) and (4.12), respectively. $\mathbf{S}^{\beta,D}$ and $\mathbf{S}^{\beta,E}$ are matries of order $11b \times N_d b$ and $11b \times N_e b$, respectively, where

$$\mathbf{S}^{\beta,D} = \begin{bmatrix} \mathbf{S}^{\beta C,D} & & & & \\ & \ddots & & & \\ & & \mathbf{S}^{\beta C,D} & & 0 \\ & & & \ddots & \\ 0 & & & & \mathbf{S}^{\beta C,D} \end{bmatrix}, \quad \mathbf{S}^{\beta,E} = \begin{bmatrix} \mathbf{S}^{\beta C,E} & & & & \\ & \ddots & & & \\ & & \mathbf{S}^{\beta C,E} & & 0 \\ & & & \ddots & \\ 0 & & & & \mathbf{S}^{\beta C,E} \end{bmatrix}. \quad (4.18, 4.19)$$

If inter-blade coupling effects are included, then the off-diagonal matrices in equations (4.18) and (4.19) become non-zero matrices.

5. MATHEMATICAL DEFINITIONS OF ROTOR TRACK AND BALANCE

5.1. BLADE TRACKING

When elastic deformations are negligible, the tip path of each blade is determined by the blade flapping angle, β_k. If elastic effects become important, then one can refer to all of the β_k as parameters that define the blade tip path itself (taking into account rigid flapping and elastic effects). Similar to the derivations in the previous sections, it is convenient to express β_k as the sum

$$\beta_k = \beta_N + \Delta\beta_k, \tag{5.1}$$

where β_N is the flapping of the nominal blade and $\Delta\beta_k$ is the perturbation. $\Delta\beta_k$ itself can be described as the sum of a natural perturbation $\Delta\beta_k^D$, and a correction perturbation $\Delta\beta_k^E$:

$$\Delta\beta_k = \Delta\beta_k^D + \Delta\beta_k^E. \tag{5.2}$$

Tracking means reducing the differences between the various $\Delta\beta_k$ to a minimum. In the present investigation it is assumed that tracking is aimed at bringing all the blades to a tip track that is as close as possible to the average of the natural perturbations of all the individual blades, $\Delta\beta_M^b$, where

$$\Delta\beta_M^b = \frac{1}{b}\sum_{k=0}^{b-1} \Delta\beta_k^D. \tag{5.3}$$

As indicated by equation (4.13), $\Delta\beta_k^D$ is defined by a vector of coefficients of order 11, $\mathbf{\Delta\beta}_k^{CD}$:

$$\mathbf{\Delta\beta}_k^{CD} = \{\Delta\beta_k^{D0}, \Delta\beta_k^{D1s}, \ldots, \Delta\beta_k^{D5s}, \Delta\beta_k^{D1c}, \ldots, \Delta\beta_k^{D5c}\}, \qquad 0 \leqslant k \leqslant b-1. \tag{5.4}$$

$\Delta\beta_M^b$ will also be defined by a vector of order 11, $\mathbf{\Delta\beta}_M^{bc}$:

$$\mathbf{\Delta\beta}_M^{bc} = \frac{1}{b}\left\{\sum_{k=0}^{b-1}\Delta\beta_k^{D0}, \sum_{k=0}^{b-1}\Delta\beta_k^{D1s}, \ldots, \sum_{k=0}^{b-1}\Delta\beta_k^{D5s}, \sum_{k=0}^{b-1}\Delta\beta_k^{D1c}, \ldots, \sum_{k=0}^{b-1}\Delta\beta_k^{D5c}\right\}. \tag{5.5}$$

The out-of-track vector of the rotor, $\mathbf{\Delta\beta}^{CT}$, is a vector of order $11b$ that is defined (see equation 4.17) as

$$\mathbf{\Delta\beta}^{CT} = \mathbf{\Delta\beta}^{C} - \mathbf{\Delta\beta}_M^{C} = \mathbf{S}^{\beta,D}\mathbf{d} + \mathbf{S}^{\beta,E}\mathbf{e} - \mathbf{\Delta\beta}_M^{C}. \tag{5.6}$$

Here $\mathbf{\Delta\beta}_M^{C}$ is a vector of order $11b$, defined as

$$\mathbf{\Delta\beta}_M^{C} = \{\mathbf{\Delta\beta}_M^{bc}, \mathbf{\Delta\beta}_M^{bc}, \ldots, \mathbf{\Delta\beta}_M^{bc}\}. \tag{5.7}$$

In the case of an ideally in-track rotor, the vector $\mathbf{\Delta\beta}^{CT}$ is the zero vector (all of its elements are equal to zero).

During rotor tracking the vector $\mathbf{d}$ is an input to the problem, while the vector of correction parameters $\mathbf{e}$ represents the vector of free variables. Usually, the order of the vector $\mathbf{e}$ is smaller than $11b$; therefore, in general, there does not exist a vector $\mathbf{e}$ that will make $\mathbf{\Delta\beta}^{CT}$ equal to zero. Instead, the quality of tracking is defined as the absolute value of the vector $\mathbf{\Delta\beta}^{CT}$. The smaller this value becomes, the closer the blades are to ideal tracking. It is conveninet to write equation (5.6) as

$$\mathbf{\Delta\beta}^{CT} = \mathbf{S}^{\beta,E}\mathbf{e} + \mathbf{g}, \tag{5.8}$$

where

$$\mathbf{g} = \mathbf{S}^{\beta,D}\,\mathbf{d} - \mathbf{\Delta\beta}^{C}_{M}. \tag{5.9}$$

Equation (5.8) is a typical least squares problem [13]: find a vector of correction perturbations $\mathbf{e}$, that will result in a minimum of the absolute value (Euclidean length) of the vector $\mathbf{\Delta\beta}^{CT}$.

5.2. BALANCING THE ROTOR

Balancing the rotor is defined as the procedure of reducing the loads that are transferred from the rotor to the hub, due to the non-uniformity of the blades. Equation (4.10) defines this procedure mathematically, and is written here again in a slightly different manner,

$$\mathbf{\Delta\ell}^{\mathrm{CH}} = \mathbf{S}^{LCH,E}\mathbf{e} + \mathbf{h}, \tag{5.10}$$

where $\mathbf{h}$ is a vector of order 55, defined as

$$\mathbf{h} = \mathbf{S}^{LCH,D}\mathbf{d}. \tag{5.11}$$

Thus, rotor balancing is the procedure of finding the vector $\mathbf{e}$ that will result in a minimum of the absolute value of the vector $\mathbf{\Delta\ell}^{CH}$.

6.3. MATHEMATICAL ASPECTS

As indicated above, in most of the practical cases the number of correction parameters is smaller than the number of equations. Thus, an overdetermined system of linear equations is obtained, that is solved by using a least squares method. As will be shown in reference [6], usually one would like to balance or track the rotor at various air speeds, a procedure that leads to a significant increase in the number of equations.

In the present investigation, the numerical method of Hanson [14] will be used in order to solve equations (5.8) or (5.10). Moreover, the method also allows a solution under certain constraints. Such constraints may include certain mathematical relations between the free variables, or certain inequalities that limit the range of variation of the free parameters. Constraints of the second kind will be applied in reference [6].

It should be pointed out that the relative importance of certain harmonics can be increased by multiplying the specific equations, representing these harmonics, by weighting coefficients that are larger than unity. The multiplication leads to smaller residual values in the case of these harmonics. An opposite trend is obtained as a result of using weighting coefficients smaller than unity.

6. CONCLUSIONS

Non-uniformity of helicopter blades results in vibrations at low frequencies. These vibrations result in increased fatigue of the crew, discomfort to passengers and maintenance and reliability problems for many helicopter parts and equipment. Non-uniformity of the blades appears as a result of manufacturing inaccuracies, but it is mainly caused by imperfections that grow during the operation of the helicopter.

Rotors have various devices that offer ways of reducing the blades non-uniformity. This is done by introducing intentional non-uniformities that are supposed to cancel the effects of the natural non-uniformities.

Since non-uniformity among blades also leads to out-of-track blades, it is common to use this fact as a means of reducing the rotor non-uniformity. Thus, instead of directly reducing the vibrations, in many cases the ground crew determines the correction parameters in such a way that the blades' out-of-track will decrease to a minimum. Unfortunately, it turns out that this may not be the optimal way of reducing vibrations.

In the present paper, a mathematical approach to track and balance was presented. Both operations have been defined mathematically in a general manner. The definitions are based on a least squares formulation, which allows any combination of equations and correction parameters. This mathematical model offers the means of a thorough investigation of tracking or balancing, and their relationship. Such an investigation is presented in the second part of this paper [6].

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APPENDIX: DETAILED EXPRESSIONS OF THE LOAD'S COMPONENTS THAT ARE TRANSFERRED FROM THE ROTOR TO THE HUB

In the following expressions, harmonics higher than the fifth have been neglected. In the case of the x_H and y_H components of the force or the moment, only the constant term and sine harmonics are presented. In the case of the force component in the z_H direction, all the terms are presented:

$$\Delta F_x^{H0} = \tfrac{1}{2}(-\Delta F_{x0}^{R1c} + \Delta F_{y0}^{R1s} - \Delta F_{x1}^{R1c} + \Delta F_{y1}^{R1s} - \Delta F_{x2}^{R1c} + \Delta F_{y2}^{R1s} - \Delta F_{x3}^{R1c} + \Delta F_{y3}^{R1s}), \tag{A1a}$$

$$\begin{aligned}\Delta F_x^{H1s} = \tfrac{1}{2}(&-\Delta F_{x0}^{R2s} + 2\Delta F_{y0}^{R0} - \Delta F_{y0}^{R2c} + 2\Delta F_{x1}^{R0} + \Delta F_{x1}^{R2c} - \Delta F_{y1}^{R2s} + \Delta F_{x2}^{R2s}\\ &- 2\Delta F_{y2}^{R0} + \Delta F_{y2}^{R2c} - 2\Delta F_{x3}^{R0} - \Delta F_{x3}^{R2c} + \Delta F_{y1}^{R2s}),\end{aligned} \tag{A1b}$$

$$\begin{aligned}\Delta F_x^{H2s} = \tfrac{1}{2}(&-\Delta F_{x0}^{R1s} - \Delta F_{x0}^{R3s} + \Delta F_{y0}^{R1c} - \Delta F_{y0}^{R3c} + \Delta F_{x1}^{R1s} + \Delta F_{x1}^{R3s} - \Delta F_{y1}^{R1c} + \Delta F_{y1}^{R3c}\\ &- \Delta F_{x2}^{R1s} - \Delta F_{x2}^{R3s} + \Delta F_{y2}^{R1c} - \Delta F_{y2}^{R3c} + \Delta F_{x3}^{R1s} + \Delta F_{x3}^{R3s} - \Delta F_{y3}^{R1c} + \Delta F_{y3}^{R3c}),\end{aligned} \tag{A1c}$$

$$\begin{aligned}\Delta F_x^{H3s} = -\tfrac{1}{2}(&\Delta F_{x0}^{R2s} + \Delta F_{x0}^{R4s} - \Delta F_{y0}^{R2c} + \Delta F_{y0}^{R4c} + \Delta F_{x1}^{R2c} + \Delta F_{x1}^{R4c} + \Delta F_{y1}^{R2s} - \Delta F_{y1}^{R4c}\\ &- \Delta F_{x2}^{R2s} - \Delta F_{x2}^{R4s} + \Delta F_{y2}^{R2c} - \Delta F_{y2}^{R4c} - \Delta F_{x3}^{R2c} - \Delta F_{x3}^{R4c} - \Delta F_{y3}^{R2s} + \Delta F_{y3}^{R4s}),\end{aligned} \tag{A1d}$$

$$\begin{aligned}\Delta F_x^{H4s} = -\tfrac{1}{2}(&\Delta F_{x0}^{R3s} + \Delta F_{x0}^{R5s} - \Delta F_{y0}^{R3c} + \Delta F_{y0}^{R5c} + \Delta F_{x1}^{R3s} + \Delta F_{x1}^{R5s} - \Delta F_{y1}^{R3c} + \Delta F_{y1}^{R5c}\\ &+ \Delta F_{x2}^{R3s} + \Delta F_{x2}^{R5s} - \Delta F_{y2}^{R3c} + \Delta F_{y2}^{R5c} + \Delta F_{x3}^{R3s} + \Delta F_{x3}^{R5s} - \Delta F_{y3}^{R3c} + \Delta F_{y3}^{R5c}),\end{aligned} \tag{A1e}$$

$$\Delta F_x^{H5s} = -\tfrac{1}{2}(\Delta F_{x0}^{R4s} - \Delta F_{y0}^{R4c} - \Delta F_{x1}^{R4c} - \Delta F_{y1}^{R4s} - \Delta F_{x2}^{R4s} + \Delta F_{y2}^{R4c} + \Delta F_{x2}^{R4c} + \Delta F_{y2}^{R4s}); \tag{A1f}$$

$$\Delta F_y^{H0} = -\tfrac{1}{2}(\Delta F_{x0}^{R1s} + \Delta F_{y0}^{R1c} + \Delta F_{x1}^{R1s} + \Delta F_{y1}^{R1c} + \Delta F_{x2}^{R1s} + \Delta F_{y2}^{R1c} + \Delta F_{x3}^{R1s} + \Delta F_{y3}^{R1c}), \tag{A2a}$$

$$\begin{aligned}\Delta F_y^{H1s} = -\tfrac{1}{2}(&2\Delta F_{x0}^{R0} - \Delta F_{x0}^{R2c} + \Delta F_{y0}^{R2s} - \Delta F_{x1}^{R2s} - 2\Delta F_{y1}^{R0} - \Delta F_{y1}^{R2c} - 2\Delta F_{x2}^{R0} + \Delta F_{x2}^{R2c}\\ &- \Delta F_{y2}^{R2s} + \Delta F_{x3}^{R2s} + 2\Delta F_{y3}^{R0} + \Delta F_{y1}^{R2c}),\end{aligned} \tag{A2b}$$

$$\begin{aligned}\Delta F_y^{H2s} = \tfrac{1}{2}(&-\Delta F_{x0}^{R1c} + \Delta F_{x0}^{R3c} - \Delta F_{y0}^{R1s} - \Delta F_{y0}^{R3s} + \Delta F_{x1}^{R1c} - \Delta F_{x1}^{R3c} + \Delta F_{y1}^{R1s} + \Delta F_{y1}^{R3s}\\ &- \Delta F_{x2}^{R1c} + \Delta F_{x2}^{R3c} - \Delta F_{y2}^{R1s} - \Delta F_{y2}^{R3s} + \Delta F_{x3}^{R1c} - \Delta F_{x3}^{R3c} + \Delta F_{y3}^{R1s} + \Delta F_{y3}^{R3s}),\end{aligned} \tag{A2c}$$

$$\begin{aligned}\Delta F_y^{H3s} = \tfrac{1}{2}(&-\Delta F_{x0}^{R2c} + \Delta F_{x0}^{R4c} - \Delta F_{y0}^{R2s} - \Delta F_{y0}^{R4s} + \Delta F_{x1}^{R2s} - \Delta F_{x1}^{R4s} - \Delta F_{y1}^{R2c} - \Delta F_{y1}^{R4c}\\ &- \Delta F_{x2}^{R2c} - \Delta F_{x2}^{R4c} + \Delta F_{y2}^{R2s} + \Delta F_{y2}^{R4s} - \Delta F_{x3}^{R2s} + \Delta F_{x3}^{R4s} + \Delta F_{y3}^{R2c} + \Delta F_{y3}^{R4c}),\end{aligned} \tag{A2d}$$

$$\begin{aligned}\Delta F_y^{H4s} = -\tfrac{1}{2}(&\Delta F_{x0}^{R3c} - \Delta F_{x0}^{R5c} + \Delta F_{y0}^{R3s} + \Delta F_{y0}^{R5s} + \Delta F_{x1}^{R3c} - \Delta F_{x1}^{R5c} + \Delta F_{y1}^{R3s} + \Delta F_{y1}^{R5s}\\ &+ \Delta F_{x2}^{R3c} - \Delta F_{x2}^{R5c} + \Delta F_{y2}^{R3s} + \Delta F_{y2}^{R5s} + \Delta F_{x3}^{R3c} - \Delta F_{x3}^{R5c} + \Delta F_{y3}^{R3c} + \Delta F_{y3}^{R5s}),\end{aligned} \tag{A2e}$$

$$\Delta F_y^{H5s} = -\tfrac{1}{2}(\Delta F_{x0}^{R4c} + \Delta F_{y0}^{R4s} + \Delta F_{x1}^{R4s} - \Delta F_{y1}^{R4c} - \Delta F_{x2}^{R4c} - \Delta F_{y2}^{R4s} - \Delta F_{x2}^{R4s} + \Delta F_{y2}^{R5c}); \tag{A2f}$$

$$\Delta F_z^{H0} = \Delta F_{z0}^{R0} + \Delta F_{z1}^{R0} + \Delta F_{z2}^{R0} + \Delta F_{z3}^{R0}, \tag{A3a}$$

$$\Delta F_z^{H1s} = \Delta F_{z0}^{R1s} - \Delta F_{z1}^{R1c} - \Delta F_{z2}^{R1s} + \Delta F_{z3}^{R1c}, \tag{A3b}$$

$$\Delta F_z^{H2s} = \Delta F_{z0}^{R2s} - \Delta F_{z1}^{R2c} + \Delta F_{z2}^{R2s} - \Delta F_{z3}^{R2s}, \tag{A3c}$$

$$\Delta F_z^{H3s} = \Delta F_{z0}^{R3s} + \Delta F_{z1}^{R3c} - \Delta F_{z2}^{R3s} - \Delta F_{z3}^{R3c}, \tag{A3d}$$

$$\Delta F_z^{H4s} = \Delta F_{z0}^{R4s} + \Delta F_{z1}^{R4s} + \Delta F_{z2}^{R4s} + \Delta F_{z3}^{R4s}, \tag{A3e}$$

$$\Delta F_z^{H5s} = \Delta F_{z0}^{R5s} - \Delta F_{z1}^{R5c} - \Delta F_{z2}^{R5s} + \Delta F_{z3}^{R5c}, \tag{A3f}$$

$$\Delta F_z^{H1c} = \Delta F_{z0}^{R1c} + \Delta F_{z1}^{R1s} - \Delta F_{z2}^{R1c} - \Delta F_{z3}^{R1s}, \tag{A3g}$$

$$\Delta F_z^{H2c} = \Delta F_{z0}^{R2c} - \Delta F_{z1}^{R2c} + \Delta F_{z2}^{R2c} - \Delta F_{z3}^{R2c}, \tag{A3h}$$

$$\Delta F_z^{H3c} = \Delta F_{z0}^{R3c} - \Delta F_{z1}^{R3s} - \Delta F_{z2}^{R3c} + \Delta F_{z3}^{R3s}, \tag{A3i}$$

$$\Delta F_z^{H4c} = \Delta F_{z0}^{R4c} + \Delta F_{z1}^{R4c} + \Delta F_{z2}^{R4c} + \Delta F_{z3}^{R4c}, \tag{A3j}$$

$$\Delta F_z^{H5c} = \Delta F_{z0}^{R5c} + \Delta F_{z1}^{R5s} - \Delta F_{z2}^{R5c} - \Delta F_{z3}^{R5s}. \tag{A3k}$$

In the case of the moment components only contributions due to the offset are presented:

$$\Delta M_x^{H0} = \tfrac{1}{2} eR(\Delta F_{z0}^{R1s} + \Delta F_{z1}^{R1s} + \Delta F_{z2}^{R1s} + \Delta F_{z3}^{R1s}), \tag{A4a}$$

$$\Delta M_x^{H1s} = -\tfrac{1}{2} eR(2\Delta F_{z0}^{R0} - \Delta F_{z0}^{R2c} - \Delta F_{z1}^{R2s} - 2\Delta F_{z2}^{R0} + \Delta F_{z2}^{R2c} + \Delta F_{z3}^{R2s}), \tag{A4b}$$

$$\Delta M_x^{H2s} = -\tfrac{1}{2} eR(\Delta F_{z0}^{R1c} - \Delta F_{z0}^{R3c} - \Delta F_{z1}^{R1c} + \Delta F_{z1}^{R3c} + \Delta F_{z2}^{R1c} - \Delta F_{z2}^{R3c} - \Delta F_{z3}^{R1c} + \Delta F_{z3}^{R3c}), \tag{A4c}$$

$$\Delta M_x^{H3s} = -\tfrac{1}{2} eR(\Delta F_{z0}^{R2c} - \Delta F_{z0}^{R4c} - \Delta F_{z1}^{R2s} + \Delta F_{z1}^{R4s} - \Delta F_{z2}^{R2c} + \Delta F_{z2}^{R4c} + \Delta F_{z3}^{R2s} - \Delta F_{z3}^{R4s}), \tag{A4d}$$

$$\Delta M_x^{H4s} = \tfrac{1}{2} eR(-\Delta F_{z0}^{R3c} + \Delta F_{z0}^{R5c} - \Delta F_{z1}^{R3c} + \Delta F_{z1}^{R5c} - \Delta F_{z2}^{R3c} + \Delta F_{z2}^{R5c} - \Delta F_{z3}^{R3c} + \Delta F_{z3}^{R5c}), \tag{A4e}$$

$$\Delta M_x^{H5s} = \tfrac{1}{2} eR(-\Delta F_{z0}^{R4c} - \Delta F_{z1}^{R4s} + \Delta F_{z2}^{R4c} - \Delta F_{z3}^{R4s}), \tag{A4f}$$

$$\Delta M_y^{H0} = \tfrac{1}{2} eR(\Delta F_{z0}^{R1c} + \Delta F_{z1}^{R1c} + \Delta F_{z2}^{R1c} + \Delta F_{z3}^{R1c}), \tag{A5a}$$

$$\Delta M_y^{H1s} = \tfrac{1}{2} eR(\Delta F_{z0}^{R2s} - 2\Delta F_{z1}^{R0} - \Delta F_{z1}^{R2c} - \Delta F_{z2}^{R2s} + 2\Delta F_{z3}^{R0} + \Delta F_{z3}^{R2c}), \tag{A5b}$$

$$\Delta M_y^{H2s} = -\tfrac{1}{2} eR(\Delta F_{z0}^{R1s} - \Delta F_{z0}^{R3s} + \Delta F_{z1}^{R1s} + \Delta F_{z1}^{R3s} - \Delta F_{z2}^{R1s} - \Delta F_{z2}^{R3s} + \Delta F_{z3}^{R1s} + \Delta F_{z3}^{R3s}), \tag{A5c}$$

$$\Delta M_y^{H3s} = -\tfrac{1}{2} eR(\Delta F_{z0}^{R2s} - \Delta F_{z0}^{R4s} - \Delta F_{z1}^{R2c} - \Delta F_{z1}^{R4c} + \Delta F_{z2}^{R2s} + \Delta F_{z2}^{R4s} + \Delta F_{z3}^{R2c} + \Delta F_{z3}^{R4c}), \tag{A5d}$$

$$\Delta M_y^{H4s} = \tfrac{1}{2} eR(\Delta F_{z0}^{R3s} + \Delta F_{z0}^{R5s} + \Delta F_{z1}^{R3s} + \Delta F_{z1}^{R5s} + \Delta F_{z2}^{R3s} + \Delta F_{z2}^{R5s} + \Delta F_{z3}^{R3s} + \Delta F_{z3}^{R5s}), \tag{A5e}$$

$$\Delta M_y^{H5s} = \tfrac{1}{2} eR(\Delta F_{z0}^{R4s} - \Delta F_{z1}^{R4c} - \Delta F_{z2}^{R4s} + \Delta F_{z3}^{R4c}). \tag{A5f}$$