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LETTERS TO THE EDITOR

COMMENTS ON "DYNAMIC BEHAVIOR OF BEAMS AND RECTANGULAR PLATES UNDER MOVING LOADS"

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1. Introduction

In a recent article, a theory for dynamic analysis of elastic structures subjected to moving mass load was developed [1]. The solution technique is based on the modified generalized finite integral transforms and the modified Struble's method. While the development is interesting, the concluding remarks drawn based on the very limited set of analysis results are inconclusive and may be misleading. For dynamic analysis of elastic structures traversed by moving mass load, the effects of two important parameters, the moving speed and the mass ratio between the moving load and the support structure, must be closely examined to extract useful information and to establish sound conclusions from the analytical work. This examination is absent in [1], which results in misleading conclusions due to insufficient analysis information. This concern will be addressed in the next section of this letter.

The claim of the developed theory being applicable to the analysis of thick beams by including the rotary inertia term is also questionable. As is well known, the shear effect cannot be neglected for accurate analysis of thick beam structures. It has been shown that the correction due to shear can be more than three times larger than the correction due to rotary inertia for computing natural frequencies of a beam with rectangular cross-section [2]. Hence, the inclusion of the rotary inertia term only may not yield accurate analysis results for thick beams. Note that the radius of gyration term is incorrectly given in Eq. (2.1). The term H_2 should be H_2^2 instead. It is worth noting that, as presented in Figure 4, while the consideration of rotary inertia effect for beam with $H_2 = 0{\cdot}031$ has little improvement in analysis accuracy when compared to the classical Bernoulli–Euler beam theory, the applicability of the Rayleigh beam theory for analysis of a beam with $H_2 = 8$ is to be challenged. A radius of gyration so large corresponds to the situation of thickness to length ratio being equal to 4·619 for a beam with rectangular cross section! The validity of the Rayleigh beam theory for this case needs to be established first by comparing the analysis results with those obtained using higher dimensional theory or experimental work, otherwise the results shown in Figure 4 are imaginary and do not represent the physical facts. Some investigations [3–6] have been made to validate the Timoshenko beam theory, which not only includes the rotary inertia term, but also considers the shearing deformations. However, applicability of the theory for a beam with such a high thickness to length ratio has not been established.

2. Analysis

The recent development of finite element techniques for dynamic analysis of elastic structures traversed by moving loads is not appropriately discussed in [1]. In fact, the

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differences between the moving force and the moving mass problems can be easily identified using the finite element approach. The moving mass problem can be conveniently analyzed by applying the approach as shown in the work by Lin and Trethewey [7], where a general treatment of moving load problems was presented. The general equation of motion developed in [7] can be easily reduced to a form suitable for dynamic analysis of elastic structures traversed by moving mass load. The governing equation of motion can be written as:

$$[[\mathrm{M}] + [m^*]]\{\ddot{d}\} + [[\mathrm{C}] + [c^*]]\{\dot{d}\} + [[\mathrm{K}] + [k^*]]\{d\} = \lfloor N \rfloor^{\mathrm{T}} \mathrm{mg}, \tag{1}$$

where [M], [C], and [K] are the structural mass, damping, and stiffness matrices respectively. m is the mass of the moving load and g is the gravitational constant. $\lfloor N \rfloor^{\mathrm{T}}$ is the transposition of the shape functions evaluated at the position of the moving load. $\{d\}$, $\{\dot{d}\}$, and $\{\ddot{d}\}$, denote the nodal displacement, velocity, and acceleration vectors respectively. And

$$[m]^* = m\lfloor N \rfloor^{\mathrm{T}}\lfloor N \rfloor, \qquad [c]^* = 2m\dot{x}\lfloor N \rfloor^{\mathrm{T}}\lfloor N \rfloor_x, \tag{2, 3}$$

$$[k]^* = m\dot{x}^2\lfloor N \rfloor^{\mathrm{T}}\lfloor N \rfloor_{xx} + m\ddot{x}\lfloor N^{\mathrm{T}} \rfloor\lfloor N \rfloor_x. \tag{4}$$

in which x is the longitudinal co-ordinate. A subscript denotes differentiation with respect to space and an overdot represents differentiation with respect to time. Note that if the effects of shearing deformations and rotary inertia are to be considered, the appropriate forms for the structural matrices, as developed in [8], can be applied. Apparently, the qualitative differences between the moving mass and the moving force problems is that the moving mass problem involves the time and space dependent moving sub-matrices $[m^*]$, $[c^*]$, and $[k^*]$, in addition to the external moving force shown in the right side of equation (1). Therefore, if the effects of the moving sub-matrices are small, the simple moving force problem will be a good approximation to the complicated moving mass problem. This situation occurs when the mass of the moving mass load is relatively small in comparison with the mass of the support structure, and the velocity and acceleration of the moving load are low.

To demonstrate the inappropriate conclusions drawn in [1], the quantitative analysis results obtained from both the moving mass and the moving force problems are presented subsequently. In Figure 1 are shown the normalized dynamic displacements at the moving load position for a simply supported beam under various moving loads with a constant speed. The abscissa denotes the normalized position of the moving load, in which τ is the time required for the moving load to across the beam span. The moving speed parameter T_f/τ is 1, where T_f is the fundamental period of the simply supported beam. For $\beta = 0{\cdot}01$, where β denotes the mass ratio between the moving load and the support beam, there is little deviation between the moving mass and the moving force analysis results. Significant deviation is observed as the mass ratio is increased. For the case of higher mass ratios, the peak responses tend to get closer to the right end of the support beam and the normalized magnitudes become lower. This may lead to a misconception that a higher mass ratio results in less dynamic impact on the support structure. On the contrary, as illustrated in Figure 2 where the normalized dynamic displacements at the beam center are plotted, the case with a higher mass ratio results in significantly larger dynamic impact on the support structure than that with a lower one. It is worth noting that while the response of the beam traversed by a moving force is primarily due to the contribution of the fundamental mode [2], the response of the beam excited by a moving mass with a high mass ratio, e.g. $\beta = 5$, clearly demonstrates the significant participation of higher modes. The free response is found to be of greater interest than the forces response. While response

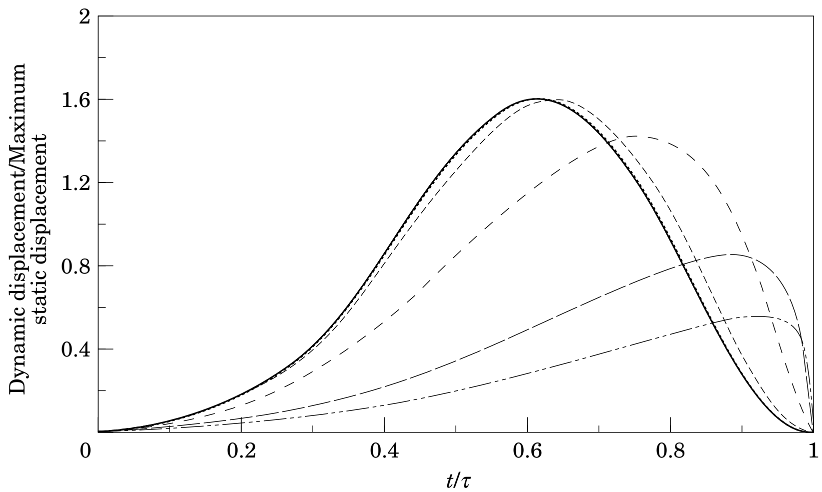

Figure 1. Normalized dynamic displacements at the moving load position for a simply supported beam under various moving loads with a constant speed, $T_f/\tau = 1$. ——, moving force, β: · · ·, 0·01; - - - -, 0·1; – – – –, 1·0; — — —, 5·0; — - - —, 10·0.

analysis at the moving load position may be useful for high speed machining applications, such as ballistic machining, it not only does not accurately represent the severity of impact of the moving load on the support structure, but also may result in misleading conclusions. The concluding remarks about the response amplitude of a simply supported Bernoulli–Euler beam for the moving mass problem being less than that for the moving force problem, as made in [1], do not hold good in general. A similar analysis can also be conducted for the case of a support beam with fixed-fixed ends and the case of a support plate.

Another important issue to be addressed concerning the moving load problems is the effect of the moving speed, which is absent in [1]. In Figure 3, the maximum dynamic displacements at the beam center normalized by the corresponding maximum static

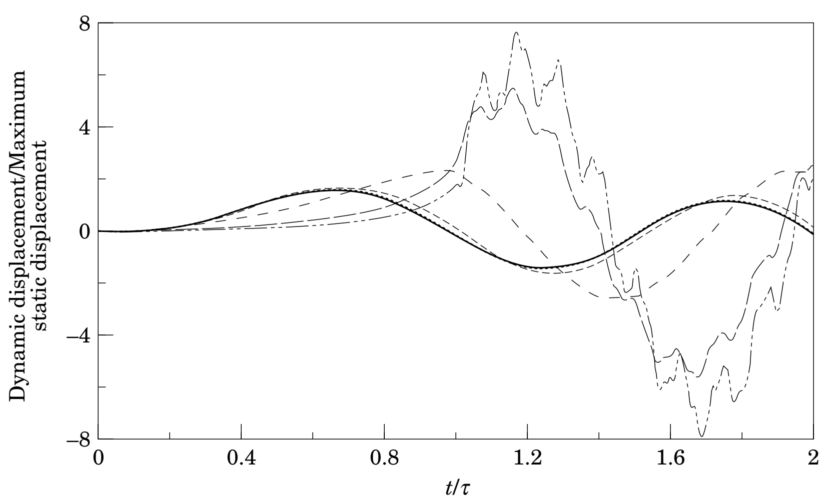

Figure 2. Normalized dynamic displacements at the beam center for a simply supported beam under various moving loads with a constant speed, $T_f/\tau = 1$. Key as Figure 1.

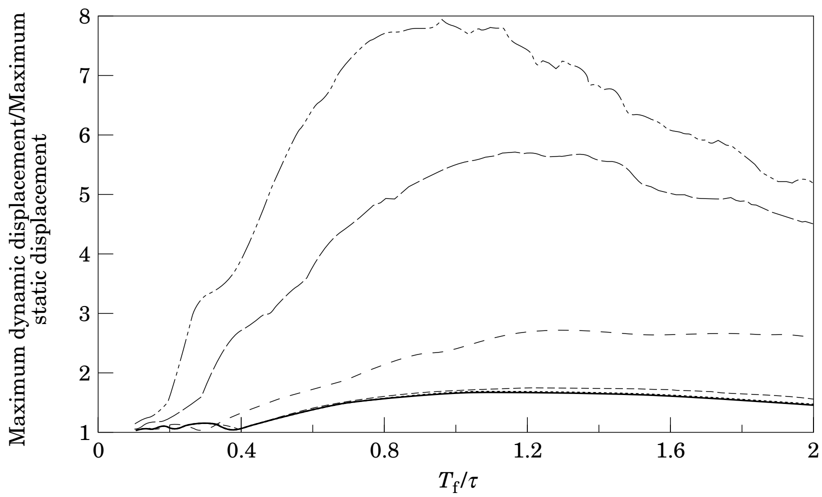

Figure 3. Normalized maximum dynamic displacements at the beam center for a simply supported beam under various moving loads. Key as Figure 1.

displacements are plotted against the moving speed parameter, T_f/τ. Note that in determining the maximum dynamic displacements, the free response region must be taken into account since vibration of the support beam may be more severe in this region than in that of the forced vibration for the case of a moving load with a high mass ratio. Evidently, there are not many differences between the moving mass and moving force problems when the moving speed is very low or when the mass ratio is small, e.g., $\beta = 0{\cdot}01$. With the increase of moving load speed, the dynamic impact on the support structure increases, reaches a maximum, and then decreases. The response curves become more irregular for higher mass ratios. The quantitative results presented here agree with the qualitative assessment made previously.

3. CONCLUSIONS

Dynamic response of elastic structures traversed by moving mass loads is a very complicated function of both the mass ratio between the moving mass and the support structure and the speed of the moving load. Any analysis of moving load problems must address fully the effects of these parameters to extract useful information for engineering design and to avoid possible misleading conclusions. For the case of low moving speed or low mass ratio, the simple moving force model can be a good approximation to the complex moving mass problems. For the case of a high mass ratio and a high moving speed, the moving force model cannot be applied and the complicated time-variant system analysis for the moving mass problems must be conducted to ensure an accurate investigation. For thick beam analysis, both the effects of shearing deformations and rotary inertia should be considered for accurate analysis. The range of applicability of this advanced beam theory should be carefully scrutinized to avoid its misuse.

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AUTHOR'S REPLY

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The authors appreciate the interest shown in the commented manuscript by Professor Lin. His criticism centers mainly on the numerical analysis (and in particular on the concluding remarks drawn from the analysis) presented in section 6 of the article. The criticism is based on the following few points:

(1) The claim made on p. 694 of the paper [1] namely, "This technique is applicable to . . . , as well as both thin and thick beams", by including the rotatory inertia term, is questionable.

(2) The concluding remarks drawn from the numerical analysis may be misleading as they are based on few numerical results.

(3) The numerical analysis in which a radius of gyration $H_2 = 8$ (whose thickness to length radius equals 4·619) is used and whose result is presented in Figure 4 of [1] is to be challenged, as the applicability of the theory for a beam with such high thickness to length ratio has not been established. Also the radius of gyration term in equation (2.1) of reference [1] is not correct. In fact, the term H_2 should be H_2^2.

The author's reactions (responses) are trivial and are as follows:

(a) The technique developed is, indeed, applicable to Bernoulli–Euler beams, Rayleigh beams and Timoshenko (thick) beams. The commented article gives a detailed account of the applicability of the technique to the first two types of beams. The corresponding detailed account for Timoshenko beams is reported in [2].

To demonstrate the applicability of the technique to Timoshenko beams, a brief qualitative analysis presented in [2] is hereby given.

The coupled equations for the deflection $y(x, t)$ and the bending slope $\Phi(x, t)$ of a uniform Timoshenko beam are given as [3].

$$KAG(\phi_{,x} - y_{,xx}) + \rho A y_{,tt} = P(x, t), \qquad EI\phi_{,xx} - KAG(\phi - y_{,x}) - \rho I \phi_{,tt} = 0 \quad (1, 2)$$

in which E = modulus of elasticity, G = modulus of rigidity, I = moment of inertia of the cross-section, A = cross-sectional area, ρ = *mass per unit volume*, K = numerical shape factor for cross-section, t = time, x = the position co-ordinate in the axial direction, $P(x, t)$ = the applied moving load. Subscripts following a comma denote partial differentiation. Any of the following classical boundary conditions

$$\text{Hinged} \quad y(x, t) = 0 = \phi_{,x}(x, t), \qquad \text{Clamped} \quad y(x, t) = 0 = \phi(x, t) \tag{3}$$

may be considered.

The associated initial conditions are:

$$y(x, 0) = 0 = y_{,t}(x, 0), \qquad \phi(x, 0) = 0 = \phi_{,t}(x, 0) \tag{4}$$

To solve the above initial boundary value problem (1)–(4), reference [1] is followed and thereby the following generalized finite integral transforms are introduced:

$$\bar{y}(m, t) = \int_0^L y(x, t) y_m(x)\,\mathrm{d}x; \qquad y(x, t) = \sum_{m=1}^{\infty} \frac{\rho A}{G_m} \bar{y}(m, t) y_m(x) \tag{5a}$$

$$\bar{\phi}(m, t) = \int_0^L \phi(x, t)\phi_m(x)\,\mathrm{d}x; \qquad \phi(x, t) = \sum_{m=1}^{\infty} \frac{\rho A}{F_m} \bar{\phi}(m, t)\phi_m(x), \quad m = 1, 2, 3, \ldots, \tag{5b}$$

where

$$G_m = \int_0^L \rho A y_m^2(x)\,\mathrm{d}x; \qquad F_m = \int_0^L \rho A \phi_m^2(x)\,\mathrm{d}x,$$

$$y_m(x) = A_m \cosh \lambda_m x + B_m \sinh \lambda_m x + C_m \cos \beta_m x + P_m \sin \beta_m x,$$

$$\phi_m(x) = D_m \cosh \lambda_m x + E_m \sinh \lambda_m x + H_m \cos \beta_m x + Q_m \sin \beta_m x$$

are the mth normal modes [2] of a uniform Timoshenko beam of length L. The constants $A_m, B_m, \ldots, Q_m$ are related as follows;

$$A_m = (1/d\gamma)[1 - d^2q^2(\gamma^2 + u^2)]D_m, \qquad B_m = (1/d\gamma)[1 - d^2q^2(\gamma^2 + u^2)]E_m,$$

$$C_m = -(1/d\varepsilon)[1 + d^2q^2(\varepsilon^2 - u^2)]H_m, \qquad P_m = -(1/d\varepsilon)[1 + d^2q^2(\varepsilon^2 - u^2)]Q_m,$$

where

$$d^2 = \rho A \omega_m^2 / EI, \qquad q^2 = EI/KAG, \qquad u^2 = I/A, \qquad \omega_m = \text{the circular frequency},$$

$$\begin{matrix}\gamma \\ \varepsilon\end{matrix} = (1/\sqrt{2})[\mp(u^2 + q^2) + [(u^2 - q^2)^2 + 4/d^2]^{1/2}]^{1/2},$$

and it is assumed that

$$[(u^2 - q^2)^2 + 4/d^2]^{1/2} > (u^2 + q^2).$$

The value of the constants λ_m and β_m are obtained by using the relevant boundary conditions.

Upon applying the generalized finite integral transform (5a), equation (1) becomes

$$\rho A \bar{y}_{,tt}(m, t) + \rho A \omega^2 \bar{y}(m, t) - KAGg_1(t) - kAGg_2(t) = \bar{P}(m, t), \tag{6}$$

where

$$g_1(t) = \sum_{k=1}^{\infty} \frac{\rho A}{G_k} \Delta_1(m, k)\bar{y}(k, t), \qquad g_2(t) = \sum_{k=1}^{\infty} \frac{\rho A}{F_k} \Delta_2(m, k)\bar{\phi}(k, t),$$

$$\Delta_1(m, k) = \int_0^L \left[\frac{\mathrm{d}\phi_m(x)}{\mathrm{d}x}\right] y_k(x)\,\mathrm{d}x, \qquad \Delta_2(m, k) = \int_0^L \left[\frac{\mathrm{d}y_m(x)}{\mathrm{d}x}\right] \phi_k(x)\,\mathrm{d}x$$

and $\bar{P}(m, t)$ is the generalized integral transform of the moving load $P(x, t)$. On the other hand, applying the generalized finite integral transform (5b), equation (2) reduces to

$$\rho I \bar{\phi}_{,tt}(m, t) + \rho I \omega^2 \bar{\phi}(m, t) + KAGg_1(t) + KAGg_2(t) = 0. \quad (7)$$

It should be noted, at this juncture, that each of equations (6) and (7) is similar to either equation (3.5) or (3.6) of reference [1]. In particular each of them is coupled. Thus, each of them could be solved using arguments similar to those in [1]. In other words, they could be solved (see [2]) by resorting to the approximate analytical method which is a modification of the asymptotic method due to Struble [1]. Hence, the technique developed in [1] is also applicable to thick beams.

(b) It should be pointed out, at this juncture, that the manuscript subject to comment presents a new analytical technique developed for the dynamic analysis of finite elastic Rayleigh beams and compares it with existing techniques applied to numerically identical models taking, *for the purpose of comparison*, the *mass ratio* per unit length to be 0·2 and speed $v = 6$ m/s as in [4–7]. This numerical analysis (based on these two values and of course, values of other specified parameters) yields results, from which certain observations, which were summarised as part of the concluding remarks, were made. It is interesting to note that Professor Lin's numerical results for the mass ratio chosen in [1] agree with the findings in the latter. Besides, a whole section is devoted to a qualitative discussion of the critical speeds (apart from the speed $v = 6$ m/s). As a matter of fact, it was known to the authors [7–9] that for different values of mass ratio and speed the behaviour of the moving load problem is different and hence lead to different concluding remarks.

Consequently, the authors welcome the extension of the numerical analysis in [1] involving the consideration of the values of some other mass ratios and speeds, provided by Professor Lin, as well as the resulting concluding remarks.

(c) The term H_2 in equation (2.1) of reference [1] is correct. In other words, it should not be H_2^2. Nevertheless, to the author's regret, some mistakes exist which went unnoticed in the article subject to comment. These are (i) the phrase "H_2 is the radius of gyration of the cross-section" (see p. 678, third paragraph, 8th line of [1]) has to be either "H_2 is the rotatory inertia term" (as correctly stated on p. 693, 694) or "H_2 is the square of radius of gyration of the cross-section" and (ii) "a radius of gyration of the cross-section $H_2 = 0{\cdot}03096$ m" of p. 690 section 6, line 2 should read "the rotatory inertia term $H_2 = 0{\cdot}03096$ m^2". The authors request the readers' pardon. As a matter of fact, while dealing with $H_2 = 8$ m^2 in Figure 4, one had in mind a radius of gyration whose value is 2·8 m.

Finally, it is interesting to note that, in general, Professor Lin's article agrees with [1] in the sense that a moving force problem is an approximation to a moving mass problem and that the former replaces the latter only under certain conditions. Furthermore similar qualitative differences between the moving force and moving mass problems stated in his article can also be deduced from equation (2.1) of [1].

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