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VIBRATION SUPPRESSION OF A NON-LINEAR AXIALLY MOVING STRING BY BOUNDARY CONTROL

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1. INTRODUCTION

Axially moving string-like continua, such as threads, wires, magnetic tapes, belts, band-saws, chains and cables, have been subjects for study by researchers in recent years; see survey papers [1–3] for extensive lists of references. Researchers have derived and studied different linear and non-linear mathematical models which describe the dynamics of such systems; see, e.g., references [4–9]. Recently, the important problem of vibration suppression of axially moving string-like continua has received attention by researchers; see, e.g., references [10–15]. Most of the controllers, except that in reference [11], are designed on the basis of linear models of axially moving strings. Our goal in this note is to design a controller for a non-linear model of axially moving strings.

In this note, we consider the axially moving string in Figure 1. The string is pulled at a constant speed through two eyelets, which are distanced from each other by one unit of length. One of the eyelets is fixed and the other one can move freely in the direction of the Y-axis. A control input force, denoted by u in Figure 1, can be applied to the free-to-move eyelet transversally: i.e., in the direction of Y.

The dynamics of the string in Figure 1 can be represented by the following non-linear partial differential equation (see, e.g., references [1, 4, 11]):

$$y_{tt}(x, t) + 2avy_{xt}(x, t) = (1 - a^2v^2 + \tfrac{3}{2}by_x^2(x, t))y_{xx}(x, t), \qquad \text{(1a)}$$

for all $x \in (0, 1)$ and $t \geqslant 0$. In equation (1a), $y(\cdot, \cdot) \in \mathbb{R}$ denotes the transversal displacement of the string, $y_x := \partial y/\partial x$, $y_{xx} := \partial^2 y/\partial x^2$, $y_{tt} := \partial^2 y/\partial t^2$, $y_{xt} := \partial^2 y/\partial x\, \partial t$, $a > 0$ and $b > 1$ are constant real numbers, and $v > 0$ is proportional to the speed of the string through the eyelets. In realistic physical situations, $av < 1$.

The tension in the string represented by equation (1a) is *not* constant, and is given by

$$T(x, t) = 1 + \tfrac{1}{2}by_x^2(x, t),$$

for all $x \in [0, 1]$ and $t \geqslant 0$ (see reference [16]). With the tension T, we have the following boundary conditions:

$$y(0, t) = 0, \qquad (1 - a^2v^2 + \tfrac{1}{2}by_x^2(1, t))y_x(1, t) = u(t), \qquad \text{(1b, c)}$$

for all $t \geqslant 0$. The boundary condition in equation (1b) states that the string is fixed at $x = 0$. The boundary condition in equation (1c) represents the balance of forces applied to the string at $x = 1$ in the direction of Y.

The initial displacement and velocity of the string are, respectively,

$$y(x, 0) = f(x), \qquad y_t(x, 0) = g(x), \qquad \text{(1d)}$$

for all $x \in (0, 1)$, where $y_t := \partial y/\partial t$. We assume that $f \in C^1[0, 1]$, and that at least one of the functions f and g is not identically zero over [0, 1].

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The control input u in equation (1c) is commonly known as the *boundary control*. In this note, we study the stabilization of the string in equation (1a) by u. More precisely, we study a u that results in $y(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$. As a stabilizing control input, we propose

$$u(t) = -ky_t(1, t), \tag{2}$$

for all $t \geqslant 0$, where $k > 0$ is a constant real number. With this choice of u, the boundary control is the negative feedback of the transversal velocity of the string at $x = 1$, with the gain k. It is known that fixed *linear* strings represented by equations (1), in which $v = 0$ and $b = 0$, can be stabilized by the control law in equation (2); see, e.g., references [17–22]. Also, it is known that axially moving *linear* strings represented by equations (1), in which $v > 0$ and $b = 0$, can be stabilized by the control law in equation (2); see references [10, 13]. Roughly speaking, the boundary control in equation (2) provides a dissipative effect in linear strings, because it is of the form of negative velocity feedback. This is in accordance with the well known fact that the negative velocity feedback increases damping in most finite dimensional inertial systems, such as large flexible systems and robotic manipulators.

Our goal in this note is to show that the boundary control u in equation (2) stabilizes the non-linear axially moving non-linear string in equations (1), i.e., u results in $y(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$. To the best of our knowledge, no such result exists.

2. STABILIZATION BY BOUNDARY CONTROL

Our plan to establish the stability of the non-linear string represented by equations (1) and (2) is as follows. We define an energy-like (Lyapunov) function of time for the string and denote it by $t \mapsto V(t)$. We show that V tends to zero exponentially.

We define the scalar-valued function V as

$$V(t) := E(t) + \gamma \int_0^1 [xy_t(x, t)y_x(x, t) + avxy_x^2(x, t)]\, \mathrm{d}x, \tag{3}$$

for all $t \geqslant 0$, where $\gamma > 0$ is a constant real number,

$$E(t) := \tfrac{1}{2} \int_0^1 [\, y_t^2(x, t) + (1 - a^2v^2)y_x^2(x, t)]\, \mathrm{d}x + \frac{b}{8} \int_0^1 y_x^4(x, t)\, \mathrm{d}x, \tag{4}$$

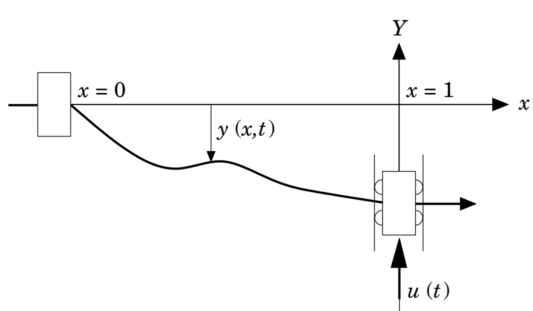

Figure 1. The string is pulled at a constant speed through two eyelets. The eyelet at $x = 0$ is fixed and the one at $x = 1$ can move freely in the direction of the Y-axis. The control input force $u(t) = -ky_t(1, t)$, for all $t \geqslant 0$, where $k > 0$ is a constant real number, is applied to the free-to-move eyelet in the direction of Y.

and $y(\cdot,\cdot)$ satisfies equations (1) and (2). From equations (3), (4) and (1d), we obtain

$$E(0)=\tfrac{1}{2}\int_0^1 [g^2(x)+(1-a^2v^2)f_x^2(x)]\,\mathrm{d}x+\frac{b}{8}\int_0^1 f_x^4(x)\,\mathrm{d}x, \tag{5a}$$

$$V(0)=E(0)+\gamma\int_0^1 [xg(x)f_x(x)+avxf_x^2(x)]\,\mathrm{d}x, \tag{5b}$$

where $f_x(x):=\mathrm{d}f(x)/\mathrm{d}x$. Since at least one of the functions f and g is not identically zero over [0, 1], we have $E(0)>0$.

Now, we prove a property of V.

Lemma 2.1. Let γ in equation (3) satisfy

$$\gamma<\frac{1-a^2v^2}{1+2av}. \tag{6}$$

Then, the function V satisfies

$$0\leqslant K_1E(t)\leqslant V(t)\leqslant K_2E(t), \tag{7}$$

for all $t\geqslant 0$, where $K_1>0$ and $K_2>0$ are constant real numbers, given by

$$K_1=1-\frac{\gamma(1+2av)}{1-a^2v^2},\qquad K_2=1+\frac{\gamma(1+2av)}{1-a^2v^2}. \tag{8a, b}$$

Proof. See Appendix A. □

Remarks. (1) Since $(1+2av)/(1-a^2v^2)>1$ for all $0<av<1$, it is clear that γ in inequality (6) is less than 1.

(2) Let γ satisfy inequality (6). Then, by inequality (7) and the fact that $E(0)>0$, it is concluded that $V(0)>0$. □

Next, we substitute equation (2) into equation (1c) and rewrite the boundary conditions as

$$y(0,t)=0,\qquad y_x(1,t)=-\frac{ky_t(1,t)}{1-a^2v^2+\frac{1}{2}by_x^2(1,t)}, \tag{9a, b}$$

for all $t\geqslant 0$. We now prove some identities for the functions satisfying equations (9).

Lemma 2.2. let $y(\cdot,\cdot)$ satisfy the boundary conditions in equations (9). Then,

$$2\int_0^1 y_{xt}y_t\,\mathrm{d}x=y_t^2(1,t),\qquad \int_0^1 (y_{xx}y_t+y_{xt}y_x)\,\mathrm{d}x=-\frac{ky_t^2(1,t)}{1-a^2v^2+\frac{1}{2}by_x^2(1,t)}, \tag{10a, b}$$

$$\int_0^1 (3y_{xx}y_x^2y_t+y_x^3y_{xt})\,\mathrm{d}x=-\frac{k^3y_t^4(1,t)}{[1-a^2v^2+\frac{1}{2}by_x^2(1,t)]^3}, \tag{10c}$$

$$\int_0^1 xy_{xt}y_t\,\mathrm{d}x = \tfrac{1}{2}y_t^2(1,t) - \tfrac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x, \tag{10d}$$

$$\int_0^1 xy_{xx}y_x\,\mathrm{d}x = \frac{k^2y_t^2(1,t)}{2[1-a^2v^2+\frac{1}{2}by_x^2(1,t)]^2} - \tfrac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x, \tag{10e}$$

$$\int_0^1 xy_{xx}y_x^3\,\mathrm{d}x = \frac{k^4y_t^4(1,t)}{4[1-a^2v^2+\frac{1}{2}by_x^2(1,t)]^4} - \tfrac{1}{4}\int_0^1 y_x^4\,\mathrm{d}x, \tag{10f}$$

for all $t \geqslant 0$.

Proof. See Appendix A. □

Next, we compute the time derivative of the function E.

Lemma 2.3. The time derivative of the function E in equation (4), along the solution of the system (1a), (1c), and (9) (equivalently, the system (1) and (2)) satisfies

$$\dot{E}(t) = -avy_t^2(1,t) - \frac{k(1-a^2v^2)y_t^2(1,t)}{1-a^2v^2+\frac{1}{2}by_x^2(1,t)} - \frac{k^3by_t^4(1,t)}{2[1-a^2v^2+\frac{1}{2}by_x^2(1,t)]^3} \leqslant 0, \tag{11}$$

for all $t \geqslant 0$.

Proof. See Appendix A. □

Using the preliminary results obtained thus far, we next prove that the functions V and E tend to zero exponentially.

Theorem 2.4. Let γ in equation (3) satisfy

$$\gamma < \min\left\{\frac{1-a^2v^2}{1+2av}, 2av, \frac{4(1-a^2v^2)}{3k}\right\}. \tag{12}$$

Then, the functions V and E, along the solution of the system (1a), (1c) and (9) (equivalently, the system (1) and (2)) satisfy

$$0 \leqslant V(t) \leqslant V(0)\,\mathrm{e}^{-\gamma t/K_2}, \qquad 0 \leqslant E(t) \leqslant \frac{V(0)}{K_1}\,\mathrm{e}^{-\gamma t/K_2}, \tag{13a, b}$$

for all $t \geqslant 0$, where K_1 and K_2 are given in equations (8).

Proof. From equation (3), we obtain

$$\dot{V}(t) = \dot{E}(t) + \gamma\int_0^1 (xy_{tt}y_x + xy_ty_{xt} + 2avxy_{xt}y_x)\,\mathrm{d}x, \tag{14}$$

for all $t \geqslant 0$. Substituting y_{tt} from equation (1a) into equation (14), we obtain

$$\dot{V}(t) = \dot{E}(t) + \gamma\int_0^1 [xy_{xt}y_t + (1-a^2v^2)xy_{xx}y_x + \tfrac{3}{2}bxy_{xx}y_x^3]\,\mathrm{d}x, \tag{15}$$

for all $t \geqslant 0$. Substituting equations (11), (10d), (10e) and (10f) into equation (15), we obtain

$$\dot{V}(t) \leqslant -\gamma E(t) - F(t), \tag{16}$$

for all $t \geqslant 0$, where

$$F(t) := \left(av - \frac{\gamma}{2}\right) y_t^2(1, t) + \left(1 - \frac{\gamma k}{2(1 - a^2v^2)}\right) \frac{k(1 - a^2v^2) y_t^2(1, t)}{1 - a^2v^2 + \frac{1}{2} b y_x^2(1, t)}$$

$$+ \left(1 - \frac{3\gamma k}{4(1 - a^2v^2)}\right) \frac{k^3 b y_t^4(1, t)}{2[1 - a^2v^2 + \frac{1}{2} b y_x^2(1, t)]^3}. \tag{17}$$

From inequality (12), we have

$$\gamma < \min\left\{2av, \frac{4(1 - a^2v^2)}{3k}\right\}, \tag{18}$$

by which we conclude that $F(t) \geqslant 0$ for all $t \geqslant 0$. Using the non-negativeness of F in inequality (16), we obtain

$$\dot{V}(t) \leqslant -\gamma E(t), \tag{19}$$

for all $t \geqslant 0$. Also, from inequality (12), we conclude that inequality (6) and hence inequality (7), hold. Using inequality (7) in inequality (19), we obtain the differential inequality

$$\dot{V}(t) \leqslant -\frac{\gamma}{K_2} V(t), \tag{20}$$

for all $t \geqslant 0$, with the initial condition $V(0)$ given in equation (5b). By a comparison theorem given in references [23, p. 29] or [24, p. 30], we conclude that V in inequality (20) satisfies $V(t) \leqslant V(0)\,\mathrm{e}^{-\gamma t/K_2}$ for all $t \geqslant 0$. Note that, by inequality (7), we have $V(t) \geqslant 0$ for all $t \geqslant 0$. Thus, inequality (13a) holds. By inequalities (7) and (13a), we conclude that inequality (13b) holds. □

Finally, we show that the boundary control u in equation (2) stabilizes the non-linear string in equations (1).

Corollary 2.5. The solution of the system (1a), (1c) and (9) (equivalently, the system (1) and (2)), $y(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$.

Proof. For the system (1a), (1c) and (9), we choose the Lyapunov function V in equation (3), and let γ in equation (3) satisfy inequality (12). Then, by Theorem 2.4, the function E tends to zero exponentially. From equation (4), we conclude that $y_x(x, t) \to 0$ as $t \to \infty$ for all $x \in [0, 1]$. Since $y(0, t) = 0$ for all $t \geqslant 0$, we conclude that $y(x, t) \to 0$ as $t \to \infty$, for all $x \in [0, 1]$. □

3. CONCLUSIONS

In this note, we have proved that the non-linear axially moving string represented by equations (1) can be stabilized by the linear boundary control in equation (2). The boundary control is the negative feedback of the transversal velocity of the string at one end.

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APPENDIX A: PROOFS

A.1. *Proof of Lemma* 2.1.

For the integral terms in equation (3), the coefficient of which is γ, we have (the argument (x, t) of the functions is deleted)

$$\int_0^1 xy_t y_x \,\mathrm{d}x \leqslant \int_0^1 x|y_t\|y_x|\,\mathrm{d}x \leqslant \tfrac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x + \tfrac{1}{2}\int_0^1 y_x^2\,\mathrm{d}x, \qquad \int_0^1 avxy_x^2\,\mathrm{d}x \leqslant av\int_0^1 y_x^2\,\mathrm{d}x, \tag{A1a, b}$$

for all $t \geqslant 0$. Thus,

$$\int_0^1 (xy_t y_x + avxy_x^2)\,\mathrm{d}x \leqslant \tfrac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x + \frac{1+2av}{2(1-a^2v^2)}\int_0^1 (1-a^2v^2)y_x^2\,\mathrm{d}x, \tag{A2}$$

for all $t \geqslant 0$. Since

$$\frac{1+2av}{1-a^2v^2} \geqslant 1, \tag{A3}$$

for all $0 < av < 1$, we conclude that

$$\int_0^1 (xy_t y_x + avxy_x^2)\,\mathrm{d}x \leqslant \frac{1+2av}{1-a^2v^2}\left(\tfrac{1}{2}\int_0^1 [\,y_t^2 + (1-a^2v^2)y_x^2]\,\mathrm{d}x\right) \leqslant \frac{1+2av}{1-a^2v^2}E(t), \tag{A4a}$$

for all $t \geqslant 0$. Similarly, we obtain

$$\int_0^1 (xy_t y_x + avxy_x^2)\,\mathrm{d}x \geqslant -\frac{1+2av}{1-a^2v^2}E(t), \tag{A4b}$$

for all $t \geqslant 0$. Using equations (A4) in equation (3), we obtain equation (7). □

A.2. Proof of Lemma 2.2.

From equation (9a), we have $y_t(0, t) = 0$ for all $t \geqslant 0$. Thus, we obtain

$$2\int_0^1 y_{xt} y_t\,\mathrm{d}x = \int_0^1 (\,y_t^2)_x\,\mathrm{d}x = y_t^2(1, t), \tag{A5}$$

for all $t \geqslant 0$. That is, equation (10a) holds.

Having $y_t(0, t) = 0$ for all $t \geqslant 0$, we next obtain

$$\int_0^1 (\,y_{xx} y_t + y_{xt} y_x)\,\mathrm{d}x = \int_0^1 (\,y_x y_t)_x\,\mathrm{d}x = y_x(1, t)y_t(1, t), \tag{A6}$$

for all $y \geqslant 0$. Using equation (9b) in equation (A6), we obtain equation (10b).

Having $y_t(0, t) = 0$ for all $t \geqslant 0$, we next obtain

$$\int_0^1 (3y_{xx} y_x^2 y_t + y_x^3 y_{xt})\,\mathrm{d}x = \int_0^1 (\,y_x^3 y_t)_x\,\mathrm{d}x = y_x^3(1, t)y_t(1, t), \tag{A7}$$

for all $t \geqslant 0$. Using equation (9b) in equation (A7), we obtain equation (10c).

Next, we write

$$\int_0^1 xy_{xt} y_t\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 (xy_t^2)_x\,\mathrm{d}x - \tfrac{1}{2}\int_0^1 y_t^2\,\mathrm{d}x, \tag{A8}$$

for all $t \geqslant 0$. Thus, equation (10d) follows.

Next, we write

$$\int_0^1 x y_{xx} y_x \, \mathrm{d}x = \tfrac{1}{2} \int_0^1 (x y_x^2)_x \, \mathrm{d}x - \tfrac{1}{2} \int_0^1 y_x^2 \, \mathrm{d}x = \tfrac{1}{2} y_x^2(1, t) - \tfrac{1}{2} \int_0^1 y_x^2 \, \mathrm{d}x, \tag{A9}$$

for all $t \geqslant 0$. Using equation (9b) in equation (A9), we obtain equation (10e).

Finally, we write

$$\int_0^1 x y_{xx} y_x^3 \, \mathrm{d}x = \tfrac{1}{4} \int_0^1 (x y_x^4)_x \, \mathrm{d}x - \tfrac{1}{4} \int_0^1 y_x^4 \, \mathrm{d}x = \tfrac{1}{4} y_x^4(1, t) - \tfrac{1}{4} \int_0^1 y_x^4 \, \mathrm{d}x, \tag{A10}$$

for all $t \geqslant 0$. Using equation (9b) in equation (A10), we obtain equation (10f). □

A.3. *Proof of Lemma* 2.3.

From equation (4), we obtain

$$\dot{E}(t) = \int_0^1 [\, y_{tt} y_t + (1 - a^2 v^2) y_{xt} y_x] \, \mathrm{d}x + \frac{b}{2} \int_0^1 y_{xt} y_x^3 \, \mathrm{d}x, \tag{A11}$$

for all $t \geqslant 0$. Substituting y_{tt} from equation (1a) into equation (A11), we obtain

$$\dot{E}(t) = -2av \int_0^1 y_{xt} y_t \, \mathrm{d}x + (1 - a^2 v^2) \int_0^1 (\, y_{xx} y_t + y_{xt} y_x) \, \mathrm{d}x$$

$$+ \frac{b}{2} \int_0^1 (3 y_{xx} y_x^2 y_t + y_x^3 y_{xt}) \, \mathrm{d}x, \tag{A12}$$

for all $t \geqslant 0$. Using equations (10a), (10b) and (10c) in equation (A12), we obtain equation (11). □