Journal of Sound and Vibration (1997) **201**(1), 67–84

A HARMONIC PROBING ALGORITHM FOR THE MULTI-INPUT VOLTERRA SERIES

K. Worden, G. Manson and G. R. Tomlinson

Department of Mechanical Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, England

(*Received* 16 *January* 1996, *and in final form* 23 *August* 1996)

It is shown how the conventional harmonic probing algorithm of Bedrosian and Rice can be extended to deal with the multi-input multi-output form of the Volterra functional series. Example calculations are presented for both continuous-time and discrete-time multi-input non-linear systems subject to multiple sinusoidal inputs.

1. INTRODUCTION

The single-input Volterra series is by now established as a powerful tool in the analysis of non-linear systems. Individual papers are too numerous to cite; a useful review by Billings [1] and the fairly recent review of Korenburg and Hunter [2] account for all references of importance up to 1990. Rugh's book [3] covers material up to 1981. Possibly the most significant developments in the theory since are the generating series approach of Fliess and his co-workers [4], and several new methods of estimating the series kernels and kernel transforms [2, 5, 6]. In contrast, the multi-input version of the series appears to have received little attention since its inception (in the guise of the closely related Wiener series) in the paper of Marmarelis and Naka in 1974 [7]. In reference [8], the situation is discussed. The more restricted case in which several independent signals are input to a system at the same point has received some attention from Bussgang *et al*. [9].

In the case of non-linear structures, as encountered in mechanical and civil engineering, the motivation for applying the Volterra series is often the need to determine how energy is transferred from harmonic inputs to sum and difference frequencies in the output; for example, Worden *et al*. [10] considered the example of non-linear wave loading on offshore structures. In this case, it was most convenient to work with the Fourier transforms of the Volterra kernels—the so-called kernel transforms or Higher-order Frequency Response Functions (HFRFs). If the differential equations of motion of the system, or a discrete-time model are available, there are a number of methods of obtaining the kernel transforms for the single-input series; arguably the most direct is the *harmonic probing*† algorithm of Bedrosian and Rice [11] (extended to discrete-time systems by Billings and Tsang [5]).

The objective of the current paper is to extend the harmonic probing algorithm to the multi-input Volterra series so that the additional cross-kernel transforms can be determined. This is a novel method of establishing the kernels; previous work seems to

† In the literature relating to control engineering or system theory, the Laplace s-domain is often used in preference to the Fourier ω-domain and it is more correct to refer to the kernel transforms as Higher-order Transfer Functions (HTFs). In that case, the method of harmonic probing is referred to as the *method of growing exponentials* [3].

0022–460X/97/110067 + 18 $25.00/0/sv960746

have been concentrated on correlation methods. The layout of the paper is as follows. In section 2 the relevant facts about the single-input Volterra series are summarized and an example is given of the use of harmonic probing. In section 3 the modifications necessary to deal with the case of multiple inputs are described, as is the new algorithm, and an example is given of how it is applied to a discrete-time multi-input non-linear system, together with a description of how it can be used to compute the response of a non-linear system under multiple sinusoidal excitation. In section 4 this theoretical work is validated via a numerical example.

2. THE SINGLE-INPUT VOLTERRA SERIES AND HARMONIC PROBING

The theory described in this section can be found elsewhere in the literature; it is included here in an attempt to make the paper as self-contained as possible.

The Volterra series is essentially a generalization of the well-known input–output relation for linear systems,

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $x(t)$ is the system input and $y(t)$, the output. The above equation is sometimes referred to as Duhamel's integral. The system is specified uniquely by its impulse response function $h(t)$. The Fourier transform $\mathscr{F}$ of equation (1) yields the familiar frequency domain expression

$$Y(\omega) = H(\omega)X(\omega), \tag{2}$$

where $X(\omega)$, $Y(\omega)$ and $H(\omega)$ are the Fourier transforms of $x(t)$, $y(t)$ and $h(t)$, respectively, and $H(\omega)$ is the system Frequency Response Function or FRF. All information about the system is encoded in either of the functions $h(t)$ or $H(\omega)$.

Equations (1) and (2) are manifestly linear and therefore cannot hold for arbitrary non-linear systems. However, both admit a generalization. The extended form of equation (1) was obtained in the early part of this century by Volterra [12] and it takes the form of an infinite series

$$y(t) = y_1(t) + y_2(t) + y_3(t) + \cdots, \tag{3}$$

where

$$y_1(t) = \int_{-\infty}^{+\infty} h_1(\tau)x(t-\tau)\,\mathrm{d}\tau, \tag{4}$$

$$y_2(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2, \tag{5}$$

$$y_3(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3. \tag{6}$$

For the systems being considered the equilibrium position is set to be $y = 0$. This means that a $y_0(t)$ DC term is not required in equation (3). This does not disallow a DC term in the output response of the system, which may still occur due to interactions between the input frequencies.

The form of the general term follows from the above equations. The functions $h_1(\tau)$, $h_2(\tau_1, \tau_2)$, $h_3(\tau_1, \tau_2, \tau_3), \dots h_n(\tau_1, \dots, \tau_n), \dots$ are generalizations of the linear impulse response function and are usually referred to as *Volterra kernels*. The use of the Volterra series in dynamic stems from the seminal paper of Barrett [13], in which the series was applied to non-linear differential equations for the first time. One can think of the series as a generalization of the Taylor series from functions to functionals. The expression (1) simply represents the lowest order truncation which is, of course, exact only for linear systems.

It was shown by Schetzen [14] that the kernels can be taken to be symmetric without loss of generality: i.e., $h_2(\tau_1, \tau_2) = h_2(\tau_2, \tau_1)$ etc. A formal argument is fairly straightforward: consider the expression for $y_2(t)$,

$$y_2(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2(\tau_1, \tau_2) \Pi_2(\tau_1, \tau_2; t)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \tag{7}$$

with the newly defined

$$\Pi_2(\tau_1, \tau_2; t) = x(t - \tau_1)x(t - \tau_2), \tag{8}$$

and note that Π_2 is manifestly symmetric in its arguments τ_1 and τ_2.

Even if h_2 has no particular symmetries, it still has a canonical decomposition into symmetric and antisymmetric parts,

$$h_2(\tau_1, \tau_2) = h_2^{sym}(\tau_1, \tau_2) + h_2^{asym}(\tau_1, \tau_2), \tag{9}$$

where

$$h_2^{sym}(\tau_1, \tau_2) = \tfrac{1}{2}(h_2(\tau_1, \tau_2) + h_2(\tau_2, \tau_1)),$$
$$h_2^{asym}(\tau_1, \tau_2) = \tfrac{1}{2}(h_2(\tau_1, \tau_2) - h_2(\tau_2, \tau_1)). \tag{10}$$

Now, consider the contribution to $y_2(t)$ from the antisymmetric component of the kernel,

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2^{asym}(\tau_1, \tau_2) \Pi_2(\tau_1, \tau_2; t)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2. \tag{11}$$

Any (infinitesimal) contribution to this "summation", say at $\tau_1 = v$, $\tau_2 = w$, will cancel with the corresponding contribution at $\tau_2 = v$, $\tau_1 = w$, as

$$h_2^{asym}(v, w)\Pi_2(v, w; t) = -h_2^{asym}(w, v)\Pi_2(w, v; t) \tag{12}$$

and the overall integral will vanish. This is purely because of the "contraction" or summation against the symmetric quantity $\Pi_2(\tau_1, \tau_2; t)$. Because h_2^{asym} makes no contribution to the quantity $y_2(t)$, it may be disregarded and the kernel h_2 can be assumed to be symmetric. Essentially, the h_2 picks up all the symmetries of the quantity Π_2. This argument may be generalized to the kernel $h_n(\tau_1, \dots, \tau_n)$. This type of argument will surface in the following section when the multi-input series is discussed.

As in the linear case, there exists a dual frequency domain representation for non-linear systems. The *higher order FRFs* or *Volterra kernel transforms* $H_n(\omega_1, \dots, \omega_n)$,

$n = 1, \ldots, \infty$, are defined as the multi-dimensional Fourier transforms of the kernels, i.e.,

$$H_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n(\tau_1, \ldots, \tau_n)\, \mathrm{e}^{-\mathrm{i}(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n, \tag{13}$$

$$h_n(\tau_1, \ldots, \tau_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H_n(\omega_1, \ldots, \omega_n)\, \mathrm{e}^{+\mathrm{i}(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_n. \tag{14}$$

It may be shown that symmetry of the kernels implies symmetry of the kernel transforms, so that, for example, $H_2(\omega_1, \omega_2) = H_2(\omega_2, \omega_1)$.

The frequency domain dual of the expression (3) may be written as

$$Y(\omega) = Y_1(\omega) + Y_2(\omega) + Y_3(\omega) + \cdots, \tag{15}$$

where

$$Y_1(\omega) = H_1(\omega)X(\omega), \tag{16}$$

$$Y_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_2(\omega_1, \omega - \omega_1) X(\omega_1) X(\omega - \omega_1)\, \mathrm{d}\omega_1, \tag{17}$$

$$Y_3(\omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H_3(\omega_1, \omega_2, \omega - \omega_1 - \omega_2) X(\omega_1) X(\omega_2) X(\omega - \omega_1 - \omega_2)\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2. \tag{18}$$

Again, the form of the general term follows.

In order to determine the analytical form of the kernel transforms, the method of harmonic probing was introduced by Bedrosian and Rice in [11] specifically for systems with continuous-time equations of motion. The method was extended to discrete-time systems by Billings and Tsang [15]. An alternative, recursive approach to probing was presented by Peyton-Jones and Billings [16]. In order to explain how the procedure works, it is necessary to determine how a system responds to a harmonic or periodic input in terms of its Volterra series.

First consider a periodic excitation composed of a single harmonic

$$x(t) = \mathrm{e}^{\mathrm{i}\Omega t}. \tag{19}$$

Substituting this expression into equations (4)–(6) and forming the total response as in (3) gives, after a relatively straightforward calculation,

$$y(t) = H_1(\Omega)\, \mathrm{e}^{\mathrm{i}\Omega t} + H_2(\Omega, \Omega)\, \mathrm{e}^{\mathrm{i}2\Omega t} + H_3(\Omega, \Omega, \Omega)\, \mathrm{e}^{\mathrm{i}3\Omega t} + \cdots. \tag{20}$$

The important point here is that the component in the output at the forcing frequency is multiplied by $H_1(\Omega)$. However, probing the system with a single harmonic yields only information about the values of the FRFs on the diagonal line in the frequency space (Ω_1, Ω_2). In order to obtain information elsewhere in this space, one should use multi-frequency excitations. With this in mind, consider the "two-tone" input

$$x(t) = \mathrm{e}^{\mathrm{i}\Omega_1 t} + \mathrm{e}^{\mathrm{i}\Omega_2 t}. \tag{21}$$

Substituting this into equations (4)–(6) and thence into equation (3) yields, after a slightly more involved calculation,

$$\begin{aligned} y(t) = {} & H_1(\Omega_1)\,\mathrm{e}^{\mathrm{i}\Omega_1 t} + H_1(\Omega_2)\,\mathrm{e}^{\mathrm{i}\Omega_2 t} \\ & + H_2(\Omega_1, \Omega_1)\,\mathrm{e}^{\mathrm{i}2\Omega_1 t} + 2H_2(\Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t} + H_2(\Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}2\Omega_2 t} \\ & + H_3(\Omega_1, \Omega_1, \Omega_1)\,\mathrm{e}^{\mathrm{i}3\Omega_1 t} + 3H_3(\Omega_1, \Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(2\Omega_1+\Omega_2)t} \\ & + 3H_3(\Omega_1, \Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1+2\Omega_2)t} + H_3(\Omega_2, \Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}3\Omega_2 t} + \cdots \end{aligned} \tag{22}$$

for the response up to third order. The important thing to note here is that the amplitude of the component at the *sum* frequency for the excitation, i.e., at $(\Omega_1 + \Omega_2)$, is twice the second order FRF $H_2(\Omega_1, \Omega_2)$. In fact, if a more general periodic excitation is used, i.e.,

$$x(t) = \mathrm{e}^{\mathrm{i}\Omega_1 t} + \cdots + \mathrm{e}^{\mathrm{i}\Omega_n t}, \tag{23}$$

it is not difficult to show that the amplitude of the output component at the frequency $(\Omega_1 + \cdots + \Omega_n)$ is $n!H_n(\Omega_1, \ldots, \Omega_n)$. This single fact is the basis of the harmonic probing algorithm. In order to find the second order FRF of a system, for example, one substitutes the expressions for the input (21) and general output (22) into the system equation of motion and extracts the coefficient of $\mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t}$; this yields an algebraic expression for H_2.

The procedure is best illustrated by choosing a concrete example. Consider the continuous-time asymmetric Duffing oscillator system.

$$mD^2y + cDy + ky + k_2y^2 + k_3y^3 = x(t), \tag{24}$$

where $D = \mathrm{d}/\mathrm{d}t$. In order to find H_1, one substitutes in the equation, the probing expressions (indicated by subscript p),

$$x(t) = x_{p_1}(t) = \mathrm{e}^{\mathrm{i}\Omega t}, \qquad y(t) = y_{p_1}(t) + \cdots, \tag{25, 26}$$

where $y_{p_1}(t)$, the output probing signal, is that portion of the Volterra response capable of generating response components at frequency Ω and is given by $y_{p_1}(t) = H_1(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega t}$. The result of these substitutions is

$$(-m\Omega^2 + \mathrm{i}c\Omega + k)H_1(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega t} + k_2H_1(\Omega)^2\,\mathrm{e}^{\mathrm{i}2\Omega t} + k_3H_1(\Omega)^3\,\mathrm{e}^{\mathrm{i}3\Omega t} + \cdots = \mathrm{e}^{\mathrm{i}\Omega t}. \tag{27}$$

Equating the coefficients of $\mathrm{e}^{\mathrm{i}\Omega t}$ on each side of this expression yields an equation for H_1,

$$(-m\Omega^2 + \mathrm{i}c\Omega + k)H_1(\Omega) = 1, \tag{28}$$

which yields the expected expression

$$H_1(\Omega) = 1/(-m\Omega^2 + \mathrm{i}c\Omega + k). \tag{29}$$

Evaluation of H_2 is only a little more complicated. One uses the probing expressions

$$x(t) = x_{p_2}(t) = \mathrm{e}^{\mathrm{i}\Omega_1 t} + \mathrm{e}^{\mathrm{i}\Omega_2 t}, \qquad y(t) = y_{p_2}(t) + \cdots, \tag{30, 31}$$

where

$$y_{p_2}(t) = H_1(\Omega_1)\,\mathrm{e}^{\mathrm{i}\Omega_1 t} + H_1(\Omega_2)\,\mathrm{e}^{\mathrm{i}\Omega_2 t} + 2H_2(\Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t}. \tag{32}$$

Note that in passing from the general output (22) to the probing expression (32), all second order terms except that at the sum frequency have been deleted. This simplification is allowed for the same reason as before: i.e., no combination of the missing terms can produce a component at the sum frequency and therefore they cannot appear in the final expression for H_2. Substituting expressions (30) and (31) into equation (24), and extracting the coefficients of $\mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t}$ yields

$$\{-m(\Omega_1 + \Omega_2)^2 + \mathrm{i}c(\Omega_1 + \Omega_2) + k\}H_2(\Omega_1, \Omega_2) + k_2H_1(\Omega_1)H_1(\Omega_2) + \cdots = 0, \tag{33}$$

so that

$$H_2(\Omega_1, \Omega_2) = -k_2 H_1(\Omega_1) H_1(\Omega_2)/(-m(\Omega_1+\Omega_2)^2 + \mathrm{i}c(\Omega_1+\Omega_2) + k)$$
$$= -k_2 H_1(\Omega_1) H_1(\Omega_2) H_1(\Omega_1+\Omega_2). \quad (34)$$

Note that the constant k_2 multiplies the whole expression for H_2, so that if the square-law term is absent from the equation of motion, H_2 vanishes. This reflects a quite general property of the Volterra series; if all non-linear terms in the equation of motion for a system are odd powers of x or y, then the associated Volterra series has no even order kernels. As a consequence, it will possess no even order kernel transforms.

It is also a general property of systems that all higher order FRFs can be expressed in terms of H_1 for the system. The exact form of the expression will of course depend on the particular system.

The harmonic probing algorithm has been established above for all continuous-time systems: i.e., those whose evolution is governed by differential equations of motion. For difference equations such as the NARMAX models of Leontaritis and Billings [17, 18], the probing algorithm requires a little modification, as in reference [15]. Consider a difference equation similar in appearance to equation (24),

$$m\,\Delta^2 y + c\,\Delta y + ky + k_2 y^2 + k_3 y^3 = x(t), \quad (35)$$

where Δ is the backward *shift* operator, defined by $\Delta z(t) = z(t - \Delta t)$, where Δt is the sampling interval. In the usual notation for difference equations, equation (35) becomes

$$my_{i-2} + cy_{i-1} + ky_i + k_2 y_i^2 + k_3 y_i^3 = x_i. \quad (36)$$

However, the form containing Δ allows the most direct comparison with the continuous-time case. The only differences for harmonic probing of discrete-time systems will be generated by the fact that the operator Δ has a different action on functions $\mathrm{e}^{\mathrm{i}\Omega t}$ to the operator D. This action may be expressed as

$$\Delta\,\mathrm{e}^{\mathrm{i}\Omega t} = \mathrm{e}^{\mathrm{i}\Omega(t-\Delta t)} = \mathrm{e}^{-\mathrm{i}\Omega\Delta t}\,\mathrm{e}^{\mathrm{i}\Omega t}. \quad (37)$$

One can carry out the harmonic probing algorithm for equation (35) exactly as for the continuous-time equation (24); the only difference will be that the Δ operator will generate a multiplier $\mathrm{e}^{-\mathrm{i}\Omega\Delta t}$ everywhere that D generated a factor $\mathrm{i}\Omega$. As a consequence, H_1 and H_2 for (35) may be computed as

$$H_1(\Omega) = 1/(-m\,\mathrm{e}^{-2\mathrm{i}\Omega\Delta t} + c\,\mathrm{e}^{-\mathrm{i}\Omega\Delta t} + k), \quad (38)$$

$$H_2(\Omega_1, \Omega_2) = -k_2 \frac{H_1(\Omega_1) H_1(\Omega_2)}{-m\,\mathrm{e}^{-2\mathrm{i}(\Omega_1+\Omega_2)\Delta t} + c\,\mathrm{e}^{-\mathrm{i}(\Omega_1+\Omega_2)\Delta t} + k}$$
$$= -k_2 H_1(\Omega_1) H_1(\Omega_1) H_1(\Omega_1+\Omega_2). \quad (39)$$

Note that the form of H_2 as a function of H_1 is identical to that for the continuous-time system.

The system in equation (36) is not NARMAX as it is a non-linear function of the most recent sampled value y_i. A NARMAX model has the general form

$$y_i = F(y_{i-1}, \ldots, y_{i-n_y}; x_{i-1}, \ldots, x_{i-n_x}; e_{i-1}, \ldots, e_{i-n_e}) + e_i, \quad (40)$$

where e_i is the noise signal. The relevant existence theorems obtained in references [17, 18] show that this form is actually general enough to represent almost all input–output systems.

It is assumed throughout this work that the Volterra series exists for all the systems considered; necessary and sufficient conditions for existence can be found in references [19, 20]. Essentially all that is required is that the non-linearity be analytic. Here, all non-linearities are polynomial and therefore satisfy this condition. Many non-linearities of interest, such as the piecewise-linear functions which arise in dealing with backlash and clearance systems, can be approximated arbitrarily closely by polynomial systems by the Stone–Weierstrass theorem [21]. Unfortunately, it is not sufficient for the Volterra series to exist for a given system; one must also have convergence over the range of excitations of interest. In practice, one requires convergence to appropriate accuracy in a few terms. There are few results on the radius of convergence for the Volterra series, Barrett [22] provided a lower bound on the radius for a Duffing oscillator and recently, in an empirical study, Tomlinson *et al*. [23] attempted to establish an upper bound. The only section of this work where convergence is an issue is section 4 in which an application to a specific dynamical system is presented.

3. MULTI-INPUT VOLTERRA SERIES

When a non-linear system is excited with more than one input, complicated intermodulation terms arise in the response. As before, the Volterra series proves equal to the task; however, it is necessary to use a more general multi-input form of the series.

Before embarking upon the theory it is first necessary to extend the notation used previously. A superscript is added to each Volterra kernel denoting the response point and the number of occurrences of each particular input relevant to the construction of that kernel is indicated: e.g., $h_5^{(j:aabbb)}(\tau_1, \ldots, \tau_5)$ represents a fifth order kernel measured at response point j and having two inputs at point a and three at point b.

To illustrate the process, consider a non-linear system excited at locations a and b with inputs $x^{(a)}(t)$ and $x^{(b)}(t)$. The expression for the response at point j is the same as equation (3) in the single-input case: i.e.,

$$y^{(j)}(t) = y_1^{(j)}(t) + y_2^{(j)}(t) + y_3^{(j)}(t) + \cdots. \tag{41}$$

For the single-input case each non-linear component $y_n^{(j)}(t)$ in equation (41) is expressed in terms of a single Volterra kernel; in the multi-input case, several kernels are needed. For the two-input case, the components are given by

$$\begin{aligned} y_n^{(j)}(t) = {} & \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n^{(j:aa\cdots aa)}(\tau_1, \ldots, \tau_n) x^{(a)}(t-\tau_1) \cdots x^{(a)}(t-\tau_n)\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \\ & + \cdots + \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n^{(j:aa\cdots bb)}(\tau_1, \ldots, \tau_n) x^{(a)}(t-\tau_1) x^{(a)}(t-\tau_2) \\ & \cdots x^{(b)}(t-\tau_{n-1}) x^{(b)}(t-\tau_n)\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n \\ & + \cdots + \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_n^{(j:b\cdots b)}(\tau_1, \ldots, \tau_n) x^{(b)} \\ & \times (t-\tau_1) \cdots x^{(b)}(t-\tau_n)\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n : \end{aligned} \tag{42}$$

i.e.,

$$y_1^{(j)}(t) = \int_{-\infty}^{+\infty} h_1^{(j:a)}(\tau)x^{(a)}(t-\tau)\,\mathrm{d}\tau + \int_{-\infty}^{+\infty} h_1^{(j:b)}(\tau)x^{(b)}(t-\tau)\,\mathrm{d}\tau \tag{43}$$

and

$$\begin{aligned} y_2^{(j)}(t) = &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2^{(j:aa)}(\tau_1,\tau_2)x^{(a)}(t-\tau_1)x^{(a)}(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2^{(j:ab)}(\tau_1,\tau_2)x^{(a)}(t-\tau_1)x^{(b)}(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2^{(j:ba)}(\tau_1,\tau_2)x^{(b)}(t-\tau_1)x^{(a)}(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2^{(j:bb)}(\tau_1,\tau_2)x^{(b)}(t-\tau_1)x^{(b)}(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2, \end{aligned} \tag{44}$$

etc.

A relabelling of the dummy variables in certain integrals allows the combination of kernels thus reducing the number of required terms; e.g., for the second order component of the response,

$$\begin{aligned} y_2^{(j)}(t) = &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2^{(j:aa)}(\tau_1,\tau_2)x^{(a)}(t-\tau_1)x^{(a)}(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{h_2^{(j:ab)}(\tau_1,\tau_2) + h_2^{(j:ba)}(\tau_2,\tau_1)\}x^{(a)}(t-\tau_1)x^{(b)}(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_2^{(j:bb)}(\tau_1,\tau_2)x^{(b)}(t-\tau_1)x^{(b)}(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \end{aligned} \tag{45}$$

The second order combination term is now re-defined as

$$\{h_2^{(j:ab)}(\tau_1,\tau_2) + h_2^{(j:ba)}(\tau_2,\tau_1)\} \to 2h_2^{(j:ab)}(\tau_1,\tau_2), \tag{46}$$

and similarly for all other combination terms.

The frequency domain dual of the response is the same as equation (15) for the single-input case,

$$Y^{(j)}(\omega) = Y_1^{(j)}(\omega) + Y_2^{(j)}(\omega) + Y_3^{(j)}(\omega) + \cdots, \tag{47}$$

where, for the two-input case, the components are now given by

$$
\begin{aligned}
Y_n^{(j)}(\omega) = &\left(\frac{1}{2\pi}\right)^{n-1} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H_n^{(j:aa\cdots aa)}(\omega_1, \omega_2 \ldots, \omega - \omega_1 - \cdots - \omega_{n-1}) \\
&\times X^{(a)}(\omega_1) X^{(a)}(\omega_2) \cdots X^{(a)}(\omega_{n-1}) X^{(a)}(\omega - \omega_1 - \cdots - \omega_{n-1})\, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} \\
&+ \cdots + \left(\frac{1}{2\pi}\right)^{n-1} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H_n^{(j:aa\cdots bb)}(\omega_1, \omega_2 \ldots, \omega - \omega_1 - \cdots - \omega_{n-1}) \\
&\times X^{(a)}(\omega_1) X^{(a)}(\omega_2) \cdots X^{(b)}(\omega_{n-1}) X^{(b)}(\omega - \omega_1 - \cdots - \omega_{n-1})\, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1} \\
&+ \cdots + \left(\frac{1}{2\pi}\right)^{n-1} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H_n^{(j:bb\cdots bb)}(\omega_1, \omega_2 \ldots, \omega - \omega_1 - \cdots - \omega_{n-1}) \\
&\times X^{(b)}(\omega_1) X^{(b)}(\omega_2) \cdots X^{(b)}(\omega_{n-1}) X^{(b)}(\omega - \omega_1 - \cdots - \omega_{n-1})\, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_{n-1}.
\end{aligned}
\tag{48}
$$

The question now arises of symmetry of the kernels under interchange of the time indices τ_i. There is no longer total symmetry under permutations of the n symbols of h_n and H_n. In fact, the kernels still have interchange invariance but under a smaller group than the group of permutations on n elements. Consider the object $y_3^{(j:aab)}$,

$$
y_3^{(j:aab)} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_3^{(j:aab)}(\tau_1, \tau_2, \tau_3) \Pi_3^{(j:aab)}(\tau_1, \tau_2, \tau_3; t)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, \mathrm{d}\tau_3, \tag{49}
$$

which contributes to the third order component of the output. The kernel is contracted with or summed against the object

$$
\Pi_3^{(j:aab)}(\tau_1, \tau_2, \tau_3; t) = x^{(a)}(t - \tau_1) x^{(a)}(t - \tau_2) x^{(b)}(t - \tau_3), \tag{50}
$$

which is symmetric on the first two indices. In the integral above, the summation of $\Pi_3^{(j:aab)}$ against $h_3^{(j:aab)}$ annihilates the part of $h_3^{(j:aab)}$ which is antisymmetric in the first two indices; the antisymmetric part can therefore be ignored. As in the single-input case, the kernel inherits in each case the symmetries of the product of inputs to which it corresponds. In general, the kernel

$$
h_{n_a+n_b}^{(j:\overbrace{a\cdots a}^{n_a}\overbrace{b\cdots b}^{n_b})}(\tau_1, \ldots, \tau_{n_a}, \tau_{n_a+1}, \ldots, \tau_{n_a+n_b}) \tag{51}
$$

is symmetric on the group of indices corresponding to the $x^{(a)}$ and $x^{(b)}$ inputs separately. The generalization to more than two inputs may be performed in a similar manner. It will be shown later that these residual symmetries in the multi-input case are just sufficient to allow the construction of a harmonic probing algorithm.

3.1. HARMONIC PROBING ALGORITHM TO OBTAIN DIRECT AND CROSS-KERNEL TRANSFORMS

The harmonic probing method introduced by Bedrosian and Rice [11] is now extended to allow calculation of the analytical form of the cross-kernel transforms alongside the direct-kernel transforms. Once again, to illustrate the process, the case of two excitation inputs shall be discussed. Extension to more inputs is straightforward.

Let the inputs to the system be $x^{(a)}(t) = \mathrm{e}^{\mathrm{i}\Omega t}$ and $x^{(b)}(t) = 0$, so that $X^{(a)}(\omega) = 2\pi\delta(\omega - \Omega)$ and $X^{(b)}(\omega) = 0$, where $\delta(\omega - \Omega)$ is the Dirac delta function.

Substituting the frequency form of the inputs into equations (47) and (48) and considering integrals composed only of non-zero inputs eventually yields, in the time domain,

$$y^{(j)}(t) = H_1^{(j:a)}(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega t} + H_2^{(j:aa)}(\Omega, \Omega)\,\mathrm{e}^{\mathrm{i}2\Omega t} + \cdots. \tag{52}$$

This is the same as equation (20) and upon substitution of the above expression into the equations of motion of the system and equating coefficients of $\mathrm{e}^{\mathrm{i}\Omega t}$ an expression for $H_1^{(j:a)}(\Omega)$ can be determined.

Similarly, setting $x^{(a)}(t) = 0$ and $x^{(b)}(t) = \mathrm{e}^{\mathrm{i}\Omega t}$ yields

$$y^{(j)}(t) = H_1^{(j:b)}(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega t} + H_2^{(j:bb)}(\Omega, \Omega)\,\mathrm{e}^{\mathrm{i}2\Omega t} + \cdots, \tag{53}$$

which then leads to an expression for $H_1^{(j:b)}(\Omega)$.

To obtain expressions for the second order direct-kernel transforms, $H_2^{(j:aa)}(\Omega_1, \Omega_2)$ and $H_2^{(j:bb)}(\Omega_1, \Omega_2)$, the "two-tone" input of equation (21) (i.e., $\mathrm{e}^{\mathrm{i}\Omega_1 t} + \mathrm{e}^{\mathrm{i}\Omega_2 t}$) is applied at points a and b respectively. The response at point j will be given by equation (22) with the appropriate kernel superscripts included; e.g., for the input at point a the response at j will be

$$\begin{aligned} y^{(j)}(t) = {} & H_1^{(j:a)}(\Omega_1)\,\mathrm{e}^{\mathrm{i}\Omega_1 t} + H_1^{(j:a)}(\Omega_2)\,\mathrm{e}^{\mathrm{i}\Omega_2 t} \\ & + H_2^{(j:aa)}(\Omega_1, \Omega_1)\,\mathrm{e}^{\mathrm{i}2\Omega_1 t} + 2H_2^{(j:aa)}(\Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t} + H_2^{(j:aa)}(\Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}2\Omega_2 t} \\ & + H_3^{(j:aaa)}(\Omega_1, \Omega_1, \Omega_1)\,\mathrm{e}^{\mathrm{i}3\Omega_1 t} + 3H_3^{(j:aaa)}(\Omega_1, \Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(2\Omega_1+\Omega_2)t} \\ & + 3H_3^{(j:aaa)}(\Omega_1, \Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1+2\Omega_2)t} + H_3^{(j:aaa)}(\Omega_2, \Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}3\Omega_2 t} + \cdots, \end{aligned} \tag{54}$$

with all a superscripts being exchanged for b superscripts for the response at point j when the input is moved to point b.

As before, it should be noted that the amplitude of the component at the *sum* frequency $(\Omega_1 + \Omega_2)$ is twice the second order FRF in each case. It has been shown that the method of obtaining the direct-kernel transforms is identical to the single-input harmonic probing method.

To obtain the cross-kernel transforms, the sum of harmonics is no longer applied at just one point but is instead split and applied at the various input points.

Let the inputs to the system be $x^{(a)}(t) = \mathrm{e}^{\mathrm{i}\Omega_1 t}$ and $x^{(b)}(t) = \mathrm{e}^{\mathrm{i}\Omega_2 t}$, so that $X^{(a)}(\omega) = 2\pi\delta(\omega - \Omega_1)$ and $X^{(b)}(\omega) = 2\pi\delta(\omega - \Omega_2)$.

Substituting these inputs into equations (47) and (48) and transferring back to the time domain yields

$$\begin{aligned} y^{(j)}(t) = {} & H_1^{(j:a)}(\Omega_1)\,\mathrm{e}^{\mathrm{i}\Omega_1 t} + H_1^{(j:b)}(\Omega_2)\,\mathrm{e}^{\mathrm{i}\Omega_2 t} \\ & + H_2^{(j:aa)}(\Omega_1, \Omega_1)\,\mathrm{e}^{\mathrm{i}2\Omega_1 t} + 2H_2^{(j:ab)}(\Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t} + H_2^{(j:bb)}(\Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}2\Omega_2 t} \\ & + H_3^{(j:aaa)}(\Omega_1, \Omega_1, \Omega_1)\,\mathrm{e}^{\mathrm{i}3\Omega_1 t} + 3H_3^{(j:aab)}(\Omega_1, \Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(2\Omega_1+\Omega_2)t} \\ & + 3H_3^{(j:abb)}(\Omega_1, \Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1+2\Omega_2)t} + H_3^{(j:bbb)}(\Omega_2, \Omega_2, \Omega_2)\,\mathrm{e}^{\mathrm{i}3\Omega_2 t} + \cdots. \end{aligned} \tag{55}$$

It can be seen that the amplitude of the component at the *sum* frequency $(\Omega_1 + \Omega_2)$ is equal to twice the second order cross-kernel transform, $H_2^{(j:ab)}(\Omega_1, \Omega_2)$, as was the case with the direct-kernel transforms. The reason for this is not due to the cross-multiplication of two inputs being applied at one point as it was in the direct-kernel case. It is due instead to the manner in which the combination terms are defined: in equation (46) it can be seen that the $h_2^{(j:ab)}(\tau_1, \tau_2)$ is preceded by a 2 under re-definition because it was constructed from two of the original kernels. The method can be generalized.

Obtaining the third order cross-kernel transform $H_3^{(j:aab)}(\Omega_1, \Omega_2, \Omega_3)$ involves probing with inputs $x^{(a)}(t) = e^{i\Omega_1 t} + e^{i\Omega_2 t}$ and $x^{(b)}(t) = e^{i\Omega_3 t}$, while the output probing expression may be greatly simplified from the general outputs, as was the case with obtaining the single-input probing expressions. This is accomplished in the same manner as before: i.e., by removing any terms which are incapable of forming a *sum* frequency component; these prove to be any kernel transform which contains repetitions of frequency components (e.g., $H_2^{(j:aa)}(\Omega_1, \Omega_1)$ and $H_3^{(j:aab)}(\Omega_1, \Omega_3, \Omega_3)$).

This gives the output probing expression for obtaining the $H_3^{(j:aab)}(\Omega_1, \Omega_2, \Omega_3)$ kernel transform as

$$\begin{aligned} y_{p3}^{(j:aab)}(t) &= H_1^{(j:a)}(\Omega_1)\, e^{i\Omega_1 t} + H_1^{(j:a)}(\Omega_2)\, e^{i\Omega_2 t} + H_1^{(j:b)}(\Omega_3)\, e^{i\Omega_3 t} + 2H_2^{(j:aa)}(\Omega_1, \Omega_2)\, e^{i(\Omega_1+\Omega_2)t} \\ &\quad + 2H_2^{(j:ab)}(\Omega_1, \Omega_3)\, e^{i(\Omega_1+\Omega_3)t} + 2H_2^{(j:ab)}(\Omega_2, \Omega_3)\, e^{i(\Omega_2+\Omega_3)t} \\ &\quad + 6H_3^{(j:aab)}(\Omega_1, \Omega_2, \Omega_3)\, e^{i(\Omega_1+\Omega_2+\Omega_3)t}. \end{aligned} \tag{56}$$

It is possible to arrive at general probing expressions after consideration of higher order terms. In general, to obtain an expression for $H_n^{(j:n_a n_b n_c \cdots)}(\Omega_1, \Omega_2, \ldots, \Omega_n)$, where n_a is the number of point a inputs, the probing inputs are found to be given by

$$x_{p_n}^{(a)}(t) = e^{i\Omega_1 t} + e^{i\Omega_{n_a} t}, \qquad x_{p_n}^{(b)}(t) = e^{i\Omega_{n_a+1} t} + e^{i\Omega_{n_a+n_b} t}, \tag{57}$$

etc., and the probing expression for the response is

$$y_{p_n}^{(j:n_a n_b n_c \cdots)}(t) = \sum_{k=1}^{k=n} \sum_{i=1}^{i=p} k!\, H_k^{(j:k_a k_b k_c \cdots)}(\Omega, \ldots, \Omega)\, e^{i(\sum \Omega)t}, \tag{58}$$

where $p = n!/k!(n-k)!$ (i.e., the number of partitions of k elements within a set of n elements) and k_a is the number of point a inputs in the kernel transform, $H_k^{(j:k_a k_b k_c \cdots)}(\Omega, \ldots, \Omega)$ and will depend upon which particular partition is being considered.

3.2. AN EXAMPLE OF HARMONIC PROBING FOR A DISCRETE-TIME SYSTEM

The objective here is to give an example of how the direct and cross-kernel transforms are obtained for a non-linear multi-input discrete-time system. The system of interest is specified by the NARMAX form

$$y_i^{(1)} = y_{i-1}^{(1)} + (y_{i-1}^{(1)})^2 + y_{i-1}^{(1)} x_{i-1}^{(1)} + x_{i-1}^{(2)}. \tag{59}$$

First, the linear FRFs are extracted. In order to obtain $H_1^{(1:1)}$, the probing expressions

$$x_{p_1}^{(1)}(t) = e^{i\Omega t}, \qquad x_{p_1}^{(2)}(t) = 0 \tag{60}$$

and

$$y_{p_1} = H_1^{(1:1)}(\Omega)\, e^{i\Omega t} \tag{61}$$

are substituted into the NARMAX model, and the coefficient of $e^{i\Omega t}$ is extracted. The result, term by term, is

$$H_1^{(1:1)}(\Omega) = H_1^{(1:1)}(\Omega)\, e^{-i\Omega \Delta t} + 0 + 0 + 0, \tag{62}$$

showing that

$$H_1^{(1:1)}(\Omega) = 0. \tag{63}$$

This is a consequence of the fact that the input $x^{(1)}(t)$ does not appear linearly in the equation. The probing expressions for the extraction of $H_1^{(1:2)}$ are

$$x_{p_1}^{(1)}(t) = 0, \qquad x_{p_1}^{(2)}(t) = \mathrm{e}^{\mathrm{i}\Omega t} \tag{64}$$

and

$$y_{p_1} = H_1^{(1:2)}(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega t}. \tag{65}$$

Substituting these expressions into the NARMAX model and equating the coefficients of $\mathrm{e}^{\mathrm{i}\Omega t}$ on each side of the equation yields

$$H_1^{(1:2)}(\Omega) = H_1^{(1:2)}(\Omega)\,\mathrm{e}^{-\mathrm{i}\Omega\Delta t} + 0 + 0 + \mathrm{e}^{-\mathrm{i}\Omega\Delta t}, \tag{66}$$

or, on rearranging,

$$H_1^{(1:2)}(\Omega) = \mathrm{e}^{-\mathrm{i}\Omega\Delta t}/(1 - \mathrm{e}^{-\mathrm{i}\Omega\Delta t}). \tag{67}$$

The second order terms are obtained next, first $H_2^{(1:11)}$ by applying the probing expressions

$$x_{p_2}^{(1)}(t) = \mathrm{e}^{\mathrm{i}\Omega_1 t} + \mathrm{e}^{\mathrm{i}\Omega_2 t}, \qquad x_{p_2}^{(2)}(t) = 0 \tag{68}$$

and

$$y_{p_2} = H_1^{(1:1)}(\Omega_1)\,\mathrm{e}^{\mathrm{i}\Omega_1 t} + H_1^{(1:1)}(\Omega_2)\,\mathrm{e}^{\mathrm{i}\Omega_2 t} + 2H_2^{(1:11)}(\Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1 + \Omega_2)t} \tag{69}$$

to the NARMAX model above. Term by term, extracting the coefficients of $\mathrm{e}^{\mathrm{i}(\Omega_1 + \Omega_2)t}$ gives

$$\begin{aligned} 2H_2^{(1:11)}(\Omega_1, \Omega_2) &= 2H_2^{(1:11)}(\Omega_1, \Omega_2)\,\mathrm{e}^{-\mathrm{i}(\Omega_1 + \Omega_2)\Delta t} + 2H_1^{(1:1)}(\Omega_1)H_1^{(1:1)}(\Omega_2)\,\mathrm{e}^{-\mathrm{i}(\Omega_1 + \Omega_2)\Delta t} \\ &\quad + (H_1^{(1:11)}(\Omega_1) + H_1^{(1:1)}(\Omega_2))\,\mathrm{e}^{-\mathrm{i}(\Omega_1 + \Omega_2)\Delta t} + 0, \end{aligned} \tag{70}$$

or, on rearranging,

$$\begin{aligned} H_2^{(1:11)}(\Omega_1, \Omega_2) &= \tfrac{1}{2}[H_1^{(1:1)}(\Omega_1) + H_1^{(1:1)}(\Omega_2) + 2H_1^{(1:1)}(\Omega_1)H_1^{(1:1)}(\Omega_2)]\,\frac{\mathrm{e}^{-\mathrm{i}(\Omega_1 + \Omega_2)\Delta t}}{1 - \mathrm{e}^{-\mathrm{i}(\Omega_1 + \Omega_2)\Delta t}} \\ &= \tfrac{1}{2}[H_1^{(1:1)}(\Omega_1) + H_1^{(1:1)}(\Omega_2) + 2H_1^{(1:1)}(\Omega_1)H_1^{(1:1)}(\Omega_2)]H_1^{(1:2)}(\Omega_1 + \Omega_2). \end{aligned} \tag{71}$$

As all the terms in the square brackets are equal to zero it follows that $H_2^{(1:11)}(\Omega_1, \Omega_2) = 0$.

The other direct-kernel transform $H_2^{(1:22)}$ comes from an application of the probing expressions

$$x_{p_2}^{(1)}(t) = 0, \qquad x_{p_2}^{(2)}(t) = \mathrm{e}^{\mathrm{i}\Omega_1 t} + \mathrm{e}^{\mathrm{i}\Omega_2 t} \tag{72}$$

and

$$y_{p_2} = H_1^{(1:2)}(\Omega_1)\,\mathrm{e}^{\mathrm{i}\Omega_1 t} + H_1^{(1:2)}(\Omega_2)\,\mathrm{e}^{\mathrm{i}\Omega_2 t} + 2H_2^{(1:22)}(\Omega_1, \Omega_2)\,\mathrm{e}^{\mathrm{i}(\Omega_1 + \Omega_2)t}. \tag{73}$$

Substituting into the system equation and collecting appropriate coefficients yields

$$2H_2^{(1:22)}(\Omega_1, \Omega_2) = 2H_2^{(1:22)}(\Omega_1, \Omega_2)\,\mathrm{e}^{-\mathrm{i}(\Omega_1 + \Omega_2)\Delta t} + 2H_1^{(1:2)}(\Omega_1)H_1^{(1:2)}(\Omega_2)\,\mathrm{e}^{-\mathrm{i}(\Omega_1 + \Omega_2)\Delta t} + 0 + 0, \tag{74}$$

or, after a little algebra,

$$H_2^{(1:22)}(\Omega_1, \Omega_2) = \tfrac{1}{2}H_1^{(1:2)}(\Omega_1)H_1^{(1:2)}(\Omega_2)H_1^{(1:2)}(\Omega_1 + \Omega_2). \tag{75}$$

Finally, at second order, the cross-kernel transform $H_2^{(1:12)}$ is obtained by applying the probing expressions

$$x_{p_2}^{(1)}(t) = \mathrm{e}^{\mathrm{i}\Omega_1 t}, \qquad x_{p_2}^{(2)}(t) = \mathrm{e}^{\mathrm{i}\Omega_2 t} \tag{76}$$

and

$$y_{p_2} = H_1^{(1:1)}(\Omega_1)\, \mathrm{e}^{\mathrm{i}\Omega_1 t} + H_1^{(1:2)}(\Omega_2)\, \mathrm{e}^{\mathrm{i}\Omega_2 t} + H_2^{(1:12)}(\Omega_1, \Omega_2)\, \mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t}. \tag{77}$$

Substituting into the system equation and extracting the coefficients of $\mathrm{e}^{\mathrm{i}(\Omega_1+\Omega_2)t}$ yields

$$\begin{aligned} H_2^{(1:12)}(\Omega_1, \Omega_2) = {} & H_2^{(1:12)}(\Omega_1, \Omega_2)\, \mathrm{e}^{-\mathrm{i}(\Omega_1+\Omega_2)\Delta t} + 2H_1^{(1:1)}(\Omega_1) H_1^{(1:2)}(\Omega_2)\, \mathrm{e}^{-\mathrm{i}(\Omega_1+\Omega_2)\Delta t} \\ & + H_1^{(1:2)}(\Omega_2)\, \mathrm{e}^{-\mathrm{i}(\Omega_1+\Omega_2)\Delta t} + 0, \end{aligned} \tag{78}$$

or, after a little effort,

$$H_2^{(1:12)}(\Omega_1, \Omega_2) = H_1^{(1:2)}(\Omega_2)[1 + 2H_1^{(1:1)}(\Omega_1)] H_1^{(1:2)}(\Omega_1 + \Omega_2). \tag{79}$$

The calculations for the third order kernel transforms proceed as above; no new features arise.

3.3. RESPONSE FROM A NON-LINEAR SYSTEM WITH DUAL SINUSOIDAL INPUT

Now that the response of a non-linear system to harmonic inputs at distinct points has been discussed, the next step is to deal with physically realizable inputs in the form of sinusoids. This may then be used to provide a means of numerical verification of the method. As a sinusoid can be represented by a sum of positive and negative frequency harmonics this is a relatively straightforward extension. Once again, the basic two-input problem will be considered. Let the inputs at points a and b be given by

$$\begin{aligned} x^{(a)}(t) &= A\cos(\Omega_a t + \phi_a) = \frac{A}{2}\left\{\cos\phi_a(\mathrm{e}^{\mathrm{i}\Omega_a t} + \mathrm{e}^{-\mathrm{i}\Omega_a t}) - \frac{\sin\phi_a}{\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega_a t} - \mathrm{e}^{-\mathrm{i}\Omega_a t})\right\}, \\ x^{(b)}(t) &= B\cos(\Omega_b t + \phi_b) = \frac{B}{2}\left\{\cos\phi_b(\mathrm{e}^{\mathrm{i}\Omega_b t} + \mathrm{e}^{-\mathrm{i}\Omega_b t}) - \frac{\sin\phi_b}{\mathrm{i}}(\mathrm{e}^{\mathrm{i}\Omega_b t} - \mathrm{e}^{-\mathrm{i}\Omega_b t})\right\}, \end{aligned} \tag{80}$$

or in terms of frequency,

$$\begin{aligned} X^{(a)}(\omega) &= \pi A\{\mathrm{e}^{\mathrm{i}\phi_a}\delta(\omega - \Omega_a) + \mathrm{e}^{-\mathrm{i}\phi_a}\delta(\omega + \Omega_a)\}, \\ X^{(b)}(\omega) &= \pi B\{\mathrm{e}^{\mathrm{i}\phi_b}\delta(\omega - \Omega_b) + \mathrm{e}^{-\mathrm{i}\phi_b}\delta(\omega + \Omega_b)\}. \end{aligned} \tag{81}$$

Substituting for the inputs into equations (47) and (48) gives the response, up to third order, at point j as,

$$\begin{aligned} y^{(j)}(t) = {} & A|H_1^{(j:a)}(\Omega_a)|\cos(\Omega_a t + \phi_a + \angle H_1^{(j:a)}(\Omega_a)) \\ & + B|H_1^{(j:b)}(\Omega_b)|\cos(\Omega_b t + \phi_b + \angle H_1^{(j:b)}(\Omega_b)) \\ & + \frac{A^2}{2}\{|H_2^{(j:aa)}(\Omega_a, \Omega_a)|\cos(2\Omega_a t + 2\phi_a + \angle H_2^{(j:aa)}(\Omega_a, \Omega_a)) + H_2^{(j:aa)}(\Omega_a, -\Omega_a)\} \\ & + \frac{AB}{2}\{|H_2^{(j:ab)}(\Omega_a, \Omega_b)|\cos((\Omega_a + \Omega_b)t + \phi_a + \phi_b + \angle H_2^{(j:ab)}(\Omega_a, \Omega_b)) \\ & + |H_2^{(j:ab)}(\Omega_a, -\Omega_b)|\cos((\Omega_a + \Omega_b)t + \phi_a - \phi_b + \angle H_2^{(j:ab)}(\Omega_a, -\Omega_b))\} \\ & + \frac{B^2}{2}\{|H_2^{(j:bb)}(\Omega_b, \Omega_b)|\cos(2\Omega_b t + 2\phi_b + \angle H_2^{(j:bb)}(\Omega_b, \Omega_b)) + H_2^{(j:bb)}(\Omega_b, -\Omega_b)\} \\ & + \frac{A^3}{4}\{|H_3^{(j:aaa)}(\Omega_a, \Omega_a, \Omega_a)|\cos(3\Omega_a t + 3\phi_a + \angle H_3^{(j:aaa)}(\Omega_a, \Omega_a, \Omega_a)) \\ & + 3|H_3^{(j:aaa)}(\Omega_a, \Omega_a, -\Omega_a)|\cos(\Omega_a t + \phi_a + \angle H_3^{(j:aaa)}(\Omega_a, \Omega_a, -\Omega_a))\} \end{aligned}$$

$$+\frac{A^2B}{4}\{|H_3^{(j:aab)}(\Omega_a,\Omega_a,\Omega_b)|\cos((2\Omega_a+\Omega_b)t+2\phi_a+\phi_b$$
$$+\angle H_3^{(j:aab)}(\Omega_a,\Omega_a,\Omega_b))+|H_3^{(j:aab)}(\Omega_a,\Omega_a,-\Omega_b)|\cos((2\Omega_a-\Omega_b)t$$
$$+2\phi_a-\phi_b+\angle H_3^{(j:aab)}(\Omega_a,\Omega_a,-\Omega_b))$$
$$+2|H_3^{(j:aab)}(\Omega_a,-\Omega_a,\Omega_b)|\cos(\Omega_b t+\phi_b+\angle H_3^{(j:aab)}(\Omega_a,-\Omega_a,\Omega_b))\}$$
$$+\frac{AB^2}{4}\{|H_3^{(j:abb)}(\Omega_a,\Omega_b,\Omega_b)|\cos((\Omega_a+2\Omega_b)t+\phi_a+2\phi_b$$
$$+\angle H_3^{(j:abb)}(\Omega_a,\Omega_b,\Omega_b))+|H_3^{(j:abb)}(-\Omega_a,\Omega_b,\Omega_b)|\cos(-\Omega_a+2\Omega_b)t-\phi_a$$
$$+2\phi_b+\angle H_3^{(j:abb)}(-\Omega_a,\Omega_b,\Omega_b))$$
$$+2|H_3^{(j:abb)}(\Omega_a,\Omega_b,-\Omega_b)|\cos(\Omega_a t+\phi_a+\angle H_3^{(j:abb)}(\Omega_a,\Omega_b,-\Omega_b))\}$$
$$+\frac{B^3}{4}\{|H_3^{(j:bbb)}(\Omega_b,\Omega_b,\Omega_b)|\cos(3\Omega_b t+3\phi_b+\angle H_3^{(j:bbb)}(\Omega_b,\Omega_b,\Omega_b))$$
$$+3|H_3^{(j:bbb)}(\Omega_b,\Omega_b,-\Omega_b)|\cos(\Omega_b t+\phi_b+\angle H_3^{(j:bbb)}(\Omega_b,\Omega_b,-\Omega_b))\}. \tag{82}$$

Results of this nature have been presented before by Gifford and Tomlinson [24] for the more restricted case in which multiple signals are input to a system at the same point.

After consideration of many higher order terms in the series it is possible to obtain, by inspection, the general expression for the response at point j for this two-input case. This is given by

$$y^{(j)}(t)=\sum_{n=1}^{n=\infty}\sum_{n_b=0}^{n_b=n}\frac{A^{(n-n_b)}B^{n_b}}{2^{(n-1)}}\left\{\sum_{p_a,p_b}\frac{n_a!n_b!}{mpa!p_b!(n_a-p_a)!(n_b-p_b)!}\right.$$
$$\times|H_b^{(j:\overbrace{a\cdots a}^{n_a}\overbrace{b\cdots b}^{n_b})}(\overbrace{\Omega_a,\ldots,\Omega_a}^{p_a},-\overbrace{\Omega_a,\ldots,-\Omega_a}^{(n_a-p_a)},\overbrace{\Omega_b,\ldots,\Omega_b}^{p_b},-\overbrace{\Omega_b,\ldots,-\Omega_b}^{(n_b-p_b)})|$$
$$\times\cos([(2p_a-n_a)\Omega_a+(2p_b-n_b)\Omega_b]t+[(2p_a-n_a)\phi_a+(2p_b-n_b)\phi_b]$$
$$\left.+\angle H_n^{(j:a\cdots ab\cdots b)}(\Omega_a,\ldots,\Omega_a,-\Omega_a,\ldots,-\Omega_a,\Omega_b,\ldots,\Omega_b,-\Omega_b,\ldots,-\Omega_b))\right\}, \tag{83}$$

where p_a is the number of positive Ω_a's in the kernel transform; and similarly for p_b. The p_a, p_b summation is repeated to give all possible frequency combinations. This results in $[(n_a+1)(n_b+1)+1]/2$ terms when n_a and n_b are even and $[(n_a+1)(n_b+1)]/2$ terms otherwise.

Also, $m=2$ if $p_a=n_a/2$ and $p_b=n_b/2$ else $m=1$.

4. VERIFICATION OF KERNEL IDENTIFICATION IN A MULTI-INPUT MULTI-OUTPUT SYSTEM

In this section a concrete example of a 2-DOF non-linear system is used to verify the work carried out in the last section and to illustrate the importance of the cross-kernel terms. Consider the non-linear system (see Figure 1) specified by the equation of motion

$$\begin{aligned} &m_1\ddot{y}^{(1)} + (c_{11} + c_{12})\dot{y}^{(1)} - c_{12}\dot{y}^{(2)} + (k_{11} + k_{12})y^{(1)} - k_{12}y^{(2)} \\ &\qquad + c_2(\dot{y}^{(1)} - \dot{y}^{(2)})^2 + c_3(\dot{y}^{(1)} - \dot{y}^{(2)})^3 + k_2(y^{(1)})^2 + k_3(y^{(1)})^3 = x^{(1)}(t), \\ &m_2\ddot{y}^{(2)} - c_{12}\dot{y}^{(1)} + (c_{12} + c_{22})\dot{y}^{(2)} - k_{12}y^{(1)} + (k_{12} + k_{22})y^{(2)} \\ &\qquad - c_2(\dot{y}^{(1)} - \dot{y}^{(2)})^2 - c_3(\dot{y}^{(1)} - \dot{y}^{(2)})^3 = x^{(2)}(t). \end{aligned} \tag{84}$$

The first part of this example requires using the harmonic probing expressions developed in section 3 to obtain the direct and cross-kernel transforms for this system. Once these have been obtained the equations obtained near the end of section 3 will then be used to arrive at expressions for the system response.

The required probing inputs are $x^{(1)}(t) = \mathrm{e}^{\mathrm{i}\Omega t}$ and $x^{(2)}(t) = 0$ and the output probing expression

$$y_p^{(j:1)}(t) = H_1^{(j:1)}(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega t}. \tag{85}$$

Substituting equation (85) into the equations of motion and equating coefficients of $\mathrm{e}^{\mathrm{i}\Omega t}$ yields expressions for $H_1^{(1:1)}(\Omega)$ and $H_1^{(2:1)}(\Omega)$. Similarly, expressions for $H_1^{(1:2)}(\Omega)$ and $H_1^{(2:2)}(\Omega)$ are obtained by setting $x^{(1)}(t) = 0$ and $x^{(2)}(t) = \mathrm{e}^{\mathrm{i}\Omega t}$.

The following matrix expression for all H_1's is obtained:

$$\begin{bmatrix} H_1^{(1:1)}(\Omega) & H_1^{(1:2)}(\Omega) \\ H_1^{(2:1)}(\Omega) & H_1^{(2:2)}(\Omega) \end{bmatrix} = \begin{bmatrix} -\Omega^2 m_1 + (k_{11} + k_{12}) & -k_2 \\ +\mathrm{i}\Omega(c_{11} + c_{12}) & -\mathrm{i}\Omega c_{12} \\ -k_{12} & -\Omega^2 m_2 + (k_{12} + k_{22}) \\ -\mathrm{i}\Omega c_{12} & +\mathrm{i}\Omega(c_{12} + c_{22}) \end{bmatrix}^{-1}. \tag{86}$$

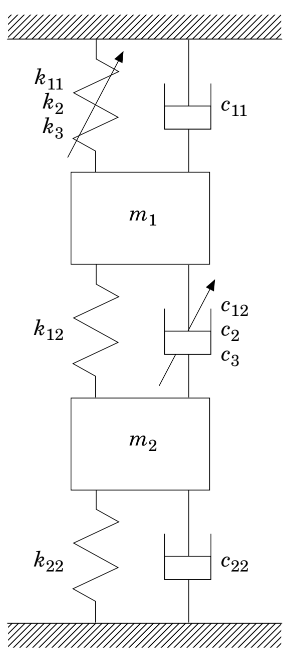

Figure 1. A 2-DOF non-linear system.

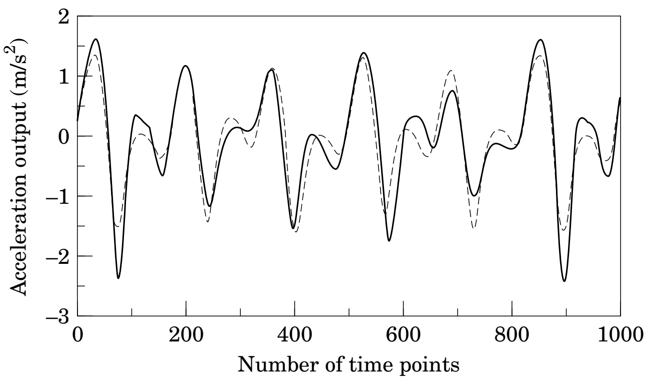

Figure 2. Acceleration time responses for mass 1 of the simulation and Volterra series approximations of the 2-DOF non-linear system. $x^{(1)}(t) = 3\cos(50t + 135°)$, $x^{(2)}(t) = 4\cos(30t + 90°)$. ——, Simulation; - - -, direct and cross-terms; – – –, direct terms only.

An expression for the matrix of second order kernel transforms (cross- and direct-) is obtained in the same manner: i.e., by substituting the appropriate probing expressions (the reduced versions of equations (54) and (55)) into the equations of motion and equating coefficients of $e^{i(\Omega_1+\Omega_2)t}$. The matrix can be written as

$$\begin{bmatrix} H_2^{(1:11)}(\Omega_1,\Omega_2) & H_2^{(1:12)}(\Omega_1,\Omega_2) & H_2^{(1:22)}(\Omega_1,\Omega_2) \\ H_2^{(2:11)}(\Omega_1,\Omega_2) & H_2^{(2:12)}(\Omega_1,\Omega_2) & H_2^{(2:22)}(\Omega_1,\Omega_2) \end{bmatrix}$$

$$= [H_1(\Omega_1+\Omega_2)]\begin{bmatrix} \alpha^{(1:11)} & \alpha^{(1:12)} & \alpha^{(1:22)} \\ \alpha^{(2:11)} & \alpha^{(2:12)} & \alpha^{(2:22)} \end{bmatrix}, \tag{87}$$

where

$$[H_1(\Omega_1+\Omega_2)] = \begin{bmatrix} H_1^{(1:1)}(\Omega_1+\Omega_2) & H_1^{(1:2)}(\Omega_1+\Omega_2) \\ H_1^{(2:1)}(\Omega_1+\Omega_2) & H_1^{(2:2)}(\Omega_1+\Omega_2) \end{bmatrix}$$

$$= \begin{bmatrix} -(\Omega_1+\Omega_2)^2 m_1 + (k_{11}+k_{12}) & -k_{12} \\ +i(\Omega_1+\Omega_2)(c_{11}+c_{12}) & -i(\Omega_1+\Omega_2)c_{12} \\ -k_{12} & -(\Omega_1+\Omega_2)^2 m_2 + (k_{12}+k_{22}) \\ -i(\Omega_1+\Omega_2)c_{12} & +i(\Omega_1+\Omega_2)(c_{12}+c_{22}) \end{bmatrix}^{-1} \tag{88}$$

and

$$\alpha^{(j:pq)} = (-1)^j m\{(1-\delta_{j2})k_2 H_1^{(1:p)}(\Omega_1)H_1^{(1:q)}(\Omega_2)$$
$$+ c_2(i\Omega_1)(i\Omega_2)(H_1^{(1:p)}(\Omega_1) - H_1^{(2:p)}(\Omega_1))(H_1^{(1:q)}(\Omega_2) - H_1^{(2:q)}(\Omega_2))\}, \tag{89}$$

where $m = 1$ if $p = q$, $m = 2$ if $p \neq q$ and δ_{ij} is the Kronecker delta [25].

This process may be extended to obtain expressions for higher order kernel transforms for this system.

The values of the constants used in this example were $m_1 = m_2 = 1$ kg, $c_{11} = c_{12} = c_{22} = 20$ N/m/s and $c_2 = 500$ N(m/s)^2, $c_3 = 1 \times 10^4$ N/(m/s)^3, $k_{11} = k_{12} = k_{22} =$

1×10^4 N/m, $k_2 = 1 \times 10^7$ N/m^2 and $k_3 = 5 \times 10^9$ N/m^3. Substituting these parameters into the above equations yields values for the higher order kernel transforms for this system. The forcing conditions used in this example were $x^{(1)}(t) = 3\cos(50t + 135^\circ)$ and $x^{(2)}(t) = 4\cos(30t + 90^\circ)$. Substituting these inputs and the previously calculated higher order kernel transforms into equation (82) yields expressions for the responses at both masses. This response at mass 1 was then calculated up to $O(X^{10})$, plotted and compared (see Figure 2) to the time responses generated via a fourth order Runge-Kutta integration of the equations of motion (84) [26]. As can be seen, the multi-input Volterra series approximation of the response gives an extremely close match to the simulated result: in fact, the responses overlie. Also shown in the figures are the time responses when the cross-kernel terms are removed. The effect on the response illustrates the importance of these terms. In Figure 3 are depicted the same results in the frequency domain. It can be seen that additional spikes at the sum and difference frequencies in Figure 3(a) are entirely due to the cross-kernel terms.

5. CONCLUSIONS

It has been shown that a simple extension of the harmonic probing algorithm of Bedrosian and Rice is sufficient to allow the calculation of both direct- and cross-kernel transforms for the multi-input Volterra series. An example calculation is given to show the importance of the cross-kernel terms and the application to response prediction is demonstrated.

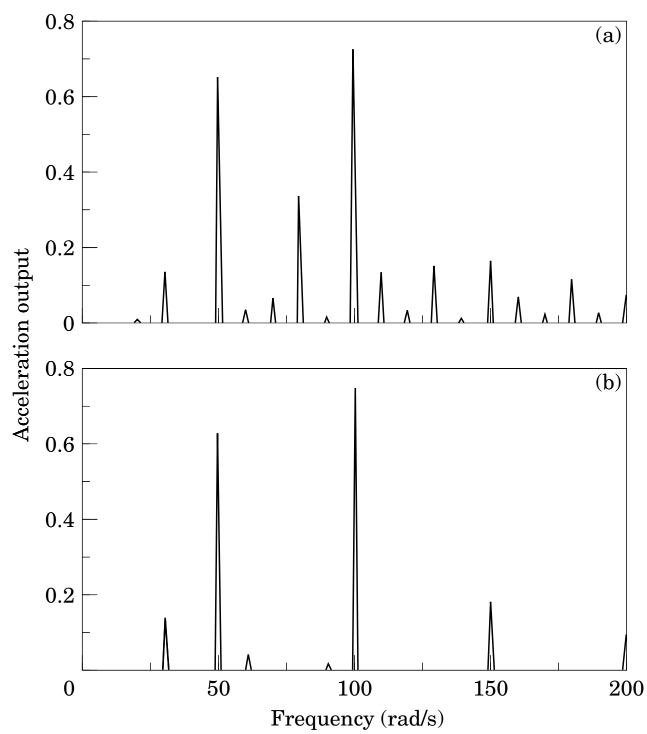

Figure 3. Acceleration frequency spectra for mass 1 of the simulation and Volterra series approximations of the 2-DOF non-linear system. $x^{(1)}(t) = 3\cos(50t + 135^\circ)$, $x^{(2)}(t) = 4\cos(30t + 90^\circ)$. (a) ——, Simulation, - - -, direct and cross-terms. (b) —, direct terms only.

ACKNOWLEDGMENTS

The authors would like to express their thanks to an anonymous referee for a number of constructive comments which we feel have improved the paper.

REFERENCES

1. S. A. Billings 1980 *Proceedings of the IEE* **127**, 272–285. Identification of nonlinear systems—a survey.
2. M. J. Korenburg and I. W. Hunter 1990 *Annals of Biomedical Engineering* **18**, 629–654. The identification of nonlinear biological systems: Wiener kernel approaches.
3. W. J. Rugh 1981 *Nonlinear System Theory—the Volterra/Wiener Approach*. Baltimore, MD: Johns Hopkins University Press.
4. M. Fliess, M. Lamnabhi and F. Lamnabhi-Lagarrigue 1983 *IEEE Transactions on Circuits and Systems* **30**, 554–570. An algebraic approach to nonlinear functional expansions.
5. S. A. Billings and K. M. Tsang 1989 *Mechanical Systems and Signal Processing* **3**, 319–339. Spectral analysis for nonlinear systems, part I: parametric non-linear spectral analysis.
6. J. Wray and G. G. R. Green 1995 *Biological Cybernetics*. Calculation of the Volterra kernels of nonlinear dynamic systems using an artificial neural network.
7. P. K. Marmarelis and K. I. Naka 1974 *IEEE Transactions on Biomedical Engineering* **21**, 88–101. Identification of multi-input biological systems.
8. P. Z. Marmarelis and V. Marmarelis 1978 *Analysis of Physiological Systems—the White Noise Approach*. New York: Plenum Press.
9. J. J. Bussgang, L. Ehrman and J. W. Graham 1974 *Proceedings of the IEEE* **62**, 1088–1119. Analysis of nonlinear systems with multiple inputs.
10. K. Worden, S. A. Billings, P. K. Stansby and G. R. Tomlinson 1992 *Journal of Fluids and Structures* **8**, 18–71. Identification of nonlinear wave forces.
11. E. Bedrosian and S. O. Rice 1971 *Proceedings of the IEEE* **59**, 1688–1707. The output properties of Volterra systems driven by harmonic and Gaussian inputs.
12. V. Volterra 1959 *Theory of Functionals and Integral Equations*. New York: Dover.
13. J. F. Barrett 1963 *Journal of Electronics and Control* **15**, 567–615. The use of functionals in the analysis of nonlinear systems.
14. M. Schetzen 1980 *The Volterra and Wiener Theories of Nonlinear Systems*. New York: Wiley–Interscience.
15. S. A. Billings and K. M. Tsang 1989 *Mechanical Systems and Signal Processing* **3**, 341–359. Spectral analysis for nonlinear systems, part II: interpretation of nonlinear frequency response functions.
16. J. C. Peyton Jones and S. A. Billings 1989 *International Journal of Control* **50**, 1925–1940. Recursive algorithm for computing the frequency response of a class of non-linear difference equation models.
17. I. J. Leontaritis and S. A. Billings 1985 *International Journal of Control* **41**, 303–328. Input–output parametric models for nonlinear systems. part I: deterministic nonlinear systems.
18. I. J. Leontaritis and S. A. Billings 1985 *International Journal of Control* **41**, 329–344. Input–output parametric models for nonlinear systems. part II: stochastic nonlinear systems.
19. G. Palm and T. Poggio 1977 *SIAM Journal on Applied Mathematics* **33**. The Volterra representation and the Wiener expansion: validity and pitfalls.
20. G. Palm and B. Pöpel 1985 *Quarterly Review of Biophysics* **18**, 135–164. Volterra representation and Wiener-like identification of nonlinear systems: scope and limitations.
21. G. F. Simmons 1963 *Introduction to Topology and Modern Analysis*. New York: McGraw-Hill.
22. J. F. Barrett 1965 *International Journal of Control* **1**, 209–216. The use of Volterra series to find region of stability of a non-linear differential equation.
23. G. R. Tomlinson, G. Manson and G. M. Lee 1996 *Journal of Sound and Vibration* **190**, 751–762. A simple criterion for establishing an upper limit of the harmonic excitation level to the Duffing oscillator using the Volterra series.
24. S. J. Gifford and G. R. Tomlinson 1989 *Journal of Sound and Vibration* **135**, 289–317. Recent advances in the application of functional series to non-linear structures.
25. W. T. Thomson 1988 *Theory of Vibration with Applications*. London: Unwin Hyman; third edition.
26. W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vetterling 1986 *Numerical Recipes—the Art of Scientific Computing*. Cambridge: Cambridge University Press.