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THE TRANSIENT AMPLITUDE OF THE VISCOELASTIC TRAVELLING STRING: AN INTEGRAL CONSTITUTIVE LAW

R.-F. FUNG, J.-S. HUANG AND Y.-C. CHEN

Department of Mechanical Engineering, Chun Yuan Christian University, Chun-Li, Taiwan 32023, *Republic of China*

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The problem considered is an initially stressed viscoelastic string subjected to steady state and harmonic variation of axial motion. The string material is assumed to be constituted by the hereditary integral type. The partial differential–integral equation of motion is derived first. Then by applying Galerkin's method, the governing equation is reduced to a set of second order non-linear differential–integral equations which are solved by finite difference numerical integration procedures. The viscoelastic string with a linear spring–dashpot model is considered as the Voigt element in series with a spring (three-parameter model). The qualitative aspect of parametric excitation due to the non-uniform travelling velocity of a viscoelastic string is investigated. Finally, the results are compared with those obtained by a constitutive law of a differential type method. The effects of elastic and viscoelastic parameters, constant and non-constant transport speeds, the wave propagation speed ratio, and the non-linear term on the transient amplitudes are also investigated.

1. INTRODUCTION

The vibration problems of initially stressed plate studied by past investigators [1, 2] have been mainly concerned with in-plane stress. The effects of arbitrary initial stress on the vibration and stability problems of a thick rectangular plate were studied by Brunelle and Robertson [3, 4]. Chen and Doong [5, 6] extended Brunelle's theory to study the effect of an arbitrary state of initial stress on a non-linear vibration problem. All of the above studies [3–6] were based on Mindlin's plate theory. To the authors' knowledge, the initially stressed theories developed in [1–6] were not extended to string vibration problems.

The non-linear vibration of axial moving strings has been studied extensively by many investigators [7–10]. Mote [7] employed the method of characteristics and demonstrated the importance of non-linear considerations even for small amplitude oscillations. Bapat and Srinivasan [8] used the method of harmonic balance to obtain approximate results of the period–tension relationship. A Mathieu–Hill type of system existed for a moving band or belt and conditions for stable operation were determined in [9]. Results were verified experimentally. In Mote's study [10], stable–unstable boundaries were predicted by application of Hsu's method for one-dimensional systems. Recently, Huang *et al*. [11] studied the dynamic stability of a moving string under three-dimensional vibration.

In all of these works, the string was assumed to be elastic. However, new materials are now widely used in industry; and many of them, such as plastics and composite materials,

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do not obey Hooke's Law. Coleman and Noll [12] proved that simple isotropic material under small deformations can be modelled by integral or differential types of linear viscoelastic models. Stevens [13] considered the stability of an initially straight, simply supported viscoelastic column subjected to an harmonically varying axial load. The column material was assumed to be adequately represented by simple spring–dashpot models.

There are many engineering designs which require viscoelastic behavior of structures; for example, creep analysis of magnetic tapes and vibration problems of conduits. In particular, damping of viscoelastic materials can be used to reduce vibration of structures. Various methods have been presented for the analysis of such problems. Lee [14] studied the stress analysis for linear viscoelastic material using an integral type of stress–strain relation and a simple finite difference numerical procedure for integration. The application of Laplace transform to viscoelastic beams was presented by Flügge [15]. Findley *et al*. [16] used the correspondence and superposition principles to solve the governing equations of the viscoelastic beam. Christenden [17] used Fourier transform to solve the transient response of viscoelastic beams. The above studies were based on the fact that the governing equations of viscoelasticity can be converted to equations of elasticity by integral transforms. The application of the finite element method to structures with complex geometry has been presented by a number of authors. White [18] used a constitutive law of hereditary integral type, in which time integration is approximated by the finite difference method, to perform a finite element analysis in a static plane strain problem. Chen and Lin [19] studied the dynamic response of a beam using a creep law of time hardening to model viscoelastic material. Recently, Chen {20] used the Laplace transform, and the associated equation was solved by the finite element method for the linear viscoelastic beam problem.

In this paper, the initially stressed theory [3] is extended to investigate the transverse vibration of an axially moving viscoelastic string. It is further assumed that the string material is represented adequately by the linear spring–dashpot model. The model considered is the Voigt element in series with a spring (three-parameter model). The constitutive law of hereditary integral type is used to lead to a set of integral equations. Galerkin's method is used to approximate the system to a set of two-order, gyroscopic and non-linear ordinary differential–integral equations. Finally, the associated equation is solved using a finite difference numerical integration procedure. Numerical results are emphasized on the effects of material parameters, the axial travelling speed, the wave speed ratio, and linear and non-linear terms.

2. EQUATION OF MOTION

The physical model of a viscoelastic travelling string is shown in Figure 1. The string is initially stressed. Following a technique described by Bolotin [21], the non-linear

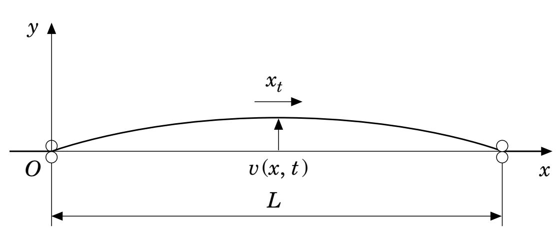

Figure 1. A model of an axially moving viscoelastic string system.

dynamic equation of motion in tensor form can be expressed in terms of Trefftz stress components as [22]

$$\frac{\partial}{\partial x_i}[(t^{ij}+\sigma^{ij})u^2_{,j}+\sigma^{is}]+X^s-\rho\ddot{u}^s=0, \qquad i, j, s=1, 2, 3, \tag{1}$$

where the x_i are material (Lagrangian) co-ordinates that originally coincided with a spatial (fixed) Cartesian co-ordinate system. t^{ij} is the initial (deformed) stress tensor, σ^{ij} and σ^{is} are the perturbed stress tensors, ρ is the mass per unit volume, u^s is the perturbed displacement tensor, and $u^s_{,j}$ is the covariant derivative notation. The contravariant indices are adopted for t^{ij}, σ^{ij}, u^s and the body force X^s in order to coincide with the tensor analysis of motion of an elastic body in a generalized coordinate system [23]. A brief derivation of these component forms of the equations from vector form was given by Brunelle and Robertson [3].

Consider that the viscoelastic string is in a state of uniform initial stress, and only transverse vibration in the y direction is taken into consideration. The body force is also neglected. From equation (1), we can obtain the equation of motion in the y direction as

$$\left(\frac{T}{A}+\sigma\right)v_{xx}+v_x\sigma_x=\rho\frac{\mathrm{d}^2v}{\mathrm{d}t^2}, \tag{2}$$

where σ is the perturbed stress and v is the displacement in the transverse direction. A is the area of cross-section of the string. The Lagrangian strain component in the x direction related to the displacement is given by

$$\varepsilon(x,t)=\tfrac{1}{2}v_x^2(x,t). \tag{3}$$

3. INTEGRAL CONSTITUTIVE LAW

We adopt the one-dimensional constitutive equation of an integral type material (called a Boltzmann material) which is given by the Boltzmann superposition principle [23] as

$$\sigma(x,t)=\varepsilon(x,t)E_0+\int_0^t\dot{E}(t-t')\varepsilon(x,t')\,\mathrm{d}t', \tag{4}$$

where $E(t)$ is the stress relaxation function while E_0 is its value at $t=0$, i.e., the initial Young's modulus of the material.

Substituting equation (3) into equation (4), we obtain the tensile stress of the string, and then substituting the tensile into equation (2), with some manipulations, we obtain

$$\begin{aligned}\rho v_{tt}(x,t)&+2\rho x_t v_{xt}(x,t)+(\rho x_1^2-T/A)v_{xx}(x,t)+x_{tt}v_x(x,t)\\&=\tfrac{3}{2}E_0v_x^2(x,t)v_{xx}(x,t)+\tfrac{1}{2}v_{xx}(x,t)\int_0^t\dot{E}(t-t')v_x^2(x,t')\,\mathrm{d}t'\\&\quad+v_x(x,t)\int_0^t\dot{E}(t-t')v_x(x,t')v_{xx}(x,t')\,\mathrm{d}t']\end{aligned} \tag{5}$$

The non-linear partial differential–integral equation (5) governs the dynamic behavior of the initially stressed, viscoelastic travelling string. In this paper, it is assumed that the initial tension T is characterized as a small periodic perturbation $T_1\cos\omega_f t$ superimposed on the steady state tension T_0, i.e., $T = T_0 + T_1 \cos \omega_f t$, which is the same as that of Huang *et al*. [11].

4. DISCRETIZATION OF LINEARIZED SYSTEM

In order to simplify the analysis, the following non-dimensional parameters are defined

$$V = v/L, \qquad \xi = x/L, \qquad \bar{v} = T_1/T_0, \qquad \Omega_f = \omega_f L/c_2, \qquad k_1 = c_1/c_2,$$
$$\bar{E}(\tau - \tau') = E(t - t')/\rho c_2^2, \qquad \tau = c_2 t/L,$$

where $c_1 = \sqrt{E_0/\rho}$ and $c_2 = \sqrt{T_0/\rho A}$. By the definitions of ξ and τ, $\xi_r = x_1/c_2$. Then the following non-dimensional equation of motion can be obtained:

$$V_{rr} + 2\xi_\tau V_{\xi\tau} + (\xi_\tau^2 - 1 - \bar{v} \sin \Omega_f \tau) V_{\xi\xi} + \xi_{\tau\tau} V_\xi$$
$$= \tfrac{3}{2} k_1^2 V_\xi^2 V_{\xi\xi} - \tfrac{1}{2} V_{\xi\xi} \int_0^\tau \frac{\partial}{\partial \tau'} \bar{E}(\tau - \tau') V_\xi^2(\xi, \tau')\, \mathrm{d}\tau'$$
$$- V_\xi \int_0^\tau \frac{\partial}{\partial \tau'} \bar{E}(\tau - \tau') V_\xi(\xi, \tau') V_{\xi\xi}(\xi, \tau')\, \mathrm{d}\tau'. \tag{6}$$

It is seen that terms related to the axial travelling motion include the velocity ξ_τ and acceleration $\xi_{\tau\tau}$. Equation (6) is used to investigate the transient amplitudes of the string with non-uniform axial motion. Equation (6) is a non-linear partial differential–integral equation. Neglecting the non-linear terms for small oscillation, the linear equation of motion becomes

$$V_{\tau\tau} + 2\xi_\tau V_{\xi\tau} + (\xi_\tau^2 - 1 - \bar{v} \sin \Omega_f \tau) V_{\xi\xi} + \xi_{\tau\tau} V_\xi = 0. \tag{7}$$

First, the spatial dependence must be eliminated from the equation of motion, yielding a set of ordinary differential–integral equations in time which can be solved for the system response. Due to the complexity of the problem, the solution for equation (6) cannot be obtained exactly. Therefore, Galerkin's method [24] is used here to separate the spatial co-ordinates from the temporal variable. Based on this method, the assumed displacement satisfies the geometric boundary conditions $V(0, \tau) = V(1, \tau) = 0$. That is,

$$V(\xi, \tau) = \sum_{n=1}^{\infty} f_n(\tau) \sin(n\pi\xi). \tag{8}$$

Substituting equation (8) into equation (5) and using the orthogonality condition, we have

$$\ddot{f}_m + \sum_{n=1}^{\infty} [2\xi_\tau A_{mn} \dot{f}_n + \xi_{\tau\tau} A_{mn} f_n + (\xi_\tau^2 - 1 - \bar{v} \sin \Omega_f \tau) B_{mn} f_n] - \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Bigg[\tfrac{3}{2} k_1^2 f_n f_i f_j C_{mnij}$$
$$- (\tfrac{1}{2} D_{mnij} + C_{mnij}) f_n \int_0^\tau \frac{\partial}{\partial \tau'} \bar{E}(\tau - \tau') f_i(\tau') f_j(\tau')\, \mathrm{d}\tau \Bigg] = 0, \qquad m = 1, 2, \ldots, \tag{9}$$

where

$$A_{mn} = \int_0^1 \phi_n' \phi_m \, \mathrm{d}\xi = \begin{cases} 0, & m = n, \\ \dfrac{2nm(\cos n\pi \cos m\pi - 1)}{n^2 - m^2}, & m \neq n, \end{cases}$$

$$B_{mn} = \int_0^l \phi_n'' \phi_m \, \mathrm{d}\xi = \begin{cases} -(n\pi)^2, & m = n, \\ 0, & m \neq n, \end{cases}$$

$$C_{mnij} = \int_0^1 \phi_n' \phi_i' \phi_j'' \phi_m \, \mathrm{d}\xi, \qquad D_{mnij} = \int_0^1 \phi_n'' \phi_i' \phi_j' \phi_m \, \mathrm{d}\xi.$$

For the elastic string vibration, the ordinary differential–integral equation (9) is reduced to an ordinary differential equation, as

$$\ddot{f}_m + \sum_{n=1}^{\infty} [2\xi_\tau A_{mn} \dot{f}_n + (\xi_\tau^2 - 1 - \bar{v} \sin \Omega_f \tau) B_{mn} f_n] - \tfrac{3}{2} k_1^2 \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_n f_i f_j C_{mnij} = 0,$$

$$\text{for } m = 1, 2, \ldots .$$

5. FINITE DIFFERENCE INTEGRATION

In order to solve the non-linear ordinary differential–integral equation (9), the finite difference numerical procedure proposed by Lee and Rogers [14] is adopted. First, the time scale is divided into intervals by the time values, τ_r, $r = 1, 2, \ldots (R+1)$, with $\tau_1 = 0$ and $\tau_{R+1} = \tau$. The last term in equation (9) can be written in the form

$$\int_{\tau_1}^{\tau_{R+1}} \frac{\partial}{\partial \tau'} \bar{E}(\tau_{R+1} - \tau') f_i(\tau') f_j(\tau') \, \mathrm{d}\tau' = \sum_{r=1}^{R} \int_{\tau_r}^{\tau_{r+1}} \frac{\partial}{\partial \tau'} \bar{E}(\tau_{R+1} - \tau') f_i(\tau') f_j(\tau') \, \mathrm{d}\tau. \quad (10)$$

Each of the integrals in (10) is transformed into the finite approximation

$$\int_{\tau_r}^{\tau_{r+1}} f_i(\tau') f_j(\tau') \frac{\partial}{\partial \tau'} \bar{E}(\tau_{R+1} - \tau') \, \mathrm{d}\tau'$$

$$\simeq \tfrac{1}{2}[f_i(\tau_{r+1}) f_j(\tau_{r+1}) + f_i(\tau_r) f_j(\tau_r)] \int_r^{\tau_{r+1}} \frac{\partial}{\partial \tau'} \bar{E}(\tau_{R+1} - \tau') \, \mathrm{d}\tau'$$

$$= \tfrac{1}{2}[f_i(\tau_{r+1}) f_j(\tau_{r+1}) + f_i(\tau_r) f_j(\tau_r)][\bar{E}(\tau_{R+1} - \tau_{r+1}) - \bar{E}(\tau_{R+1} - \tau_r)]. \quad (11)$$

Substituting equation (11) into equation (9) yields the following equation:

$$\ddot{f}_m + \sum_{n=1}^{\infty} [2\xi_\tau A_{mn} \dot{f}_n + \xi_{\tau\tau} A_{mn} f_n + (\xi_\tau^2 - 1 - \bar{v} \sin \Omega_f \tau) B_{mn} f_n] - \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Big\{ [\tfrac{3}{2} k_1^2 f_n f_i f_j C_{mnij}$$

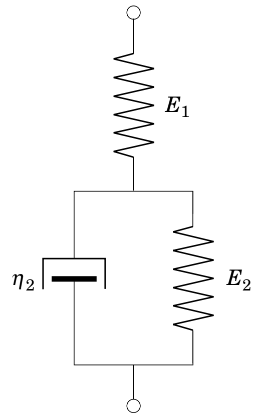

Figure 2. Three-parameter model of the viscoelastic string material.

$$- \left(\tfrac{1}{4}D_{mnij} + \tfrac{1}{2}C_{mnij}\right) f_n \sum_{r=1}^{R} [\, f_i(\tau_{r+1}) f_j(\tau_{r+1}) + f_i(\tau_r) f_j(\tau_r)][\bar{E}(\tau_{R+1} - \tau_{r+1})$$

$$- \bar{E}(\tau_{R+1} - \tau_r)] \Big\} = 0. \qquad (12)$$

Equation (12) is the most general form of differential–integral equation of a one-dimensional string which is made of integral constitutive law of linear viscoelastic material.

6. NUMERICAL RESULTS AND DISCUSSION

In order to obtain numerical results, the string material is considered as a three-element model, shown in Figure 2, which is the simplest spring–dashpot model. It can be used to simulate the behavior of linear viscoelastic materials of ‘‘solid’’ type with limited creep

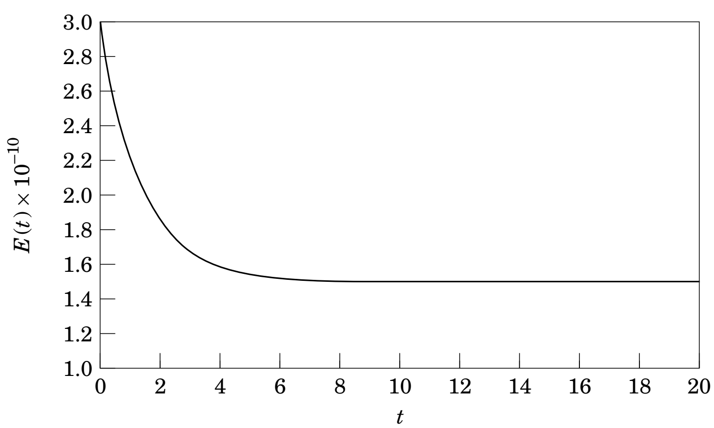

Figure 3. The relaxation modulus.

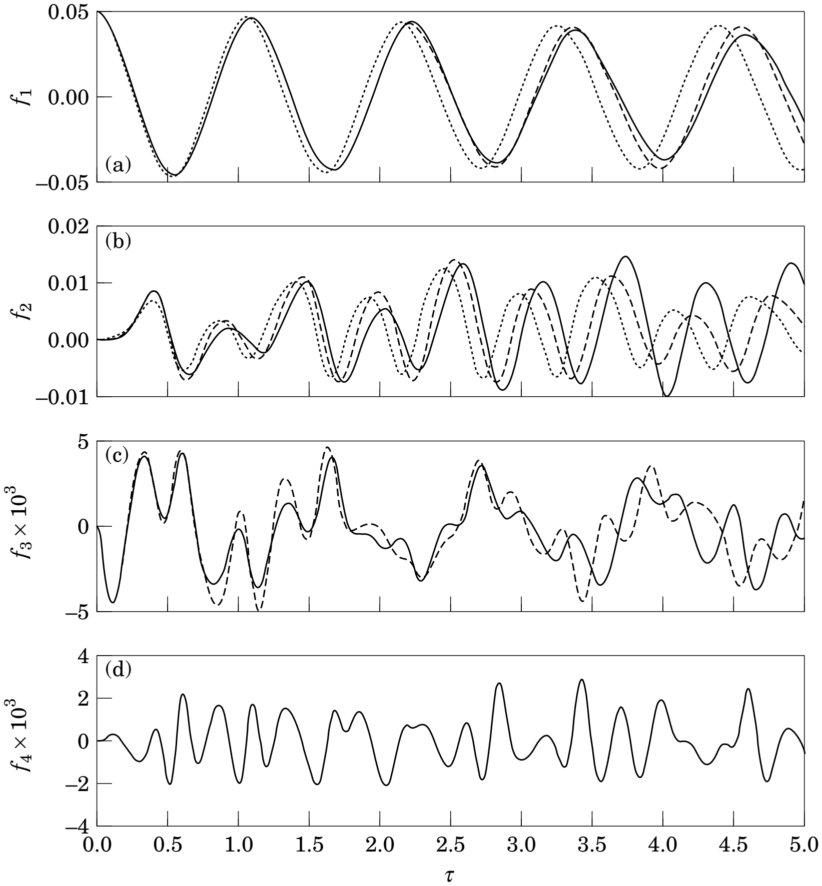

Figure 4. A comparison of the transient amplitudes of the two-, three- and four-term approximations for $k_1 = 20$, $k_2 = 0{\cdot}5$, $k_3 = 1000$, $\xi_\tau = 0{\cdot}2$, $\Omega_f = 0$ and $\bar{v} = 0$. (a) Amplitude of the first approximation; (b) amplitude of second approximation; (c) amplitude of third approximation; (d) amplitude of fourth approximation. ——, Four-term approximation; – – –, three-term approximation, · · ·, two-term approximation.

deformation when E_2 is non-zero, and of "fluid" type with unlimited viscous deformation when $E_2 = 0$. The behavior of this type of material is described by the following expression:

$$E(t) = \frac{E_1 E_2}{E_1 + E_2} + \frac{E_1^2}{E_1 + E_2}\, e^{-(E_1+E_2)/\eta_2 2}. \tag{13}$$

That has $E(0) = E_0 = E_1$. The values of $E_1 = E_2 = 30\,000$ MPa and $\eta_2 = 300$ MPa days are used here. A detailed derivation of equation (13) can be found in the Appendix. The plot of $E(t)$ is shown in Figure 3. The transient curve approaches a steady state value $E_1/2$. Using equation (13) and the definition of the non-dimensional parameter $\overline{E}$, the following equations are obtained from equation (12):

$$\ddot{f}_m + \sum_{n=1}^{\infty} [2\xi_r A_{mn} \dot{f}_n + \xi_{\tau\tau} A_{mn} f_n + (\xi_\tau^2 - 1 - \bar{v} \sin \Omega_f \tau) B_{mn} f_n] - \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Big\{ \tfrac{3}{2} k_1^2 f_n f_i f_j C_{mnij}$$

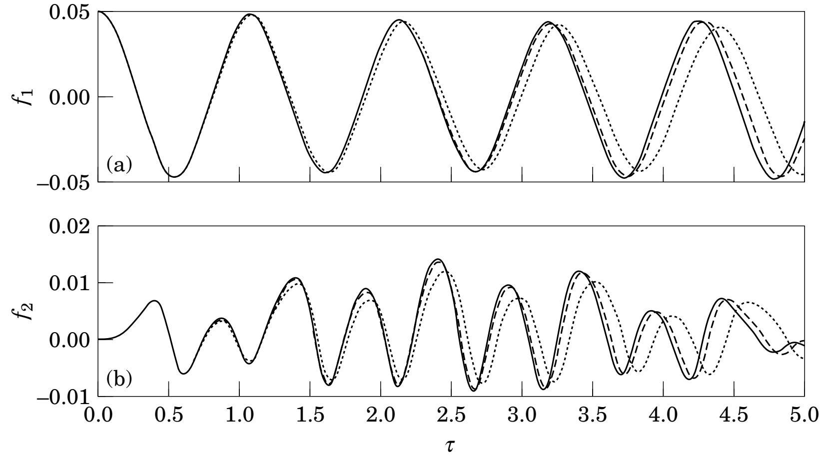

Figure 5. A comparison of the transient amplitudes on the basis of different time step. (a) Amplitude of first approximation; (b) amplitude of second approximation. ——, Differential method; – – –, $\Delta = 0{\cdot}001$; · · ·, $\Delta = 0{\cdot}01$.

$$- \left(\tfrac{1}{4}D_{mnij} + \tfrac{1}{2}C_{mnij}\right)k_1^2 k_2 f_n \sum_{r=1}^{R} \left[f_i(\tau_{r+1}) f_j(\tau_{r+1}) + f_i(\tau_r) f_j(\tau_r) \right]$$

$$\left. \times \left[\mathrm{e}^{-k_3(\tau_{R+1} - \tau_{r+1})} - \mathrm{e}^{-k_3(\tau_{R+1} - \tau_r)} \right] \right\} = 0, \tag{14}$$

where $k_2 = E_1/(E_1 + E_2)$ and $k_3 = (E_1 + E_2)/\eta_2 c_2 L$. It can be seen that the integral term in equation (9) has been reduced to an algebraic summation term in equation (14). The parameter k_3 is primarily a measure of the degree of viscoelastic behavior of the string, with a decreasing k_3 value corresponding to an increasing viscoelasticity. For the three-parameter model, the case for k_3 approaching infinity corresponds to the elastic string system.

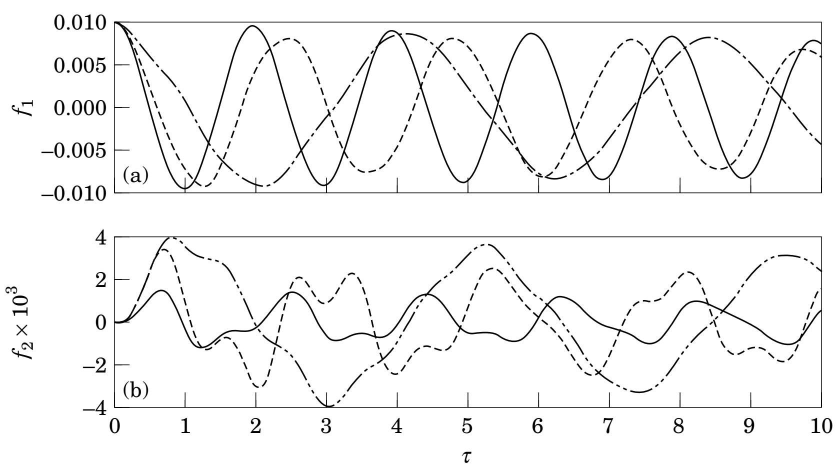

Figure 6. The influence of the transport speed ξ_τ on the transient amplitudes. The other parameters are the same as in Figure 4. (a) Amplitude of first approximation; (b) amplitude of second approximation. ——, $\xi_\tau = 0{\cdot}2$; – – –, $\xi_\tau = 0{\cdot}5$; – · –, $\xi_\tau = 0{\cdot}8$.

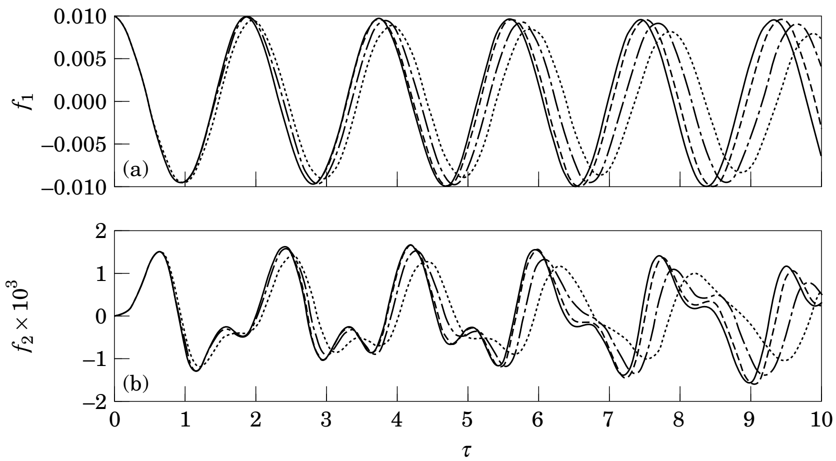

Figure 7. The influence of the material parameter k_3 on the transient amplitudes. The other parameters are the same as in Figure 4. (a) Amplitude of first approximation; (b) amplitude of second approximation. ——, Elastic; – – –; $k_3 = 0{\cdot}1$; – · –, $k_3 = 1$; · · ·, $k_3 = 10$.

The numerical results shown here are dynamic behavior of a travelling string with constant velocity and non-uniform velocity.

6.1. CONSTANT TRAVELLING VELOCITY

The examples given here are chosen to study the transient amplitudes of the viscoelastic string system. The parameter values are as follows: $T = 100$ N, $\rho = 7860$ kg/m^3 and $L = 1$ m. The Runge–Kutta numerical method with initial conditions $f_1(0) = 0{\cdot}01$ and the others zero in Figures 4–7 is used to integrate equation (14). In the present analysis, up to four-term approximations based on the eigenfunctions of the stationary string are considered (i.e., $m = 1, 2, 3, 4$). Takng four terms, the resulting four ordinary differential equations are coupled and have periodic coefficients. The transient amplitudes of the two-, three- and four-term approximations are shown and compared in Figure 4(a)–(d). While

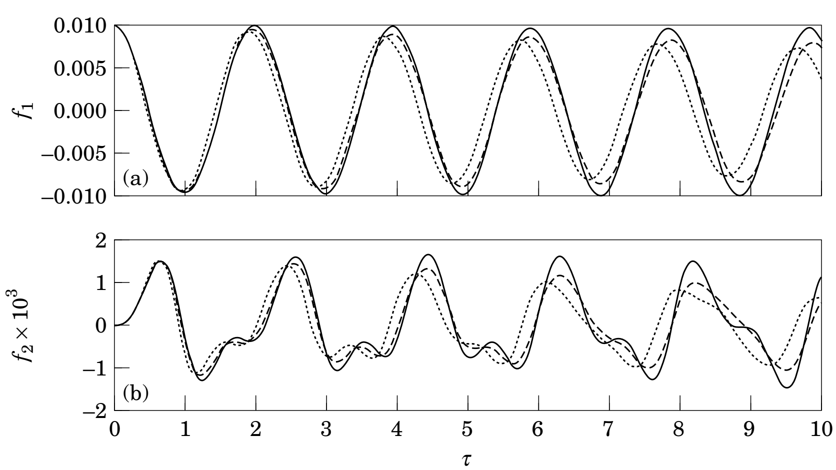

Figure 8. The influence of the wave speed ratio k_1^2 on the transient amplitudes. The other parameters are the same as in Figure 4. (a) Amplitude of first approximation; (b) amplitude of second approximation. ——, Linear case, $k_1 = k_3 = 0$; – – –, $k_1 = 20$; · · ·, $k_1 = 25$.

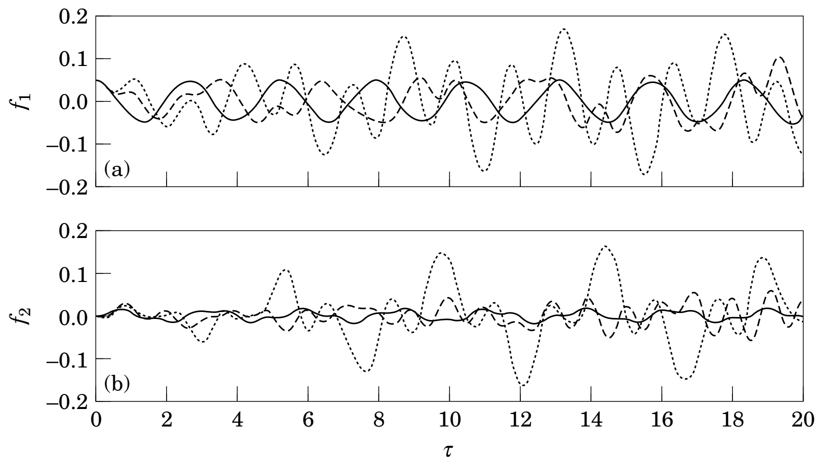

Figure 9. The effect of the axial variational velocity β_1 on the transient amplitudes for $\Omega_0 = \pi\sqrt{1-\beta_0^2}$ and $\beta_0 = 0{\cdot}5$. The other parameters are the same as in Figure 4. (a) Amplitude of first approximation; (b) amplitude of second approximation. ——, $\beta_1 = 0$; – – –, $\beta_1 = 0{\cdot}4$; · · ·, $\beta_1 = 0{\cdot}6$.

a qualitative agreement is evident among the two-term approximations, increasing the number of terms in the approximation improves the quantitative results.

In Figures 5–10, only the first two approximate amplitudes are shown in the figures, to save space. The time increment is determined by $\Delta = t_{r+1} - t_r$, which gives a constant time step. The computations are carried out for $\Delta = 0{\cdot}01$ and $0{\cdot}001$, and the results are compared to provide a practical assessment of the reduction in the truncation error with a decreasing step size. In Figure 5 are shown the results of the integral method compared with that obtained by the differential method [26]. The result clearly indicates that the error increases with increasing time step size.

In Figures 6–8 are shown the transient amplitudes, in which the effects of the transport speed ξ_τ, material parameter k_3 and wave speed ratio k_1^2 are considered, respectively. In Figure 6 are presented the different values of the transport speed $\xi_\tau = 0{\cdot}2$, $0{\cdot}5$ and $0{\cdot}8$ for

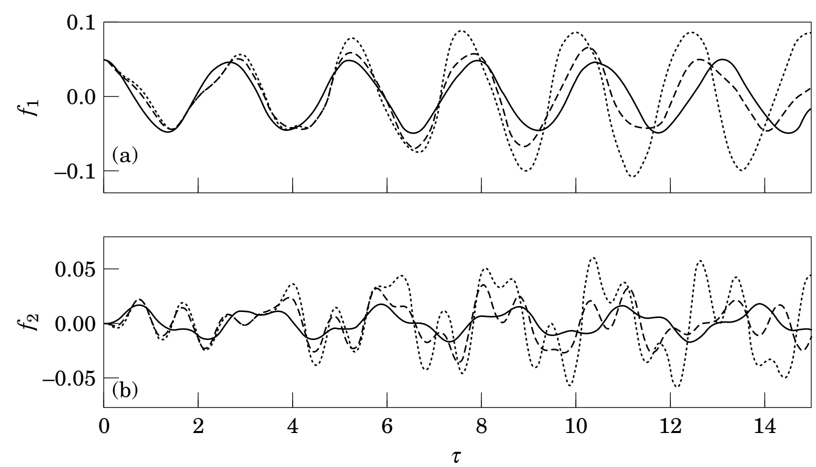

Figure 10. The effect of the axial variational velocity β_1 on the transient amplitudes for $\Omega_0 = 2\pi\sqrt{1-\beta_0^2}$ and $\beta_0 = 0{\cdot}5$. The other parameters are the same as in Figure 4. (a) Amplitude of first approximation; (b) amplitude of second approximation. ——, $\beta_1 = 0$; – – –, $\beta_1 = 0{\cdot}3$; · · ·, $\beta_1 = 0{\cdot}4$.

an axial moving viscoelastic string. From the figure, we find that an increase in transport speed is related to a decrease in vibration frequency. In Figure 7 is shown the influence of the material parameter k_3 on the frequency. From the figure, we find that an increase in the value of parameter k_3 is related to a decrease in vibration frequency.

Consider the metal strings with the wave speed ratio $k_1^2 = c_1^2/c_2^2 = EA/T = O(400\text{–}1000)$ [27]. The natural frequency depends strongly on the rigidity of the string. The higher the rigidity of the string, the higher will be the natural frequency. In Figure 8 are shown the responses of the viscoelastic system for the different value of the wave speed ratio k_1^2. The initial conditions are given by $f_1(0) = 0{\cdot}05$, and the others are zero. It can be seen that an increase in k_1^2 is related to an increase in frequency of the transient amplitude.

6.2. NON-UNIFORM TRAVELLING VELOCITY

In this section, we investigate the transverse vibration of an axial accelerating viscoelastic string. In many studies of the dynamic behavior of axial moving material systems, the axial velocity has been taken to be constant. However, when a system is subjected to acceleration, it may alter the stability of the system. The equation of motion for an accelerating elastic string, in which there is no mean velocity but it is harmonically varying at about zero value, was derived by Pakdemirli *et al*. [25]. In this paper, the interesting case is that a prescribed function of time is treated instead of constant velocity. The time-dependent axial velocity function $\xi_\tau(\tau)$ is characterized as a small periodic perturbation superimposed on the steady state velocity.

The equation of motion for the system is obtained from equation (12). Taking two terms in the series solution (8), equation (9) is reduced to a set of coupled ordinary differential–integral equations:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{f}_1 \\ \ddot{f}_2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{16}{3}\xi_\tau \\ \frac{16}{3}\xi_\tau & 0 \end{bmatrix}\begin{bmatrix} \dot{f}_1 \\ \dot{f}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} (1 + \bar{v}\cos\Omega_f\tau - \xi_\tau^2)\pi^2 & -\frac{8}{3}\xi_{\tau\tau} \\ \frac{8}{3}\xi_{\tau\tau} & (1 + \bar{v}\cos\Omega_f\tau - \xi_r^2)4\pi^2 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2}k_1^2\pi^4(\frac{1}{2}f_1^3 + 4f_1f_2^2) + k_1^2k_2\pi^4\sum_{r=1}^{R}[\mathrm{e}^{-k_3(\tau_{R+1}-\tau_{r+1})} - e^{-k_3(\tau_{R+1}-\tau_r)}]\{2f_2[f_1(\tau_{r+1})f_2(\tau_{r+1}) \\ +f_1(\tau_r)f_2(\tau_r)] + f_1[f_2^2(\tau_{r+1}) + f_2^2(\tau_r) + \frac{3}{8}f_1[f_1^2(\tau_{r+1}) + f_1^2(\tau_r)]\} \\ -\frac{3}{2}k_1^2\pi^4(4f_1^2f_2 + 8f_2^3) + k_1^2k_2\pi^4\sum_{r=1}^{R}[\mathrm{e}^{-k_3(\tau_{R+1}-\tau_{r+1})} - \mathrm{e}^{-k_3(\tau_{R+1}-\tau_r)}]\{2f_1(f_1(\tau_{r+1})f_2(\tau_{r+1}) \\ +f_1(\tau_r)f_2(\tau_r)] + f_2[f_1^2(\tau_{r+1}) + f_1^2(\tau_r)] + 6f_2[f_2^2(\tau_{r+1}) + f_2^2(\tau_r)]\} \end{bmatrix}. \tag{15}$$

Consider the string system to have an harmonic variation of axial velocity superimposed on a steady state velocity, i.e.,

$$\xi_\tau = \beta_0 + \beta_1\sin\Omega_0\tau, \tag{16a}$$

where β_0 is the steady state velocity, β_1 is the amplitude of the perturbed axial velocity and Ω_0 is the frequency of the perturbed velocity. The axial velocity represented by equation

(16a) is in non-dimensional form, and it is typically observed in physical systems. Using the identities

$$\xi_{\tau\tau} = \tfrac{1}{2}\beta_1^2 \Omega_0 \sin 2\Omega_0 \tau, \tag{16b}$$

$$\xi_r^2 = \beta_0^2 + \tfrac{1}{2}\beta_1^2 + 2\beta_0\beta_1 \sin \Omega_0 \tau - \tfrac{1}{2}\beta_1^2 \cos 2\Omega_0 \tau, \tag{16c}$$

and substituting equations (16) into equation (15), it is of interest to note that the time-dependent coefficients of equation (15) include $\sin \Omega_0 \tau$, $\sin 2\Omega_0 \tau$ and $\cos 2\Omega_0 \tau$ terms. Thus, the parametric excitation occurs at both frequencies Ω_0 and $2\Omega_0$ of the perturbed velocity.

These condition, for which the axial velocity frequency Ω_0 is equal to $\pi\sqrt{1-\beta_0^2}$ and $2\pi\sqrt{1-\beta_0^2}$, are shown in Figures 9 and 10, respectively. The initial conditions are given by $f_1(0) = 0{\cdot}05$ and the others are zero. We take β_1 to be of the same order as β_0 in order to diverge rapidly in the transient amplitude and save the time in the simulation. It can be seen that the transient amplitudes increase with time and that the string system is unstable.

For the elastic string vibration, we can set the value of η_2 to approach zero and set $E_1 = E_2 = 2E$. We then have a set of coupled equations, as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{f}_1 \\ \ddot{f}_2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{16}{3}\xi_\tau \\ \frac{16}{3}\xi_\tau & 0 \end{bmatrix}\begin{bmatrix} \dot{f}_1 \\ \dot{f}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} (1 + \bar{v}\cos \Omega_f \tau - \xi_\tau^2)\pi^2 & -\frac{8}{3}\xi_{\tau\tau} \\ \frac{8}{3}\xi_{\tau\tau} & (1 + \bar{v}\cos \Omega_f \tau - \xi_\tau^2)4\pi^2 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -(\frac{3}{16} f_1^3 + \frac{3}{2} f_1 f_2^2)\dfrac{c_1^2}{c_2^2}\pi^4 \\ -(\frac{3}{2} f_1^2 f_2 + 3 f_2^3)\dfrac{c_1^2}{c_2^2}\pi^4 \end{bmatrix} = 0. \tag{17}$$

Neglecting the non-linear terms in equation (17) and letting $\beta_0 = 0$ in equation (16a), the resulting equation will be the same as that of reference [22]. Also, it is of interest to note that equation (17) includes both $\sin \Omega_0 \tau$ and $\cos 2\Omega_0 \tau$. Thus the parametric excitation occurs at both frequencies Ω_0 and $2\Omega_0$.

7. CONCLUSIONS

In this paper, the general form of a differential–integral equation of a string is derived with the integral constitutive law of linearly viscoelastic material. The three-parameter model is adopted in the linear and non-linear vibration systems. The partial differential–integral equation governing the string behavior is discretized as gyroscopically and non-linearly coupled ordinary differential–integral equations by using Galerkin's method. The integral term of the equation is therefore reduced to a set of simultaneous algebraic equations by the use of finite difference. The vibration response of a viscoelastic string is investigated with the constant and non-uniform axial velocities.

From the above numerical results, the following conclusions can be drawn.

(1) The method of solution can be applied directly to a wide range of problems for a general linear viscoelastic material with an integral constitutive law; that is, it is not restricted only to the three-element model material.

(2) A decrease in the value of parameter k_3 is related to a decrease in vibration frequency.

(3) As the parameter k_1 of the wave propagation increases, the frequency of the transient amplitude will increase.

(4) The parametric excitation occurs at both frequencies Ω_0 and $2\Omega_0$ of the harmonic variation of the axially traveling velocity. The transient amplitudes increase with time and the string system is unstable.

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APPENDIX

The one-dimensional constitutive equation [13] of differential type material (called a Rivlin material) obeys the relation

$$\sum_{j=0}^{R} a_j \frac{\mathrm{d}^j \sigma}{\mathrm{d}t^j} = \sum_{j=0}^{P} b_j \frac{\mathrm{d}^j \varepsilon}{\mathrm{d}t^j} \qquad (a_0 \neq 0,\ b_0 = 1). \tag{A1}$$

Equation (A1) may also be written as

$$\mathbf{A}\sigma = \mathbf{B}\varepsilon, \tag{A2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are differential operators:

$$\mathbf{A} = \sum_{j=0}^{R} a_j \frac{\mathrm{d}^j}{\mathrm{d}t^j}, \qquad \mathbf{B} = \sum_{j=0}^{P} b_j \frac{\mathrm{d}^j}{\mathrm{d}t^j}.$$

The three-element model shown in Figure 2 is the simplest spring–dashpot model. The differential constitutive law of a linear viscoelastic material can be written as

$$\sigma + \frac{E_1 + E_2}{\eta_2}\sigma = E_1\varepsilon + \frac{E_1 E_2}{\eta_2}\varepsilon. \tag{A3}$$

From equation (A3) we can obtain the coefficients $a_0 = (E_1 + E_2)/\eta_2$, $a_1 = 1$, $b_0 = E_1E_2/\eta_2$, $b_1 = E_1$ and $R = P = 1$.

Applying the Laplace transform to equations (4) and (A3), one obtains

$$\bar{E}(s) = \frac{1}{s}\frac{\bar{\sigma}(s)}{\bar{\varepsilon}(s)}, \qquad a_0\bar{\sigma}(s) - a_1(s\bar{\sigma}(s) - \sigma(0)) = b_0\bar{\varepsilon}(s) - b_1(s\bar{\varepsilon}(s) - \varepsilon(0)). \tag{A4, A5}$$

From initial condition, $a_1\sigma(0) = b_1\varepsilon(0)$, so that equation (A5) can be written as

$$\frac{\bar{\sigma}(s)}{\bar{\varepsilon}(s)} = \frac{b_0 - sb_1}{a_0 - sa_1}. \tag{A6}$$

Substituting equation (A6) into equation (A4), one obtains

$$\bar{E}(s) = \frac{1}{s}\frac{b_0 - sb_1}{a_0 - sa_1}. \tag{A7}$$

Applying the inverse Laplace transform, the stress relaxation function can be determined as

$$E(t) = \frac{E_1 E_2}{E_1 + E_2} + \frac{E_1^2}{E_1 + E_2}\, \mathrm{e}^{-(E_1 + E_2)t/\eta_2}, \tag{A8}$$

and $E(0) = E_0 = E_1$.