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APPROXIMATE SOLUTIONS OF NON-CLASSICALLY DAMPED LINEAR SYSTEMS IN NORMALIZED AND PHYSICAL CO-ORDINATES

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1. INTRODUCTION

In this note, we consider n-degree-of-freedom linear second order systems represented by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t), \qquad x(0) = \dot{x}(0) = \theta_n, \tag{1}$$

for all $t \geqslant 0$. In equation (1), $x(t) \in \mathbb{R}^n$ denotes the displacement vector, $f(t) \in \mathbb{R}^n$ denotes the vector of bounded inputs, and θ_n denotes the zero vector in $\mathbb{R}^n$; the mass matrix M, the damping matrix C, and the stiffness matrix K belong to $\mathbb{R}^{n \times n}$ and are symmetric and positive definite. The displacement vector $x(\cdot)$ is the vector of *physical* co-ordinates.

Let $U \in \mathbb{R}^{n \times n}$ denote the modal matrix (see, e.g., references [1, 2]) corresponding to the system (1). The modal matrix is a non-singular matrix the columns of which are the eigenvectors of the symmetric generalized eigenvalue problem

$$Ku^{(i)} = \omega_i^2 Mu^{(i)}, \tag{2}$$

where $\omega_i^2 > 0$ and $u^{(i)}$, $i = 1, 2, \ldots, n$, are the eigenvalues (undamped natural frequencies squared) and the corresponding eigenvectors, respectively. The modal matrix is usually orthonormalized according to $U^{\mathrm{T}}MU = I_n$, (U^{T} denotes the transpose of U and I_n denotes the $n \times n$ identity matrix) and hence satisfies $U^{\mathrm{T}}KU = \operatorname{diag}[\omega_1^2, \omega_2^2, \ldots, \omega_n^2] =: \Omega^2$. It is well known that by the linear change of co-ordinates

$$x(t) = Uq(t), \tag{3}$$

the system (1) can be written in the normalized form

$$\ddot{q}(t) + \tilde{C}\dot{q}(t) + \Omega^2 q(t) = U^{\mathrm{T}}f(t), \qquad q(0) = \dot{q}(0) = \theta_n, \tag{4}$$

for all $t \geqslant 0$, where $q(t) \in \mathbb{R}^n$ denotes the vector of *normalized* co-ordinates and $\tilde{C} = U^{\mathrm{T}}CU \in \mathbb{R}^{n \times n}$. The symmetric matrix $\tilde{C}$ is called the normalized damping matrix. In general, this matrix is not diagonal. Thus, in general, the system (4) is a set of coupled second order differential equations. When $\tilde{C}$ is not a diagonal matrix, the system (1) is said to be non-classically damped.

A simple approximate technique of solving the system (4) is to neglect the off-diagonal elements of $\tilde{C}$, and hence decouple the system equations to a set of n scalar second order differential equations (see, e.g., references [1, 3]). The decoupled system is represented by

$$\ddot{q}_a(t) + \tilde{C}_d\dot{q}_a(t) + \Omega^2 q_a(t) = U^{\mathrm{T}}f(t), \qquad q_a(0) = \dot{q}_a(0) = \theta_n, \tag{5}$$

for all $t \geqslant 0$, where $\tilde{C}_d \in \mathbb{R}^{n \times n}$ is a diagonal matrix the diagonal elements of which are those of $\tilde{C}$. The solution of the decoupled system (5) is readily obtained. This solution leads to the following approximate solution for the system (1):

$$x_a(t) = Uq_a(t), \tag{6}$$

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for all $t \geqslant 0$. Clearly, when $\tilde{C}$ is not a diagonal matrix, there is an error in the approximate solution $q_a(\cdot)$. Let the vector

$$e_N(t) := q(t) - q_a(t), \tag{7}$$

in $\mathbb{R}^n$ denote the error between the exact and approximate solutions in the normalized co-ordinates for all $t \geqslant 0$. Researchers have analyzed the error vector $e_N(\cdot)$ extensively (see, e.g., references [4, 5], and the references therein). For instance, results are available that relate the norm of $e_N(\cdot)$ to the off-diagonal elements of $\tilde{C}$, the natural frequencies of the system, and the input to the system. Also conditions under which the norm of $e_N(\cdot)$ is small have been established.

It is certainly important to be able to determine the norm of the error vector $e_N(\cdot)$. However, it is more important to have an idea of the norm of the error vector in the physical co-ordinates, because displacements are measured in these co-ordinates. Let the vector

$$e_P(t) := x(t) - x_a(t) = Ue_N(t), \tag{8}$$

in $\mathbb{R}^n$ denote the error between the exact and approximate solutions in the physical co-ordinates for all $t \geqslant 0$. In this note, we study the relation between the norms of $e_N(\cdot)$ and $e_P(\cdot)$. We show that one of the following cases can arise: case (i), the norm of $e_N(\cdot)$ is small (respectively, large), and the norm of $e_P(\cdot)$ is small (large); case (ii), the norm of $e_N(\cdot)$ is large, but the norm of $e_P(\cdot)$ is small; case (iii), the norm of $e_N(\cdot)$ is small, but the norm of $e_P(\cdot)$ is large. Having shown that any of cases (i)–(iii) can arise, we conclude that the norm of $e_N(\cdot)$ by itself does not provide an accurate estimate of the norm of $e_P(\cdot)$. Thus, merely based on the closeness of $q_a(\cdot)$ and $q(\cdot)$, it is not possible to conclude that the approximate solution $x_a(\cdot)$ in equation (6) is close to the solution $x(\cdot)$ of the system (1). As we will see in the next section, the modal matrix plays an important role in determining the relation between the norms of the error vectors in the normalized and physical co-ordinates.

2. ERRORS IN THE NORMALIZED AND PHYSICAL CO-ORDINATES

In this section, we show how cases (i)–(iii) can arise and provide an example for each case. In order to have an idea of the magnitude of vectors, we use the L_2-norm of vector-valued functions, which is defined as $\|v\|_2 := \sup_{t \geqslant 0} [v^{\mathrm{T}}(t)v(t)]^{1/2}$ for $v: \mathbb{R}_+ \to \mathbb{R}^n$, given by $v(t) = [v_1(t)\ v_2(t)\ \cdots\ v_n(t)]^{\mathrm{T}}$.

The key relation between $e_N(\cdot)$ and $e_P(\cdot)$ is given by equation (8), which is rewritten as

$$e_P(t) = Ue_N(t), \tag{9}$$

for all $t \geqslant 0$. The modal matrix U has an important role in equation (9). We show in the following that different forms of U result in cases (i)–(iii).

2.1. Case (i): The norm of $e_N(\cdot)$ is small (respectively, large) and the norm of $e_P(\cdot)$ is small (large)

From equation (9), we conclude that if the modal matrix $U \in \mathbb{R}^{n \times n}$ is close to the identity matrix I_n, then $e_P(\cdot)$ is close to $e_N(\cdot)$ and case (i) arises. We demonstrate this fact by an example.

2.1.1. *Example* 1

Consider the system represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1{\cdot}05 \end{bmatrix}\begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 & -0{\cdot}5 \\ -0{\cdot}5 & 5{\cdot}6 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 4 & -0{\cdot}5 \\ -0{\cdot}5 & 16 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin 2t, \quad (10)$$

for all $t \geqslant 0$, where $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. The modal matrix corresponding to the system (10) is easily computed, and is

$$U = \begin{bmatrix} 0{\cdot}9991 & -0{\cdot}0433 \\ 0{\cdot}0423 & 0{\cdot}09750 \end{bmatrix}. \quad (11)$$

The modal matrix is close to the identity matrix I_2. By applying the change of co-ordinates $x(t) = Uq(t)$ to the system (10), we obtain the following representation of the system in the normalized co-ordinates:

$$\begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}9960 & -0{\cdot}2984 \\ -0{\cdot}2984 & 5{\cdot}3676 \end{bmatrix}\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} + \begin{bmatrix} 3{\cdot}9792 & 0 \\ 0 & 15{\cdot}2597 \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 2{\cdot}0405 \\ 0{\cdot}8884 \end{bmatrix} \sin 2t, \quad (12)$$

for all $t \geqslant 0$, where $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 0$. We decouple the system (12) by neglecting the off-diagonal elements of the normalized damping matrix. The approximately decoupled system is

$$\begin{bmatrix} \ddot{q}_{a1}(t) \\ \ddot{q}_{a2}(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}9960 & 0 \\ 0 & 5{\cdot}3676 \end{bmatrix}\begin{bmatrix} \dot{q}_{a1}(t) \\ \dot{q}_{a2}(t) \end{bmatrix} + \begin{bmatrix} 3{\cdot}9792 & 0 \\ 0 & 15{\cdot}2597 \end{bmatrix}\begin{bmatrix} q_{a1}(t) \\ q_{a2}(t) \end{bmatrix} = \begin{bmatrix} 2{\cdot}0405 \\ 0{\cdot}8884 \end{bmatrix} \sin 2t, \quad (13)$$

for all $t \geqslant 0$, where $q_{a1}(0) = q_{a2}(0) = 0$ and $\dot{q}_{a1}(0) = \dot{q}_{a2}(0) = 0$.

The system (13) can be readily solved for $q_a(\cdot) = [q_{a1}(\cdot) \;\; q_{a2}(\cdot)]^{\mathrm{T}}$, and the system (12) can be solved numerically for $q(\cdot) = [q_1(\cdot) \;\; q_2(\cdot)]^{\mathrm{T}}$. Let $e_{N1}(\cdot)$ and $e_{N2}(\cdot)$ denote the components of $e_N(\cdot) = q(\cdot) - q_a(\cdot)$, the error vector in the normalized co-ordinates. We obtained $e_{N1}(\cdot)$ and $e_{N2}(\cdot)$ and have plotted them in Figures 1(a) and (b), respectively. Since $e_{N1}(t)$ and $e_{N2}(t)$ are small for all $t \geqslant 0$, the norm of $e_N(\cdot)$ is small. Since the modal matrix U in equation (11) is close to I_2, the norm of the error in the physical co-ordinates is expected to be close to that of $e_N(\cdot)$ and hence small. Let $e_{P1}(\cdot)$ and $e_{P2}(\cdot)$ denote the components of $e_P(\cdot)$, the error vector in the physical co-ordinates. Using equations (9) and (11), we determined $e_{P1}(\cdot)$ and $e_{P2}(\cdot)$, which are plotted in Figures 1(a) and (b), respectively. It is evident that $e_{P1}(\cdot)$ and $e_{N1}(\cdot)$ are very close to each other and $e_{P2}(\cdot)$ and $e_{N2}(\cdot)$ coincide. Thus, the norm of $e_P(\cdot)$ is small and the approximate solution $x_a(\cdot) = Uq_a(\cdot)$ is close to the solution $x(\cdot)$ of the system (10).

2.2. CASE (II): THE NORM OF $e_N(\cdot)$ IS LARGE, BUT THE NORM OF $e_P(\cdot)$ IS SMALL

We again concentrate on the modal matrix $U \in \mathbb{R}^{n \times n}$. The matrix U is a non-singular matrix. However, suppose that k singular values of U, where $1 < k \leqslant n$, are close to zero (see references [6, 7] for details of the singular value decomposition of a matrix). The singular value decomposition of U is written as

$$U = Y\Sigma Z^{\mathrm{T}}, \quad (14)$$

where $Y \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{n\times n}$ are unitary matrices, and $\Sigma := \mathrm{diag}\,[\sigma_1, \sigma_2, \dots, \sigma_n] \in \mathbb{R}^{n\times n}$ is a matrix the diagonal elements of which are the singular values of U ordered as $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$. Since k singular values of U are small, it it concluded that $\sigma_{n-k+1}, \dots, \sigma_{n-1}, \sigma_n$ are small positive numbers. The last k columns of the matrix Z, denoted by $z_{n-k+1}, \dots, z_{n-1}, z_n$, are vectors in $\mathbb{R}^n$ of unit length ($\| z_i \|_2 = 1$) that satisfy (see eqn. (23))

$$\| U z_i \|_2 \ll 1, \tag{15}$$

for all $i = n-k+1, \dots, n-1, n$. The vectors z_i, $i = n-k+1, \dots, n-1, n$, are called the right singular vectors of U corresponding to σ_i, respectively.

We next define a set of vectors the members of which are close to the right singular vectors z_i, $i = n-k+1, \dots, n-1, n$, corresponding to the small singular values σ_i of U. This set is

$$N := \{ v \in \mathbb{R}^n \mid \| v - z_i \|_2 = \varepsilon \ll 1,\ i = n-k+1, \dots, n-1, n \}. \tag{16}$$

A vector $v \in N$ satisfies $v = z_i + \zeta_i$ for an $i = n-k+1, \dots, n-1, n$, where $\zeta_i \approx \theta_n$ with the norm $\| \zeta_i \|_2 = \epsilon \ll 1$. Hence, for sufficiently small ϵ, the norm $\| v \|_2 \approx 1$ and

$$\| U v \|_2 \ll 1. \tag{17}$$

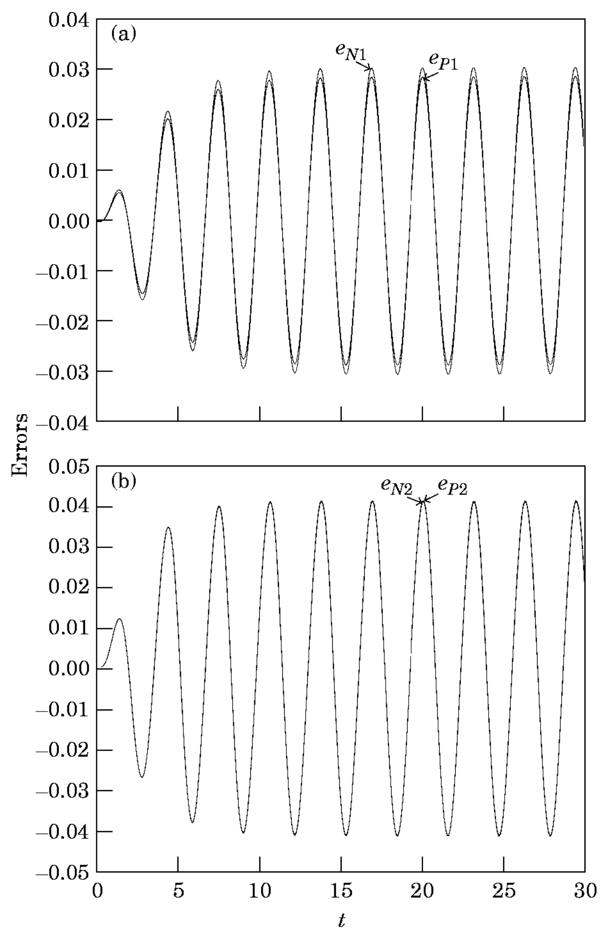

Figure 1. A comparison of the error functions in Example 1: (a) $e_{P1}(\cdot)$ and $e_{N1}(\cdot)$ are very close to each other, (b) $e_{P2}(\cdot)$ and $e_{N2}(\cdot)$ coincide.

Geometrically, a $v \in N$ is sufficiently close to a z_i for an $i = n - k + 1, \ldots, n - 1, n$, for which inequality (15) holds. Thus, v satisfies inequality (17).

Next, we consider a vector $w(t) = g(t)v$ in $\mathbb{R}^n$ for all $t \geqslant 0$, where $g: \mathbb{R}_+ \to \mathbb{R}$ is a bounded function of time and $v \in N$. Since $\| v \|_2 \approx 1$, the norm $\| w \|_2 \approx \sup_{t \geqslant 0} |g(t)|$. When U operates on $w(\cdot)$, the resultant vector satisfies

$$\| Uw \|_2 = \sup_{t \geqslant 0} |g(t)| \, \| Uv \|_2 \ll \sup_{t \geqslant 0} |g(t)|. \tag{18}$$

That is, the norm of $w(\cdot)$ decreases substantially after being operated on by U.

Now, we study the norm of the error vector in the normalized and physical co-ordinates when some of the singular values of the modal matrix U are small; that is, when N is a non-empty set. Suppose that the error vector in the normalized co-ordinates can be written as

$$e_N(t) = \sum_{i=1}^{l} g_i(t) v_i + \sum_{j=1}^{m} h_j(t) \zeta_j, \tag{19}$$

for all $t \geqslant 0$, where $g_i: \mathbb{R}_+ \to \mathbb{R}$ is a bounded function of time and $v_i \in N$ for all $i = 1, 2, \ldots, l$, and $h_j: \mathbb{R}_+ \to \mathbb{R}$ is a bounded function of time and $\zeta_j \approx \theta_n$ for all $j = 1, 2, \ldots, m$. Moreover, suppose that the norm of the error is not necessarily small. Using equation (9), we obtain the error vector in the physical co-ordinates:

$$e_P(t) = \sum_{i=1}^{l} g_i(t) U v_i + \sum_{j=1}^{m} h_j(t) U \zeta_j, \tag{20}$$

for all $t \geqslant 0$. Recalling that a $v_i \in N$ satisfies inequality (17) and noting that a $\zeta_j \approx \theta_n$ has a small norm, we conclude that $\| e_P \|_2 \ll 1$. That is, although the solution vectors $q(\cdot)$ and $q_a(\cdot)$ of the systems (4) and (5), respectively, are not necessarily close to each other, their corresponding vectors in the physical co-ordinates, $x(\cdot)$ and $x_a(\cdot)$, are. Thus, case (ii) can arise. We demonstrate the possibility of this case by an example.

2.2.1 *Example* 2

Consider the system represented by

$$\begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 200 & -4 \\ -4 & 10 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1000 & -1 \\ -1 & 100 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 500 \end{bmatrix} \sin t, \tag{21}$$

for all $t \geqslant 0$, where $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. The modal matrix corresponding to the system (21) is easily computed, and is

$$U = \begin{bmatrix} 0{\cdot}0316 & 0 \\ 0{\cdot}0003 & 1 \end{bmatrix}. \tag{22}$$

The singular value decomposition of U is easily obtained. The singular values of U are $\sigma_1 = 1$ and $\sigma_2 = 0{\cdot}0316$. The right singular vector corresponding to σ_2 is $z_2 = [1 \ \ -0{\cdot}0003]^{\mathrm{T}}$. Thus, we have

$$N = \{ v \in \mathbb{R}^2 \,|\, \| v - z_2 \|_2 = \epsilon \ll 1, \ z_2 = [1 \ \ -0{\cdot}0003]^{\mathrm{T}} \}. \tag{23}$$

The normalized form of system (21) is

$$\begin{bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}1996 & -0{\cdot}1234 \\ -0{\cdot}1234 & 10 \end{bmatrix} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}9986 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0{\cdot}1816 \\ 500 \end{bmatrix} \sin t, \tag{24}$$

for all $t \geqslant 0$, where $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 0$. The approximately decoupled system corresponding to the system (24) is

$$\begin{bmatrix} \ddot{q}_{a1}(t) \\ \ddot{q}_{a2}(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}1996 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \dot{q}_{a1}(t) \\ \dot{q}_{a2}(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}9986 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} q_{a1}(t) \\ q_{a2}(t) \end{bmatrix} = \begin{bmatrix} 0{\cdot}1816 \\ 500 \end{bmatrix} \sin t, \tag{25}$$

for all $t \geqslant 0$, where $q_{a1}(0) = q_{a2}(0) = 0$ and $\dot{q}_{a1}(0) = \dot{q}_{a2}(0) = 0$.

We solved the systems (24) and (25) for $q(\cdot) = [q_1(\cdot) \;\; q_2(\cdot)]^{\mathrm{T}}$ and $q_a(\cdot) = [q_{a1}(\cdot) \;\; q_{a2}(\cdot)]^{\mathrm{T}}$, respectively. We have plotted $e_{Ni}(\cdot) = q_i(\cdot) - q_{ai}(\cdot)$, $i = 1, 2$, the components of the error vector in the normalized co-ordinates, in Figures 2(a) and (b), respectively. From these figures, we conclude that the steady state error vector satisfies

$$e_N(t) = \begin{bmatrix} e_{N1}(t) \\ e_{N2}(t) \end{bmatrix} = (3{\cdot}1 \sin t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (\sin (t - 1{\cdot}57)) \begin{bmatrix} 0 \\ 0{\cdot}0042 \end{bmatrix}, \tag{26}$$

for all $t \geqslant 0$. The norm of $e_N(\cdot)$ is easily estimated and is close to 3·1. In equation (26), the vector $[1 \;\; 0]^{\mathrm{T}} \in N$, where N is given in equation (23), and $[0 \;\; 0{\cdot}0042]^{\mathrm{T}} \approx \theta_2$. That is, $e_N(\cdot)$ satisfies condition (19). Thus, the error vector in the physical co-ordinates is expected to have a small norm. Using equations (9) and (22), we determined $e_{P1}(\cdot)$ and $e_{P2}(\cdot)$, the components of $e_P(\cdot)$, the error vector in the physical co-ordinates, and have plotted them in Figures 2(a) and (b), respectively. It is evident that $\| e_P \|_2 \ll 1$ and much smaller than the norm of $e_N(\cdot)$. Thus, we conclude that even though $q(\cdot)$ and $q_a(\cdot)$ are not close to each other, the vector $x_a(\cdot) = Uq_a(\cdot)$ is close to the solution $x(\cdot) = [x_1(\cdot) \;\; x_2(\cdot)]^{\mathrm{T}}$ of the system (21). That is, the approximate solution $x_a(\cdot)$ is reasonably accurate.

2.3. Case (iii): the norm of $e_N(\cdot)$ is small, but the norm of $e_P(\cdot)$ is large

We search for a particular form of the modal matrix $U \in \mathbb{R}^{n \times n}$ that can make case (iii) possible. If U has large diagonal elements and small off-diagonal elements, then case (iii) can arise. We note that case (iii) is the most undesirable situation, because an accurate approximate solution in the normalized co-ordinates results in an inaccurate solution after being transformed back to the physical co-ordinates. We demonstrate the possibility of this case by an example.

2.3.1. *Example* 3

Consider the system represented by

$$\begin{bmatrix} 0{\cdot}01 & 0 \\ 0 & 0{\cdot}015 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 5 & -1 \\ -1 & 10 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 & -0{\cdot}1 \\ -0{\cdot}1 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \sin 2t, \tag{27}$$

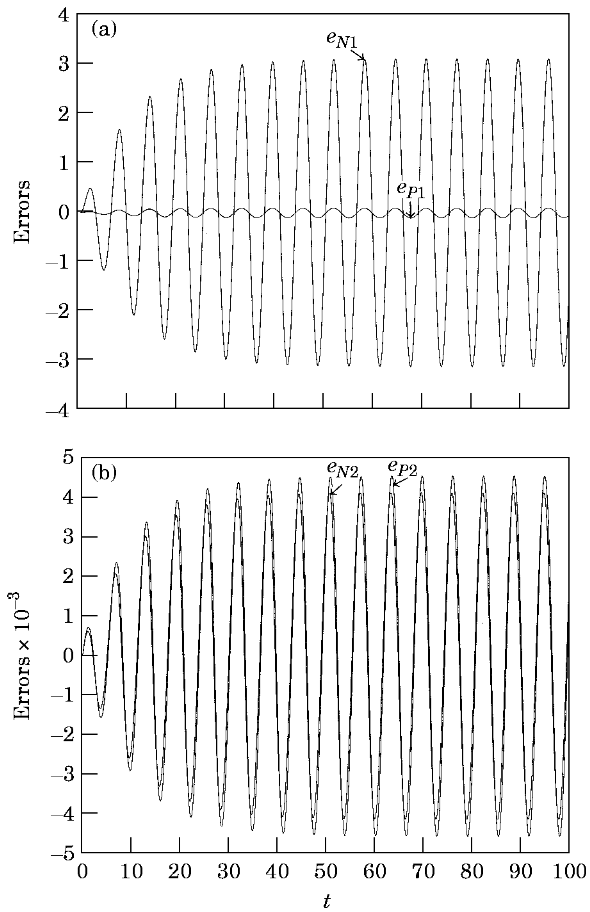

Figure 2. A comparison of the error functions in Example 2: (a) $\sup_{t\geqslant 0}|e_{P1}(t)|\ll 1$ and is much smaller than $\sup_{t\geqslant 0}|e_{N1}(t)|$; (b) $\sup_{t\geqslant 0}|e_{P2}(t)|\approx\sup_{t\geqslant 0}|e_{N2}(t)|\ll 1$.

for all $t\geqslant 0$, where $x_1(0)=x_2(0)=0$ and $\dot{x}_1(0)=\dot{x}_2(0)=0$. The modal matrix corresponding to the system (27) is easily computed, and is

$$U=\begin{bmatrix}9{\cdot}9881 & -0{\cdot}4877\\ 0{\cdot}3988 & 8{\cdot}1552\end{bmatrix}. \tag{28}$$

The modal matrix has large diagonal elements and small off-diagonal elements.

The normalized form of the system (27) is

$$\begin{bmatrix}\ddot{q}_1(t)\\ \ddot{q}_2(t)\end{bmatrix}+\begin{bmatrix}492{\cdot}4346 & -73{\cdot}0935\\ -73{\cdot}0935 & 674{\cdot}2167\end{bmatrix}\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix}+\begin{bmatrix}99{\cdot}6017 & 0\\ 0 & 267{\cdot}0625\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$$

$$=\begin{bmatrix}99{\cdot}8810\\ -4{\cdot}8870\end{bmatrix}\sin 2t, \tag{29}$$

for all $t \geqslant 0$, where $q_1(0) = q_2(0) = 0$ and $\dot{q}_1(0) = \dot{q}_2(0) = 0$. The approximately decoupled system corresponding to the system (29) is

$$\begin{bmatrix} \ddot{q}_{a1}(t) \\ \ddot{q}_{a2}(t) \end{bmatrix} + \begin{bmatrix} 492{\cdot}4346 & 0 \\ 0 & 674{\cdot}2167 \end{bmatrix} \begin{bmatrix} \dot{q}_{a1}(t) \\ \dot{q}_{a2}(t) \end{bmatrix} + \begin{bmatrix} 99{\cdot}6017 & 0 \\ 0 & 267{\cdot}0625 \end{bmatrix} \begin{bmatrix} q_{a1}(t) \\ q_{a2}(t) \end{bmatrix} = \begin{bmatrix} 99{\cdot}8810 \\ -4{\cdot}8870 \end{bmatrix} \sin 2t, \quad (30)$$

for all $t \geqslant 0$, where $q_{a1}(0) = q_{a2}(0) = 0$ and $\dot{q}_{a1}(0) = \dot{q}_{a2}(0) = 0$.

We solved the systems (29) and (30) for $q(\cdot) = [q_1(\cdot)\ \ q_2(\cdot)]^{\mathrm{T}}$ and $q_a(\cdot) = [q_{a1}(\cdot)\ \ q_{a2}(\cdot)]^{\mathrm{T}}$, respectively. We have plotted $e_{Ni}(\cdot) = q_i(\cdot) - q_{ai}(\cdot)$, $i = 1, 2$, the components of the error vector in the normalized co-ordinates, in Figures 3(a) and (b), respectively. Using equations (9) and (28), we determined $e_{P1}(\cdot)$ and $e_{P2}(\cdot)$, the components of $e_P(\cdot)$, the error vector in the physical co-ordinates, and have plotted them in Figures 3(a) and (b), respectively. It is evident that the norm of $e_P(\cdot)$ is much larger than the norm of $e_N(\cdot)$. Thus, we conclude that even though $q(\cdot)$ and $q_a(\cdot)$ are close to each other, the vector

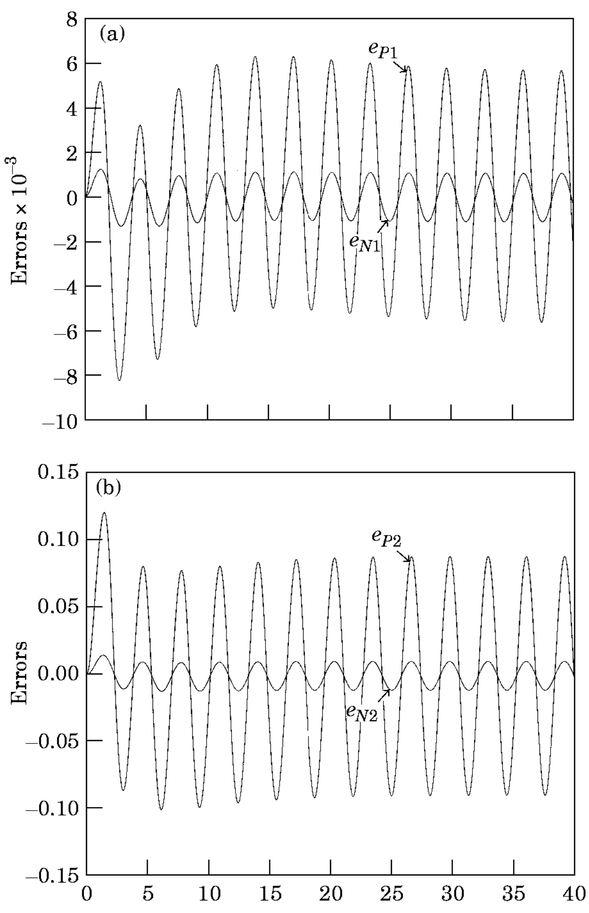

Figure 3. A comparison of the error functions in Example 3: (a) $\sup_{t \geqslant 0} |e_{P1}(t)|$ is much larger than $\sup_{t \geqslant 0} |e_{N1}(t)|$; (b) $\sup_{t \geqslant 0} |e_{P2}(t)|$ is much smaller than $\sup_{t \geqslant 0} |e_{N2}(t)|$.

$x_a(\cdot) = Uq_a(\cdot)$ is not close to the solution $x(\cdot) = [x_1(\cdot)\ \ x_2(\cdot)]^T$ of the system (27). That is, the approximate solution $x_a(\cdot)$ is inaccurate.

3. CONCLUSIONS

In this note, we considered n-degree-of-freedom linear second order systems. A commonly used approximate technique of solving such systems is as follows: (i) by a linear change of co-ordinates, the normalized form of the sytem is obtained; (ii) the normalized form is approximately decoupled to a set of n scalar second order differential equations; (iii) the approximately decoupled systems are solved to yield an approximate solution in the normalized co-ordinates; (iv) the approximate solution is transformed back to the physical co-ordinates to yield an approximate solution in the physical co-ordinates. The goal of this note was to show that the accuracy of the approximate solution in the physical co-ordinates cannot be determined easily, and hence to show that the approximate technique can possibly lead to inaccurate solutions in the physical co-ordinates.

We denoted the error vectors in the normalized and physical co-ordinates by $e_N(\cdot)$ and $e_P(\cdot)$, respectively. We carefully studied all possible relations between the norms of $e_N(\cdot)$ and $e_P(\cdot)$, and showed that cases (i)–(iii) can arise. We presented an example for each case. For instance, we showed in Example 3 that an accurate approximate solution in the normalized co-ordinates results in an inaccurate solution in the physical co-ordinates. Since any of cases (i)–(iii) can arise, we conclude that the norm of $e_N(\cdot)$ by itself does not provide an accurate estimate of the norm of $e_P(\cdot)$. However, we showed that the norm of $e_N(\cdot)$ can lead to a reliable estimate of the norm of $e_P(\cdot)$ when the modal matrix satisfies certain conditions. We provided a set of such conditions. These conditions, however, are just sufficient and unlikely to be satisfied in large scale systems or when the input vectors consist of many arbitrary functions of time. Therefore, in general, determining the accuracy of approximate solution in the physical co-ordinates is a formidable task.

It is certainly true that the modal analysis of n-degree-of-freedom linear second order systems provides useful information regarding the natural frequencies and the normal modes of the system. However, using the modal matrix in order to transform the system to the normalized co-ordinates and to obtain an approximate solution for the system is not a good approach, because the accuracy of the approximate solution in the physical co-ordinates remains unknown. We believe that with the advent of fast computers, the time-response of systems in the physical co-ordinates can be obtained more efficiently and accurately by direct numerical integration, and in particular by parallel integration algorithms (see, e.g., references [8–10] and the references therein).

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