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VIBRATION OF THIN SKEW FIBRE REINFORCED COMPOSITE LAMINATES

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Skew fibre reinforced composite laminates are important structural elements in modern engineering structures, particularly in the aerospace industry. The natural frequencies of these skew laminates are of primary significance to structural designers. As far as the author's knowledge is concerned, the references on this topic are very limited. In this paper a B-spline Rayleigh–Ritz method (RRM) is presented for free vibration analysis of thin skew fibre reinforced composite laminates which may have arbitrary lay-ups, admitting the possibility of coupling between in-plane and out-of-plane behaviour and general anisotropy. Various numerical applications are presented, and the method is shown to be accurate and efficient.

1. INTRODUCTION

Skew plates and laminates are structural elements of practical importance in applications such as building floors, bridge decks, ship superstructures and aerospace vehicles. To have an efficient and reliable design, it is essential to employ an accurate analysis method to predict the static, stability and dynamic behaviour of such structural elements. This paper is concerned with the free vibration analysis of thin skew isotropic plates and generally anisotropic laminates composed of fibre reinforced composite materials by using the B-spline Rayleigh–Ritz method (RRM).

A large number of references exist on free vibration of thin skew isotropic and orthotropic plates. An extensive literature survey has been conducted by Liew and Wang [1], and extra references can be found in Leissa's excellent reviews [2–4]. Among many others, a few examples are given here. Durvasula [5] presented the natural frequencies of thin skew isotropic plates having clamped edges using the Galerkin method with conventional beam mode functions. By extending this work, Nair and Durvasula studied the free vibration of thin skew isotropic [6] and orthotropic [7] plates having various boundary conditions using RRM with conventional beam mode functions: i.e., the analytical RRM. In reference [8], Mizusawa *et al*. presented natural frequencies of skew isotropic plates by using B-spline RRM. Cheung *et al*. [9] developed a B-spline finite strip method (FSM) for free vibration analysis of general plates with skewed shape as a special case. An integral method was used to obtain the natural frequencies of skew orthotropic plates in reference [10]. Recently, Liew and Lam [11] applied the *pb*-2 RRM for vibration analysis of skew isotropic plates. McGee *et al*. [12] studied the free vibration of cantilevered skew isotropic plates with corner stress singularities, using the algebraic polynomial RRM. Bardell [13] used a hierarchial finite element method (FEM) to determine the natural frequencies and modes of skew isotropic plates.

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Fibre reinforced composite materials are becoming increasingly important in many engineering applications, especially in the aerospace industry. Skew laminates made of these materials could be primary structural elements. However, in the open literature research works on the free vibration analysis of these skew laminates are very limited. Krishnan and Deshpande [14] carried out free vibration analysis of skew isotropic plates, single layered laminas and three-layered symmetric cross-ply laminates using FEM based on both classical plate theory (CPT) and Reissner [15]–Mindlin [16] plate theory. Kapania and Singhvi [17] developed a Chebyshev polynomials RRM based on CPT for free vibration analysis of tapered skew composite laminates. Many useful results were reported. However, it is noted that all of their reported results were concerned with canilevered laminates. It is worth noting that, in reference [18], Kamal and Durvasula studied stability problems by considering free vibrations of simply supported skew composite laminates that are subjected to both direct and shear in-plane forces using a Chebyshev polynomials RRM. As far as the author's knowledge is concerned, it seems that there is no systematic analysis in the open literature for free vibration of skew generally anisotropic composite laminates which may have arbitrary lay-ups and fibre orientations and various boundary conditions. Exact solutions for these skew laminates are very difficult, if not impossible, to obtain. Methods of an approximate nature may be the only choice for general solutions.

B-spline functions have attractive properties for use in structural analysis. Their piecewise form, high order of continuity and locally non-zero nature offer the prospect of both efficiency and versatility. In a number of research works [19–25], the author and his colleague have considered the use of B-splin RRM analyses of the free vibration of Timoshenko [26] beams and Reissner–Mindlin rectangular plates and laminates. It has been proved that the B-spline RRM is an accurate and efficient numerical analyzing tool in these applications. In this paper, the B-spline RRM is extended to embrace skew geometry. However, the laminates are assumed to have very thin geometry and, consequently, the CPT is adopted, which ignores the through-thickness shear effects. Moreover, the effects of through-thickness rotary inertia are also excluded.

In next section, the definition of the problem and the method of analysis are described, and the numerical applications are given in section 3; these include skew isotropic plates and skew generally anisotropic composite laminates. Conclusions are given in section 4.

2. METHOD OF ANALYSIS

2.1. PROBLEM DEFINITION

A skew laminate with its orthogonal and oblique co-ordinate systems, i.e., the oxy and $o\xi\eta$ systems respectively, is shown in Figure 1. The length of the skewed edge is A and the length of the other edge is B. The laminate is of uniform thickness h and, in general, is made up of a number of layers, each consisting of unidirectional fibre reinforced composite material. The lay-up of layers is arbitrary, admitting the possibility of coupling between in-plane and out-of-plane behaviour and of anisotropy. The skew angle is α, measured from the x-axis to the ξ-axis, and the fibre angle of the lth layer counted from the surface $z = -h/2$ is θ, measured from the x-axis to the fibre direction. They are defined positive when measured clockwise; o–x–y–z forms a right-hand co-ordinate system. The three fundamental displacement quantities are the three mid-surface translational displacements u, v and w along the x-, y- and z- axes, respectively. It should be noted that it becomes necessary that the two in-plane mid-surface translational displacements u and v are included in the analysis due to the coupling between in-plane and out-of-plane behaviour in laminates with non-symmetric lay-ups. Of course, in the case of laminates with symmetric lay-ups, only the out-of-plane displacement w is considered.

2.2. STRAIN AND KINETIC ENERGIES

During vibration, the three translational displacements $\bar{u}$, $\bar{v}$ and $\bar{w}$ at a general point in the laminate are assumed to have the forms

$$\bar{u}(x, y, z, t) = u(x, y, t) - zw_{,x}(x, y, t), \qquad \bar{v}(x, y, z, t) = v(x, y, t) - zw_{,y}(x, y, t),$$
$$\bar{w}(x, y, z, t) = w(x, y, t), \tag{1}$$

where t is the time dimension. The strains are

$$\varepsilon_x = u_{,x} - zw_{,xx}, \qquad \varepsilon_y = v_{,y} - zw_{,yy}, \qquad \gamma_{xy} = u_{,y} + v_{,x} - 2zw_{,xy}. \tag{2}$$

The material properties of each lamina are assumed to be orthotropic. That is, the stress–strain relationships or constitutive equations are of the form

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{21} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}, \tag{3}$$

where the subscripts 1 and 2 represent the principal axes of the material and the $\bar{Q}_{i,j}$ $(i, j = 1, 2, 6)$ are the plane-stress reduced stiffness coefficients and can be expressed in engineering notation, as

$$\bar{Q}_{11} = E_L/(1 - \nu_{LT}\nu_{TL}), \qquad \bar{Q}_{22} = E_T/(1 - \nu_{LT}\nu_{TL}),$$
$$\bar{Q}_{12} = \nu_{TL}E_L/(1 - \nu_{LT}\nu_{TL}), \qquad \bar{Q}_{21} = \bar{Q}_{12}, \qquad \bar{Q}_{66} = G_{LT}, \tag{4}$$

where L and T represent the directions parallel with and perpendicular to the fibre direction, respectively. By performing a proper co-ordinate transforamtion, the stress–strain relationships of a single lamina in the $oxyz$ co-ordinate system can be obtained as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{21} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix},$$

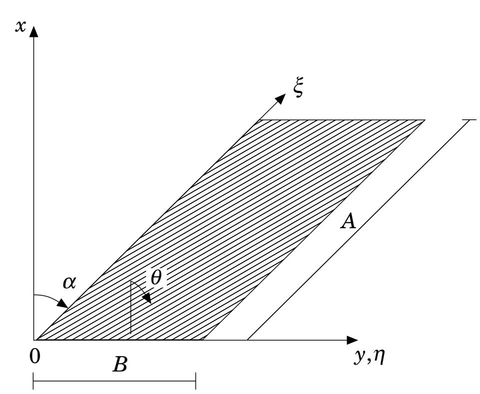

Figure 1. The geometry of a skew laminate.

where the Q_{ij} $(i, j = 1, 2, 6)$ are

$$Q_{11} = U_1 + U_2 \cos(2\theta) + U_3 \cos(4\theta), \qquad Q_{22} = U_1 - U_2 \cos(2\theta) + U_3 \cos(4\theta),$$
$$Q_{12} = U_4 - U_3 \cos(4\theta), \qquad Q_{66} = U_5 - U_3 \cos(4\theta),$$
$$Q_{16} = -\tfrac{1}{2}U_2 \sin(2\theta) - U_3 \sin(4\theta), \qquad Q_{26} = -\tfrac{1}{2}U_2 \sin(2\theta) + U_3 \sin(4\theta). \tag{6}$$

Here

$$U_1 = \tfrac{1}{8}(3\bar{Q}_{11} + 3\bar{Q}_{22} + 2\bar{Q}_{12} + 4\bar{Q}_{66}), \qquad U_2 = \tfrac{1}{2}(\bar{Q}_{11} - 2\bar{Q}_{22}),$$
$$U_3 = \tfrac{1}{8}(\bar{Q}_{11} + \bar{Q}_{22} - 2\bar{Q}_{12} - 4\bar{Q}_{66}), \qquad U_4 = \tfrac{1}{8}(\bar{Q}_{11} + \bar{Q}_{22} + 6\bar{Q}_{12} - 4\bar{Q}_{66}),$$
$$U_5 = \tfrac{1}{8}(\bar{Q}_{11} + \bar{Q}_{22} - 2\bar{Q}_{12} + 4\bar{Q}_{66}). \tag{7}$$

From equation (5) it is noted that there are interactions between the normal stresses σ_x and σ_y and the shear strain γ_{xy}. This feature makes the laminate anisotropic, although the material properties of each lamina are orthotropic.

By performing appropriate through-thickness integration upon equation (5), the constitutive equations for an arbitrary laminate are obtained as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & & & & & \\ A_{12} & A_{22} & & & & \\ A_{16} & A_{22} & A_{66} & & & \\ B_{11} & B_{12} & B_{16} & D_{11} & & \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \\ -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix} \tag{8}$$

Here N_x, N_y and N_{xy} are the membrane direct and shearing forces per unit length; M_x, M_y and M_{xy} are the bending and twisting moments per unit length. The laminate stiffness coefficients in equations (8) are defined as

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2)\,\mathrm{d}z, \qquad i, j = 1, 2, 6. \tag{9}$$

Equations (8) can be rewritten in a more compact form as

$$\boldsymbol{\sigma}^* = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \boldsymbol{\epsilon}^*. \tag{10}$$

The quantities $\boldsymbol{\sigma}^*$ and $\boldsymbol{\epsilon}^*$ are column matrices of generalized stress resultants and of strains, the definitions of which will be clear on comparing equations (8) and (10). Similarly, the definitions of the submatrices appearing in equation (10) will be clear on comparing with equation (8). It should be noted that the constitutive equations (10) or (8) are very general indeed. The existence of the **B** matrix is a major difference between a laminate and a single-layer plate, where the symmetry about the mid-surface leads the **B** to be zero. Consequently, the analysis would be more expensive where **B** exists, as the two in-plane mid-surface displacements u and v are involved. Furthermore, there are three types of anisotropy which are possible following the terms resulting from Q_{16} and Q_{26} which link normal stresses σ_x and σ_y to in-plane shear strain γ_{xy} respectively. The terms A_{16} and A_{26} form the stretching–shearing anisotropy. The terms B_{16} and B_{26} form the stretching–twisting anisotropy, while the bending–twisting anisotropy occurs due to the terms D_{16} and D_{26}.

These three types of anisotropy make laminate problems rather complicated. Not only do they prevent any attempt to obtain closed form solutions, but they also make some approximate solution methods inappropriate. For instance, the analytical RRM based on beam mode functions will give somewhat over-stiff solutions [20, 21] for some rectangular laminates due to these anisotropies.

During free vibration the fundamental quantities vary harmonically with time, with circular frequency p. Let u, v and w now be regarded as amplitudes of the motion. Then the maximum strain energy of the laminate is

$$U_{max} = \tfrac{1}{2}\int_{A_0} \boldsymbol{\sigma}^{*\mathrm{T}}\boldsymbol{\epsilon}^* \, \mathrm{d}A_0 = \tfrac{1}{2}\int_{A_0} \boldsymbol{\epsilon}^{*\mathrm{T}} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \boldsymbol{\epsilon}^* \, \mathrm{d}A_0, \tag{11}$$

or, in a full form,

$$\begin{aligned} U = \tfrac{1}{2}\int_{A_0} \{ & A_{11}(u_{,x})^2 + A_{22}(v_{,y})^2 + A_{66}(u_{,y} + v_{,x})^2 + 2A_{12}u_{,x}v_{,y} \\ & + 2A_{16}u_{,x}(u_{,y} + v_{,x}) + 2A_{26}v_{,y}(u_{,y} + v_{,x}) \\ & + 2B_{11}u_{,x}w_{,xx} - 2B_{22}v_{,y}w_{,yy} - 4B_{66}(u_{,y} + v_{,x})w_{,xy} + 2B_{12}(u_{,x}w_{,yy} + v_{,y}w_{,xx}) \\ & - 2B_{16}[2u_{,x}w_{,xy} + (u_{,y} + v_{,x})w_{,xx}] - 2B_{26}[2v_{,y}w_{,xy} + (u_{,y} + v_{,x})w_{,yy}] \\ & + D_{11}w_{,xx}^2 + D_{22}w_{,yy}^2 + 4D_{66}w_{,xy}^2 + 2D_{12}w_{,xx}w_{,yy} + 4D_{16}w_{,xx}w_{,xy} \\ & + 4D_{26}w_{,yy}w_{,xy}\} \, \mathrm{d}A_0, \end{aligned} \tag{12}$$

where A_0 is the mid-surface area.

The maximum kinetic energy is

$$T_{max} = \tfrac{1}{2}p^2 \int_{A_0} \rho h(u^2 + v^2 + w^2) \, \mathrm{d}A_0, \tag{13}$$

where ρ is the material density, which is assumed here to be uniform through the volume of the laminate.

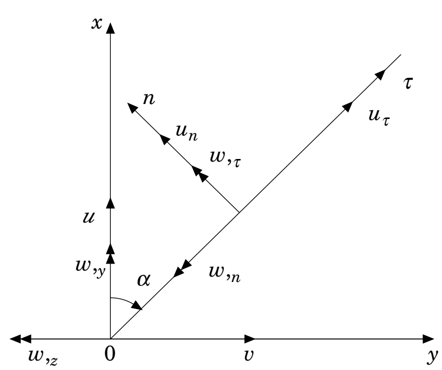

Figure 2. Displacement components at a skew edge.

The transformation between the orthogonal co-ordinate x, y and the oblique co-ordinates ξ, η is

$$x = (\cos\alpha)\xi, \qquad y = (\sin\alpha)\xi + \eta. \tag{14}$$

Suppose that $f(x, y)$ is a function defined in the region of the skew geometry. The relationships between the first and the second derivatives of $f(x, y)$ for the two co-ordinate systems are

$$f_{,x} = L_x(f), \qquad f_{,y} = L_y(f),$$

$$f_{,xx} = L_x^2(f), \qquad f_{,yy} = L_y^2(f), \qquad f_{,xy} = L_x L_y(f), \tag{15}$$

where

$$L_x = a_1\,\partial/\partial\xi - a_2\,\partial/\partial\eta, \qquad L_y = \partial/\partial\eta \tag{16}$$

are linear differential operators and

$$a_1 = 1/\cos\alpha, \qquad a_2 = \tan\alpha. \tag{17}$$

Table 1

Values of Ω for SSSS skew isotropic plates*

α (degrees)	q	Modes 1	2	3	4	5	6	7	8
0	1	2·0003	5·0146	5·0146	8·0213	14·1444	14·1444	16·6544	16·6544
	2	2·0000	5·0145	5·0145	8·0213	10·0406	10·0406	13·0424	13·0424
	3	2·0000	5·0002	5·0002	8·0003	10·1831	10·1831	13·1602	13·1602
	4	2·0000	5·0002	5·0002	8·0003	10·0036	10·0036	13·0031	13·0031
	5	2·0000	5·0000	5·0000	8·0000	10·0036	10·0036	13·0031	13·0031
	6	2·0000	5·0000	5·0000	8·0000	10·0001	10·0001	13·0005	13·0005
	7	2·0000	5·0000	5·0000	8·0000	10·0001	10·0001	13·0001	13·0001
	8	2·0000	5·0000	5·0000	8·0000	10·0000	10·0000	13·0000	13·0000
	[27]	2·0000	5·0000	5·0000	7·9999	9·9999	9·9999	12·9998	12·9998
30	1	2·5529	5·4086	7·4892	9·4004	18·6155	18·7983	19·7868	26·5319
	2	2·5447	5·3574	7·3884	8·6802	12·9543	13·4007	14·4667	19·5583
	3	2·5392	5·3352	7·3131	8·5548	12·8342	12·8551	14·5982	18·4402
	4	2·5335	5·3339	7·2989	8·5080	12·5000	12·5363	14·3006	17·6653
	5	2·5331	5·3334	7·2910	8·5000	12·4606	12·4646	14·2837	17·2960
	6	2·5315	5·3333	7·2867	8·4980	12·4473	12·4480	14·2683	17·1805
	7	2·5302	5·3333	7·2837	8·4972	12·4450	12·4451	14·2620	17·1533
	8	2·5293	5·3333	7·2815	8·4967	12·4445	12·4446	14·2579	17·1481
	[27]	2·5294	5·3333	7·2821	8·4966	12·4442	12·4442	14·2850	17·1471
45	1	3·6980	7·0662	11·7664	12·8059	26·5788	28·2498	30·8390	42·8119
	2	3·6567	6·8078	10·9397	11·4095	17·3526	18·8943	22·3173	30·2607
	3	3·6321	6·7318	10·4183	11·1543	15·5982	18·1535	21·7197	24·9748
	4	3·6145	6·7189	10·2429	11·0702	14·6754	17·3770	20·1706	23·1098
	5	3·6020	6·7159	10·1940	11·0254	14·3801	17·1574	19·2945	22·5801
	6	3·5927	6·7155	10·1817	11·0011	14·2909	17·0780	18·9275	22·3932
	7	3·5856	6·7154	10·1779	10·9848	14·2713	17·0575	18·8128	22·3248
	8	3·5800	6·7154	10·1759	10·9724	14·2675	17·0530	18·7841	22·2957
	[27]	3·5800	6·7153	10·1756	10·9754	14·2662	17·0518	18·7806	22·2955

2.3. B-SPLINE DISPLACEMENT FIELD AND BOUNDARY CONDITIONS

The displacement field is assumed in the oblique co-ordinate system $o\xi\eta z$ and is of the form

$$u(\xi, \eta) = (\bar{\boldsymbol{\theta}}_k \otimes \bar{\boldsymbol{\beta}}_k)\mathbf{d}_1, \qquad v(\xi, \eta) = (\bar{\boldsymbol{\theta}}_k \otimes \bar{\boldsymbol{\beta}}_k)\mathbf{d}_2, \qquad w(\xi, \eta) = (\bar{\boldsymbol{\theta}}_k \otimes \bar{\boldsymbol{\beta}}_k)\mathbf{d}_3, \tag{18}$$

where $\bar{\boldsymbol{\theta}}_k$ are the modified B-spline basis functions [20–22] in the ξ-direction. They contain $q_\xi + k$ B-spline functions, where q_ξ and k are the number of spline sections in the ξ-direction and the polynomial order of B-spline functions, respectively. The $\bar{\boldsymbol{\beta}}_k$ are similarly defined in the η-direction. The number of spline sections in the η-direction is q_η. The $\mathbf{d}_i$ $(i = 1, 2, 3)$ are column matrices of generalized displacement parameters.

In the case of rectangular laminates, this displacement field can satisfy any prescribed geometric boundary conditions in a straightforward manner [20–22]. When skew laminates are considered, however, an explanation of the introduction of boundary conditions is required.

The boundary conditions at the two non-skew edges, i.e., $\xi = 0, A$, will not be considered, since they are identical to those in the case of rectangular laminates. Taking one of the skew edges, i.e., the edge $\eta = 0$, as an example, one defines the boundary conditions as follows.

TABLE 2

Values of Ω^* for CCCC *isotropic plates*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	3·6476	7·5278	7·5278	11·0026	—	—	—	—
	2	3·6467	7·5171	7·5171	11·0026	13·5603	13·6317	16·8413	16·8413
	3	3·6465	7·4375	7·4375	10·9664	14·0397	14·1198	17·2124	17·2124
	4	3·6462	7·4395	7·4395	10·9696	13·3362	13·3998	16·7244	16·7244
	5	3·6461	7·4367	7·4367	10·9654	13·3583	13·4224	16·7388	16·7388
	6	3·6461	7·4364	7·4364	10·9469	13·3354	13·3988	16·7211	16·7211
	7	3·6461	7·4364	7·4364	10·9647	13·3326	13·3959	16·7188	16·7188
	8	3·6461	7·4364	7·4364	10·9647	13·3321	13·3954	16·7183	16·7183
	[27]	3·6460	7·4362	7·4362	10·9644	13·3315	13·3947	16·7174	16·7174
30	1	4·7555	9·0261	11·8021	15·9440	—	—	—	—
	2	4·6816	8·4857	11·1913	13·1824	18·6042	19·8623	20·5937	27·5626
	3	4·6734	8·2844	10·7458	12·3717	18·3054	18·4664	20·1339	25·7796
	4	4·6710	8·2742	10·6877	12·1370	16·9809	17·1003	18·9646	23·4932
	5	4·6703	8·2686	10·6627	12·0959	16·8250	16·8466	18·9401	22·5994
	6	4·6700	8·2680	10·6578	12·0852	16·7394	16·7688	18·8808	22·2636
	7	4·6699	8·2679	10·6567	12·0834	16·7210	16·7539	18·8695	22·1415
	8	4·6699	8·2679	10·6561	12·0830	16·7176	16·7511	18·8665	22·1139
	[27]	4·6698	8·2677	10·6554	12·0825	16·7159	16·7496	18·8644	22·1064
45	1	7·1248	13·3435	19·0269	25·6419	—	—	—	—
	2	6·7736	11·7279	17·6072	19·0573	28·6706	31·4509	31·4745	45·3370
	3	6·6924	11·0166	16·4948	16·5019	25·7108	27·1897	31·2769	40·0063
	4	6·6663	10·8518	15·4997	16·1855	21·9828	24·4788	28·5719	33·3027
	5	6·6583	10·8017	15·1544	16·0269	20·6643	23·7847	27·1412	30·4010
	6	6·6550	10·7923	15·0560	15·9698	20·1369	23·4209	25·9273	29·5868
	7	6·6534	10·7907	15·0344	15·9493	19·9841	23·2990	25·3947	29·3251
	8	6·6525	10·7903	15·0297	15·9413	19·9484	23·2646	25·2288	29·2399
	[27]	6·6519	10·7898	15·0276	15·9342	19·9365	23·2526	25·1799	29·2107

Clamped edge:

$$u(x, y)|_{\eta=0} = u(\xi, 0) = 0, \qquad v(x, y)|_{\eta=0} = v(\xi, 0) = 0,$$
$$w(x, y)|_{\eta=0} = w(\xi, 0) = 0, \qquad w_{,n}(x, y)|_{\eta=0} = 0. \tag{19}$$

The $w_{,n}(x, y)$ in equations (19) is the normal rotation of the edge, as shown in Figure 2, and it can be expressed as

$$w_{,n} = -w_{,x} \sin\alpha + w_{,y} \cos\alpha. \tag{20}$$

By using equations (15) and noting that $w_{,\xi} = 0$ at a clamped edge, it follows that

$$w_{,n} = -\cos\alpha w_{,\eta}. \tag{21}$$

Thus, the fourth condition of equations (19) is of the form

$$w_{,\eta}(x, y)|_{\eta=0} = w_{,\eta}(\xi, 0) = 0. \tag{22}$$

Therefore, it can be seen that the boundary conditions at a clamped edge are the same as those in the case of a rectangular laminate.

Simply supported edge: the standard simply supported boundary conditions are

$$u_\tau(x, y)|_{\eta=0} = u_\tau(\xi, 0) = 0, \qquad w(x, y)|_{\eta=0} = w(\xi, 0) = 0, \tag{23}$$

Table 3

Values of Ω for SSSS *skew composite laminates with five symmetric cross-ply layers*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	1·9146	3·9878	6·6839	7·6872	11·8666	13·7337	21·8628	22·2937
	2	1·9141	3·9876	6·6838	7·6872	8·1878	10·6710	14·8530	15·4489
	3	1·9141	3·9747	6·6571	7·6568	8·3152	10·7545	14·3427	15·0972
	4	1·9141	3·9747	6·6571	7·6568	8·1543	10·6277	14·7886	14·9102
	5	1·9141	3·9745	6·6567	7·6564	8·1543	10·6275	14·2062	14·7886
	6	1·9141	3·9745	6·6567	7·6564	8·1515	10·6253	14·2032	14·7833
	7	1·9141	3·9745	6·6567	7·6564	8·1511	10·6250	14·1890	14·7825
	8	1·9141	3·9745	6·6567	7·6564	8·1511	10·6249	14·1863	14·7824
30	1	2·8902	5·3923	9·7640	10·3213	17·0364	19·5677	29·9806	32·1387
	2	2·8630	5·2710	8·9039	9·4955	12·7993	13·9755	20·4970	23·9885
	3	2·8495	5·2152	8·6323	9·3381	12·5730	12·9342	18·8787	18·9566
	4	2·8408	5·2027	8·5124	9·2937	12·2239	12·3630	17·5677	18·6219
	5	2·8348	5·1967	8·4897	9·2747	12·1561	12·1897	16·8036	17·6957
	6	2·8306	5·1933	8·4848	9·2662	12·1260	12·1428	16·5615	17·5444
	7	2·8273	5·1909	8·4839	9·2611	12·1142	12·1327	16·4944	17·4899
	8	2·8248	5·1891	8·4836	9·2574	12·1070	12·1301	16·4804	17·4778
45	1	4·7699	8·0037	15·6248	16·3919	31·6570	32·9777	46·8190	57·4490
	2	4·6558	7·4286	12·1928	15·6200	20·4852	22·2890	31·6316	39·1680
	3	4·5981	7·2230	11·1073	15·0462	17·2223	20·9526	26·6890	31·0008
	4	4·5582	7·1591	10·6656	14·6933	15·5780	19·5443	22·7040	28·2140
	5	4·5300	7·1349	10·5103	14·3680	15·0393	18·7915	20·7278	25·3921
	6	4·5086	7·1241	10·4655	14·1820	14·8730	18·2845	19·9600	23·9293
	7	4·4920	7·1173	10·4544	14·1199	14·8104	18·0466	19·6897	23·1147
	8	4·4786	7·1121	10·4512	14·1024	14·7797	17·9628	19·6031	22·7778

where u_τ is the tangential in-plane displacement at the skew edge $\eta = 0$, as shown in Figure 2. By applying the rotational co-ordinate transformation, u_τ is related to u and v as follows:

$$u_\tau = u \cos \alpha + v \sin \alpha. \tag{24}$$

Thus the standard simply supported boundary conditions can be expressed as

$$u(\xi, 0) = -\tan \alpha v(\xi, 0), \qquad w(\xi, 0) = 0. \tag{25}$$

Free edge: there are no geometric boundary conditions to be applied at a free edge.

2.4. EIGENVALUE EQUATIONS

By substituting the B-spline displacement field (18) into the energy expressions set out in equations (12) and (13), and in conjunction with the linear differential operators given in equations (15), one will obtain the total potential energy of the laminate in terms of the generalized displacement parameters in the $o\xi\eta$ co-ordinate system. Then, applying Hamilton's principle results in the eigenvalue equations

$$(\mathbf{K} - p^2\mathbf{M})\mathbf{D} = \mathbf{0}. \tag{26}$$

In equation (26), **K** and **M** are the stiffness and consistent mass matrices of the laminate, respectively. The details of these two matrices can be found in reference [21] and are not

TABLE 4

Values of Ω for CCCC *skew composite laminates with five symmetric cross-ply layers*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	4·2501	6·7822	10·6510	11·9967	—	—	—	—
	2	4·2379	6·7775	10·6508	11·6639	11·9967	15·4283	20·6138	21·4343
	3	4·2380	6·6917	10·4518	11·7846	12·1017	15·6377	18·9498	21·4810
	4	4·2378	6·6940	10·4573	11·4408	11·7907	15·1501	20·1636	20·4420
	5	4·2378	6·6916	10·4516	11·4617	11·7845	15·1619	18·2039	20·2054
	6	4·2378	6·6914	10·4512	11·4403	11·7840	15·1458	18·2812	20·1626
	7	4·2378	6·6913	10·4511	11·4377	11·7839	15·1439	18·2033	20·1574
	8	4·2378	6·6913	10·4511	11·4372	11·7839	15·1435	18·1899	20·1565
30	1	5·8645	9·6964	15·9796	19·8235	—	—	—	—
	2	5·6802	8·7374	14·4362	15·1005	19·3219	23·7651	29·3435	35·7310
	3	5·6469	8·3932	13·1762	14·3817	18·4194	19·7706	26·2506	28·6349
	4	5·6359	8·3438	12·5618	14·2274	17·3317	17·7180	25·0393	26·3018
	5	5·6326	8·3287	12·4425	14·1550	17·0236	17·1104	22·8532	23·5547
	6	5·6314	8·3257	12·4052	14·1342	16·8087	16·9973	22·0547	23·2169
	7	5·6309	8·3249	12·3980	14·1281	16·7508	16·9675	21·7720	22·9948
	8	5·6308	8·3246	12·3960	14·1260	16·7383	16·9593	21·6956	22·9389
45	1	9·5577	16·7280	26·9708	35·4504	—	—	—	—
	2	8·8742	13·9992	23·9234	24·7401	35·0525	41·8036	46·9419	63·7236
	3	8·6360	12·6177	19·7874	22·9087	31·9659	32·3071	45·3810	47·3506
	4	8·5288	12·1111	17·6437	22·2476	25·9239	28·8321	38·1970	43·8306
	5	8·4855	11·8960	16·7142	21·8372	23·1157	27·4796	32·3324	37·6119
	6	8·4677	11·8256	16·3268	21·6097	21·7530	26·5542	29·0313	35·1138
	7	8·4594	11·8055	16·2000	21·1337	21·5508	25·9947	27·4378	33·2650
	8	8·4550	11·7993	16·1667	20·9381	21·5054	25·6527	26·8443	32·0557

given here. p is the circular frequency (rad/s), and $\mathbf{D}$ is a column matrix of generalized displacement parameters, which is defined as

$$\mathbf{D} = \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{Bmatrix}. \tag{27}$$

Before the introduction of any boundary conditions, the number of degrees of freedom is $3(q_\xi + k)(q_\eta + k)$. After the boundary conditions are introduced, the eigenvalue equation (26) can be solved in a number of ways to obtain the natural frequencies. In this paper, the Sturm sequence method is used. Numerical applications are reported in the next section.

3. NUMERICAL APPLICATIONS

Mainly, free vibrations of skew fibre reinforced composite laminates are considered in this section. The laminates in sections 3.2–3.5 are selected in such a way that the coupling between in-plane and out-of-plane behaviour either exists or is absent by choosing either symmetric or anti-symmetric lay-ups. Furthermore, in either case the fibre orientations may be either cross-ply or angle-ply in order to examine the effect of different material

TABLE 5

Values of Ω for SSSS skew composite laminates with five symmetric angle-ply layers

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	2·4421	4·9955	6·2421	9·1048	14·3750	15·1158	16·4501	21·4692
	2	2·4395	4·9943	6·2209	8·5360	10·4110	11·7325	13·2716	16·8514
	3	2·4372	4·9874	6·1918	8·5286	10·4523	11·8345	13·0115	15·7619
	4	2·4359	4·9867	6·1879	8·4917	10·2661	11·6641	12·8905	15·5529
	5	2·4351	4·9865	6·1851	8·4882	10·2589	11·6584	12·8373	15·2642
	6	2·4345	4·9865	6·1836	8·4872	10·2544	11·6507	12·8283	15·2303
	7	2·4341	4·9865	6·1825	8·4871	10·2537	11·6480	12·8264	15·2194
	8	2·4339	4·9865	6·1818	8·4870	10·2536	11·6464	12·8260	15·2173
30	1	2·6259	5·7161	6·9081	10·1011	16·4985	18·4204	19·0854	23·5530
	2	2·6196	5·7029	6·8810	9·5366	12·1187	13·3410	14·9413	18·2857
	3	2·6162	5·6928	6·8433	9·5219	12·1327	13·4909	14·5495	17·8724
	4	2·6146	5·6918	6·8386	9·4828	11·9122	13·2510	14·3823	17·5595
	5	2·6134	5·6911	6·8354	9·4791	11·9010	13·2457	14·3006	17·3807
	6	2·6127	5·6907	6·8336	9·4778	11·8931	13·2389	14·2854	17·3509
	7	2·6122	5·6904	6·8324	9·4775	11·8908	13·2367	14·2817	17·3407
	8	2·6119	5·6902	6·8316	9·4773	11·8900	13·2355	14·2809	17·3382
45	1	3·3851	7·0728	10·1771	12·3219	24·9026	25·2309	27·5401	36·2550
	2	3·3623	6·9520	9·9490	11·0927	17·4261	17·4556	19·8013	26·4198
	3	3·3466	6·9096	9·7825	10·8407	16·3495	17·0378	19·9215	24·6479
	4	3·3361	6·9047	9·7386	10·7462	15·7679	16·4088	19·4594	22·5241
	5	3·3292	6·9027	9·7172	10·7284	15·5933	16·2344	19·4109	21·7005
	6	3·3244	6·9016	9·7056	10·7236	15·5448	16·1678	19·3767	21·4178
	7	3·3209	6·9008	9·6972	10·7218	15·5340	16·1499	19·3593	21·3245
	8	3·3182	6·9002	9·6908	10·7206	15·5318	16·1447	19·3481	21·3005

anisotropies on the present B-spline RRM. The material properties of each lamina are identical and have the following values:

$$E_L/E_T = 40{\cdot}0, \qquad G_{LT}/E_T = 0{\cdot}6, \qquad G_{TT}/E_T = 0{\cdot}5, \qquad \nu_{LT} = 0{\cdot}25. \tag{28}$$

The thickness of each lamina is assumed to be the same, and the laminates are assumed to have rhombic geometry, i.e., $A = B$, although the general case $A \neq B$ can be studied without any complications. Due to the lack of comparative solutions, all of the present results are presented in a manner of convergence studies with the number of spline sections $q = q_\xi = q_\eta$ ranging from 1 to 8, and are given in a non-dimensional frequency parameter defined as

$$\Omega = p(B^2/\pi^2 h)(\rho/E_T)^{1/2}. \tag{29}$$

Throughout these exercises, the first eight modes of vibration are considered and the polynomial order k of B-spline functions is kept to be quintic: i.e., $k = 5$. The main purposes of these exercises are twofold. One is to demonstrate the accuracy and efficiency of the proposed B-spline RRM, and the other is to produce some results which may be regarded as benchmark solutions for other academic research workers and design engineers. The arrangement of the applications is as follows: skew isotropic plates, skew composite laminates with five symmetric cross-ply layers, skew composite laminates with five symmetric angle-ply layers, skew composite laminates with four anti-symmetric cross-ply layers and skew composite laminates with four anti-symmetric angle-ply layers.

TABLE 6

Values of Ω for CCCC *skew composite laminates with five symmetric angle-ply layers*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	3·9220	7·3271	9·1978	13·2512	—	—	—	—
	2	3·9187	7·2047	8·7218	11·4655	14·2408	15·2966	17·5855	22·2031
	3	3·9037	7·1509	8·4808	11·3483	14·1241	15·4777	16·7004	20·5827
	4	3·9015	7·1498	8·4681	11·2257	13·3682	14·7940	16·2781	19·5579
	5	3·9011	7·1473	8·4608	11·2193	13·3548	14·7818	16·1654	18·9452
	6	3·9009	7·1466	8·4592	11·2130	13·3269	14·7501	16·1390	18·8660
	7	3·9009	7·1464	8·4587	11·2116	13·3226	14·7443	16·1294	18·8235
	8	3·9009	7·1464	8·4585	11·2112	13·3216	14·7425	16·1271	18·8145
30	1	4·5533	8·6805	10·3318	14·5308	—	—	—	—
	2	4·5525	8·5066	10·0826	13·2456	16·5739	17·8645	20·1048	23·9594
	3	4·5449	8·3883	9·8937	13·0634	16·6707	18·3806	19·4134	23·4630
	4	4·5434	8·3854	9·8894	12·8751	15·7844	17·5090	18·6724	22·4660
	5	4·5432	8·3824	9·8826	12·8627	15·7420	17·5276	18·4177	22·0867
	6	4·5431	8·3820	9·8814	12·8550	15·6987	17·4947	18·3623	21·9913
	7	4·5431	8·3819	9·8812	12·8536	15·6919	17·4900	18·3435	21·9463
	8	4·5431	8·3819	9·8810	12·8533	15·6906	17·4889	18·3396	21·9364
45	1	6·4551	12·2232	16·1890	21·6989	—	—	—	—
	2	6·3272	11·3130	15·3676	17·6311	25·3428	27·3906	27·8825	37·6794
	3	6·3130	10·9055	14·7236	16·2112	24·5490	24·8386	27·7741	35·0754
	4	6·3084	10·8464	14·5905	15·6676	22·1952	22·8907	26·0900	31·6833
	5	6·3064	10·8245	14·5212	15·5219	21·4677	22·4802	26·0121	29·2082
	6	6·3055	10·8206	14·5019	15·4790	21·1713	22·2067	25·9137	28·2425
	7	6·3050	10·8197	14·4967	15·4708	21·0829	22·1058	25·8920	27·8261
	8	6·3048	10·8193	14·4949	15·4692	21·0620	22·0759	25·8849	27·6869

In each category two types of boundary conditions, i.e., fully simply supported (SSSS) and fully clamped (CCCC), and three skew angles, i.e., $\alpha = 0^\circ$, 30° and 45°, are considered. Finally, free vibrations of cantilevered skew (CFFF) composite laminates with two unsymmetric layers are studied in section 3.6, where details of the laminates are given. The symbols C, S and F denote clamped, simply supported and free, respectively. The four boundaries are counted from $\xi = 0$ and clockwise.

3.1. SKEW ISOTROPIC PLATES

Due to the lack of comparative results for skew composite laminates, two skew isotropic plates, i.e., SSSS and CCCC plates, are considered first, so that comparisons can be made with earlier published solutions. In this application only the out-of-plane displacement w is considered in the displacement field (18), of course. The results are recorded in Tables 1 and 2 in a non-dimensional frequency parameter, defined as

$$\Omega^* = p(B^2/\pi^2)(\rho h/D)^{1/2}, \tag{30}$$

where $D = Eh^3/[12(1-\nu^2)]$, in which E is Young's modulus, and the Poisson ratio ν is taken to be 0·3. It is observed that the rates of convergence are very satisfactory for all three skew angles, although the rates slow down with the increase of the skew angle. Very close agreements are found between the present converged results and the comparative solutions [27] which are obtained by using the *pb*-2 RRM based on Mindlin plate theory for a very thin geometry, i.e., $h/B = 0{\cdot}001$, in which the through-thickness shear effects

TABLE 7

Values of Ω *for* SSSS *skew composite laminates with four anti-symmetric cross-ply layers*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	1·7543	5·0284	5·0284	7·0435	16·0392	16·0392	16·9131	16·9131
	2	1·7539	5·0283	5·0283	7·0435	10·9516	10·9516	12·1573	12·1573
	3	1·7539	5·0095	5·0095	7·0161	11·1203	11·1203	12·3033	12·3033
	4	1·7539	5·0093	5·0093	7·0159	10·9000	10·9000	12·1020	12·1020
	5	1·7539	5·0091	5·0091	7·0155	10·8959	10·8959	12·0983	12·0983
	6	1·7539	5·0090	5·0090	7·0154	10·8917	10·8917	12·0944	12·0944
	7	1·7539	5·0090	5·0090	7·0154	10·8910	10·8910	12·0937	12·0937
	8	1·7539	5·0090	5·0090	7·0154	10·8908	10·8908	12·0936	12·0936
30	1	2·4914	5·6131	7·5733	9·4631	18·3842	19·2028	22·2334	26·9083
	2	2·4782	5·5457	7·4308	8·8265	12·9175	13·7425	15·5108	19·4891
	3	2·4709	5·5187	7·3383	8·6625	12·9039	13·1883	15·5116	18·6438
	4	2·4663	5·5168	7·3170	8·6191	12·5998	12·8275	15·1044	17·7920
	5	2·4632	5·5162	7·3068	8·6095	12·5678	12·7532	15·0398	17·4779
	6	2·4610	5·5160	7·3014	8·6063	12·5552	12·7340	15·0095	17·3639
	7	2·4594	5·5158	7·2977	8·6047	12·5519	12·7302	14·9989	17·3303
	8	2·4583	5·5158	7·2950	8·6036	12·5505	12·7294	14·9941	17·3204
45	1	3·8830	7·2788	12·3950	13·3995	26·8856	27·4045	34·5946	45·0804
	2	3·8349	7·0342	11·4218	11·9244	18·1674	18·6687	24·3343	31·6849
	3	3·8050	6·9598	10·8954	11·5838	16·4201	17·9311	23·4889	26·2337
	4	3·7835	6·9462	10·7151	11·4439	15·5178	17·0882	21·8346	24·3714
	5	3·7680	6·9419	10·6645	11·3740	15·2268	16·7889	20·9280	23·4366
	6	3·7567	6·9403	10·6511	11·3407	15·1351	16·6618	20·5678	22·7438
	7	3·7482	6·9394	10·6468	11·3212	15·1124	16·6210	20·4524	22·4533
	8	3·7415	6·9388	10·6445	11·3070	15·1070	16·6083	20·4228	22·3572

virtually disappear. These close agreements serve to verify the present approach and to establish the foundation for its application into skew composite laminates where no comparative solutions are available.

3.2. SKEW COMPOSITE LAMINATES WITH FIVE SYMMETRIC CROSS-PLY LAYERS

The stacking sequence of these laminates is $0°/90°/0°/90°/0°$. There is no coupling between in-plane and out-of-plane behaviour, i.e., $\mathbf{B} = \mathbf{0}$, due to the symmetric lay-ups and hence only the out-of-plane displacement w in equation (18) is involved. Furthermore, there is no bending–twisting anisotropy either, i.e., $D_{16} = D_{26} = 0$, due to the cross-ply lay-ups. The results are presented in Tables 3 and 4 for SSSS and CCCC laminates respectively. In both cases the manner of convergence is very satisfactory. Of course, with the increase in skew angle, more spline sections are needed to obtain accurate solutions, as expected.

3.3. SKEW COMPOSITE LAMINATES WITH FIVE SYMMETRIC ANGLE-PLY LAYERS

The stacking sequence of these laminates is $45°/45°/-45°/-45°/45°$. Similarly, there is no coupling between in-plane and out-of-plane behaviour due to the symmeric lay-ups in these laminates, and only w in equation (18) needs consideration. However, due to the symmetric angle-ply lay-ups there exists bending–twisting anisotropy: i.e., $D_{16} \neq 0$ and $D_{26} \neq 0$. This anisotropy makes the conventional analytical RRM inappropriate, even in the case of rectangular laminates [20, 21]. To test the present B-spline RRM, the same task as in

TABLE 8

Values of Ω for CCCC *skew composite laminates with four anti-symmetric cross-ply layers*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	3·8793	8·1474	8·1474	10·9625	—	—	—	—
	2	3·8685	8·1459	8·1459	10·9625	15·2898	15·3056	17·0803	17·0803
	3	3·8686	8·0103	8·0103	10·7748	15·8929	15·9096	17·5680	17·5680
	4	3·8685	8·0128	8·0128	10·7785	14·9645	14·9784	16·7448	16·7448
	5	3·8685	8·0088	8·0088	10·7729	14·9776	14·9916	16·7548	16·7548
	6	3·8685	8·0084	8·0084	10·7722	14·9459	14·9598	16·7267	16·7267
	7	3·8685	8·0083	8·0083	10·7722	14·9409	14·9548	16·7222	16·7222
	8	3·8685	8·0083	8·0083	10·7721	14·9400	14·9539	16·7215	16·7215
30	1	4·9504	9·3289	12·2109	15·9607	—	—	—	—
	2	4·8792	8·8866	11·6755	13·6672	18·2862	20·8932	21·8062	27·6586
	3	4·8728	8·7019	11·2355	12·8562	18·3942	19·1434	22·0292	26·4880
	4	4·8709	8·6885	11·1662	12·6221	17·3068	17·7934	20·5632	24·2678
	5	4·8704	8·6813	11·1340	12·5584	17·2234	17·4836	20·4417	23·4295
	6	4·8702	8·6803	11·1265	12·5425	17·1591	17·3642	20·3359	22·9382
	7	4·8701	8·6801	11·1247	12·5395	17·1452	17·3353	20·3059	22·7680
	8	4·8701	8·6801	11·1241	12·5388	17·1424	17·3294	20·2977	22·7235
45	1	7·4551	13·5952	20·1612	26·8305	—	—	—	—
	2	7·0886	11·9603	18·6274	19·6983	28·1656	32·6356	34·6846	47·6851
	3	7·0014	11·2810	17·1535	17·4191	26·5955	26·7377	34·1261	40·1535
	4	6·9724	11·1330	16·1358	17·0380	22·9132	24·3010	30·8734	34·9118
	5	6·4631	11·0889	15·7888	16·8247	21·6107	23·5949	28·9695	32·2820
	6	6·9594	11·0803	15·6812	16·7305	21·0885	23·1672	27·6838	31·3417
	7	6·9575	11·0786	15·6544	16·6928	20·9229	22·9814	27·1372	30·4514
	8	6·9564	11·0782	15·6482	16·6786	20·8790	22·9147	26·9540	30·0230

sections 3.1 and 3.2 is carried out here and the numerical results are presented in Tables 5 and 6. It can be seen that this anisotropy does not have any significant effect on the present B-spline RRM. The convergence rates for both SSSS and CCCC laminates are very satisfactory indeed.

3.4. SKEW COMPOSITE LAMINATES WITH FOUR ANTI-SYMMETRIC CROSS-PLY LAYERS

The stacking sequence of these laminates is $0°/90°/0°/90°$. Thus there exists coupling between in-plane and out-of-plane behaviour due to B_{11} and B_{22}. The full displacement field (18) should be applied, and hence the solution of these problems would be more expensive. Numerical results are recorded in Tables 7 and 8 for the respective SSSS and CCCC cases. Also, very good convergence manner is observed. It should be pointed out that the frequency parameter Ω in this application depends on the thickness-to-width ratio, i.e., h/B, due to the existence of the coupling matrix $\mathbf{B}$. Here, h/B is taken to be 0·001.

3.5. SKEW COMPOSITE LAMINATES WITH FOUR ANTI-SYMMETRIC ANGLE-PLY LAYERS

The stacking sequence of these laminates is $45°/-45°/45°/-45°$. Coupling between in-plane and out-of-plane behaviour occurs due to the B_{16} and B_{26} terms. Similarly, the full displacement field (18) should be used. It should be noted that bending–twisting anisotropy is absent in these laminates due to $D_{16} = D_{26} = 0$. However, stretching–twisting anisotropy exists due to B_{16} and B_{26}. As the D_{16} and D_{26} terms make the analytical RRM inappropriate in these applications, these B_{16} and B_{26} terms have the same effect on the analytical RRM

TABLE 9

Values of Ω for SSSS *skew composite laminates with four anti-symmetric angle-ply layers*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	2·4821	5·4769	5·4769	9·6749	13·4660	13·8734	18·0767	18·0767
	2	2·4809	5·4751	5·4751	9·6676	10·0671	10·1904	15·0569	15·0569
	3	2·4804	5·4632	5·4632	9·6491	10·1800	10·3063	15·1477	15·1477
	4	2·4801	5·4627	5·4627	9·6477	10·0333	10·1491	15·0149	15·0149
	5	2·4800	5·4622	5·4622	9·6467	10·0311	10·1461	15·0122	15·0122
	6	2·4799	5·4620	5·4620	9·6462	10·0283	10·1426	15·0092	15·0092
	7	2·4799	5·4619	5·4619	9·6460	10·0279	10·1419	15·0085	15·0085
	8	2·4798	5·4618	5·4618	9·6458	10·0278	10·1416	15·0082	15·0082
30	1	2·8736	5·6074	8·1292	10·1669	17·9377	19·4655	22·0370	29·0288
	2	2·8560	5·5661	7·9604	9·2269	12·8196	14·3139	16·0001	21·3274
	3	2·8438	5·5476	7·8509	9·1121	12·6572	13·6922	15·9983	19·1229
	4	2·8360	5·5460	7·8260	9·0587	12·3381	13·3921	15·5910	18·3072
	5	2·8308	5·5454	7·8112	9·0501	12·2961	13·3094	15·5117	17·7968
	6	2·8271	5·5452	7·8018	9·0477	12·2815	13·2894	15·4676	17·6634
	7	2·8244	5·5451	7·7948	9·0469	12·2780	13·2851	15·4497	17·6250
	8	2·8224	5·5450	7·7894	9·0464	12·2767	13·2842	15·4402	17·6156
45	1	3·8830	7·2788	12·3950	13·3995	26·8856	27·4045	34·5946	45·0804
	2	3·8349	7·0342	11·4218	11·9244	18·1674	18·6687	24·3343	31·6849
	3	3·8050	6·9598	10·8954	11·5838	16·4201	17·9311	23·4889	26·2337
	4	3·7835	6·9462	10·7151	11·4439	15·5178	17·0882	21·8346	24·3714
	5	3·7680	6·9419	10·6645	11·3740	15·2268	16·7889	20·9280	23·4366
	6	3·7567	6·9403	10·6511	11·3407	15·1351	16·6618	20·5678	22·7438
	7	3·7482	6·9394	10·6468	11·3212	15·1124	16·6210	20·4524	22·4533
	8	3·7415	6·9388	10·6445	11·3070	15·1070	16·6083	20·4228	22·3572

[21]. From the results presented in Tables 9 and 10 it can be concluded that the present B-spline RRM is not affected by these coupling terms and that the convergence rate is really good. Similarly, as the frequency parameter Ω depends on the thickness-to-width ratio, i. e., h/B, in this application due to the existence of the coupling matrix $\mathbf{B}$, h/B is taken to be a fixed value of 0·001. It is noted that the values of Ω with skew angles 45° in Tables 9 and 10 are the same as those in Tables 7 and 8, respectively, since at this particular angle the problems become the same.

3.6. SKEW COMPOSITE LAMINATES WITH TWO UNSYMMETRIC LAYERS

When using the present B-spline RRM for free vibration analysis a cantilevered (CFFF) skew composite laminate presents less complexity than either a fully simply supported (SSSS) or a fully clamped (CCCC) one does since either SSSS or CCCC boundary condition gives more constraints than CFFF boundary condition does. However, cantilevered skew composite laminates may have practical importance. For instance, they may be used to approximate aircraft wings and stabilizers. In this final numerical application, the cantilevered skew composite laminates studied in reference [17] are reconsidered. The stacking sequence of these laminates is $\alpha/(\alpha - 22{\cdot}5^\circ)$, where α is the skew angle and the thickness of each lamina is the same. Due to the arbitrary lay-ups there exist couplings between in-plane and out-of-plane behaviour and all types of material anisotropy. The length B is 0·2032 m (8·0 in) and the thickness h is $0{\cdot}125 \times 10^{-2}$ m ($0{\cdot}492 \times 10^{-1}$ in). The mid-surface area of the laminates is $0{\cdot}413 \times 10^{-1}$ m^2 (64·0 in^2). The

TABLE 10

Values of Ω for CCCC *skew composite laminates with four anti-symmetric angle-ply layers*

		Modes							
α (degrees)	q	1	2	3	4	5	6	7	8
0	1	3·7530	7·5966	7·5966	12·0377	1254·51	—	—	—
	2	3·7375	7·5381	7·5381	12·0377	13·0359	13·1049	18·1919	18·1919
	3	3·7362	7·4986	7·4986	12·0348	13·3847	13·4655	18·3810	18·3810
	4	3·7347	7·4954	7·4954	12·0321	12·8656	12·9322	18·1155	18·1155
	5	3·7344	7·4923	7·4923	12·0226	12·8723	12·9394	18·1123	18·1123
	6	3·7342	7·4916	7·4916	12·0206	12·8530	12·9196	18·0946	18·0946
	7	3·7341	7·4913	7·4913	12·0197	12·8498	12·9161	18·0904	18·0904
	8	3·7341	7·4912	7·4912	12·0194	12·8491	12·9152	18·0891	18·0891
30	1	4·9985	9·1728	12·7722	17·3763	1255·10	—	—	—
	2	4·9113	8·5915	11·9195	13·7769	18·3429	21·8687	21·9646	30·2793
	3	4·8967	8·4168	11·3239	12·9362	17·8361	19·2939	21·8855	26·6840
	4	4·8919	8·4103	11·2072	12·6526	16·6521	17·9483	20·4531	24·6495
	5	4·8900	8·4062	11·1614	12·6074	16·4971	17·6348	20·3055	23·2242
	6	4·8893	8·4055	11·1506	12·5935	16·4198	17·5374	20·1780	22·7942
	7	4·8890	8·4053	11·1474	12·5907	16·4030	17·5114	20·1347	22·6260
	8	4·8889	8·4053	11·1461	12·5901	16·3995	17·5057	20·1206	22·5803
45	1	7·4551	13·5952	20·1612	26·8305	1502·72	—	—	—
	2	7·0886	11·9603	18·6274	19·6983	28·1656	32·6356	34·6846	47·6851
	3	7·0014	11·2810	17·1535	17·4191	26·5955	26·7377	34·1261	40·1535
	4	6·9724	11·1330	16·1358	17·0380	22·9132	24·3010	30·8734	34·9118
	5	6·6931	11·0889	15·7888	16·8247	21·6107	23·5949	28·9695	32·2820
	6	6·9594	11·0803	15·6812	16·7305	21·0886	23·1672	27·6838	31·3417
	7	6·9575	11·0786	15·6544	16·6928	20·9229	22·9814	27·1372	30·4514
	8	6·9564	11·0782	15·6482	16·6786	20·8790	22·9147	26·9540	30·0230

material properties of each lamina are $E_L = 160{\cdot}54$ GPa ($23{\cdot}3 \times 10^6$ psi), $E_T = 12{\cdot}48$ GPa ($1{\cdot}81 \times 10^6$ psi), $G_{LT} = 6{\cdot}72$ GPa ($0{\cdot}976 \times 10^6$ psi), $\nu_{LT} = 0{\cdot}22$ and the density $\rho = 1909{\cdot}99$ kg m^{-3} ($0{\cdot}069$ lb in^{-3}). The first eight frequencies (Hz) are recorded in Table 11 with various skew angles, and they are obtained by using $q = q_\xi = q_\eta = 8$ and $k = 5$. A close agreement is found between the present results and those in reference [17]. The slight difference may be due to neglecting the in-plane inertias in the analysis of reference [17].

4. CONCLUSIONS

A B-spline RRM is presented for the study of free vibrations of thin skew fibre reinforced composite laminates with various boundary conditions. The laminates may have arbitrary lay-ups, which may include couplings between in-plane and out-of-plane behaviour and any types of material anisotropy due to the interaction terms Q_{16} and Q_{26} between the normal stresses σ_x and σ_y and the shear strain γ_{xy}.

Numerical applications include skew isotropic plates and various composite laminates. For the skew isotropic plates, very close agreement is found between the present results and the comparative solutions. This serves to verify the present method and to establish the foundation for the analysis of skew composite laminates. Due to lack of comparative solutions for skew composite laminates, in sections 3.2–3.5 all of the numerical results are presented in a manner of convergence studies. The laminates are selected in such a way that the coupling between in-plane and out-of-plane behaviour either exists or is absent by choosing either symmetric or anti-symmetric lay-ups. Furthermore, in either case the fibre orientations may be either cross-ply or angle-ply, in order to examine the effect of the material anisotropy on the present B-spline RRM. All of these applications demonstrate that the B-spline RRM developed is accurate and efficient in all the cases considered. Unlike the analytical RRM, the B-spline RRM gives accurate solutions no matter what types of material anisotropy exist. It is hoped that the tabulated results may

TABLE 11

Frequencies (Hz) *of* CFFF *skew composite laminates with two unsymmetric layers*

	Modes							
α (degrees)	1	2	3	4	5	6	7	8
−45	16·86	47·52	104·56	141·41	206·28	281·52	308·42	371·01
[17]	16·94	47·75	105·04	142·07	207·27	—	—	—
−30	24·67	52·04	136·87	157·05	203·57	294·94	357·31	433·93
	24·78	52·27	137·48	157·75	204·49	—	—	—
−15	30·96	54·93	129·25	194·05	228·01	279·39	331·71	484·07
	31·09	55·17	129·82	194·89	229·00	—	—	—
0	34·12	56·08	122·86	215·24	245·46	262·29	338·45	448·69
	34·26	56·34	123·40	216·17	246·57	—	—	—
15	33·31	54·93	120·18	211·86	245·24	257·76	346·96	411·63
	33·45	55·18	120·71	212·78	246·32	—	—	—
30	28·45	56·09	120·83	184·16	238·67	245·43	362·07	385·43
	28·57	51·34	121·36	185·06	239·73	—	—	—
45	20·32	44·85	115·17	143·93	218·96	234·86	330·70	392·20
	20·42	45·09	115·75	144·76	219·94	—	—	—

be useful to engineers and designers and may also serve as benchmark solutions for other academic research workers. In the final numerical application, i.e., section 3.6, cantilevered skew composite laminates are considered. Close agreements are observed between the present results and those in reference [17].

Finally, it is worth noting that the present B-spline RRM could be extended to study free vibration of thick skew composite laminates based on shear deformation laminate theory. This study will be reported in another paper.

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