Journal of Sound and Vibration (1997) **201**(3), 353–363

TRANSVERSE VIBRATIONS OF SHORT BEAMS: FINITE ELEMENT MODELS OBTAINED BY A CONDENSATION METHOD

S. Corn, N. Bouhaddi and J. Piranda

Laboratoire de Mécanique Appliquée R. Chaléat, U.F.R. Sciences et Techniques, 24 rue de l'Epitaphe 25030 Besançon Cedex, France

(*Received* 20 *May* 1996, *and in final form* 6 *September* 1996)

This paper is concerned with the dynamic behaviour of Timoshenko beams. A new method for simply and systematically constructing finite beam elements is then proposed. The continuous model, which takes into account both rotary inertia and transverse shear deformation, is presented as a tutorial review. It allows certain vibratory phenomena characteristic of short beams to be demonstrated. A method is proposed for constructing a two-node finite element based on Guyan condensation that leads to the results of classical formulations, but in a simple and systematic manner. This element is verified with numerical and experimental tests. The proposed method is then generalized in order to obtain new improved three-node finite elements.

1. INTRODUCTION

In structural mechanics, the Euler–Bernoulli formulation represents the most widely used theory for modelling the dynamic flexural behaviour of beams. This theory was extended by Timoshenko [1] in order to account for rotary inertia and transverse shear effects, often considered to be secondary. This extension leads to small corrections of the predictions of the Euler–Bernoulli model in the case of slender beams. However, it can lead to significant differences in the case of short beams.

First, we present in this paper a review of the continuous Timoshenko beam theory, and specifically the particular phenomena due to the introduction of shear effects [2]. The existence of "shear modes" is made apparent by direct integration of the differential equations of motion, and comfirmed by numerical tests.

Second, the Timoshenko model is discretized in beam finite elements having two d.o.f. (degrees of freedom) per node: one displacement and one rotation angle. The hypothesis and calculation made by Davis [3] are briefly presented in order to obtain a two-node finite element from a cubic interpolation of the transverse displacement. A new method is then proposed, based on Guyan condensation [4], leading to stiffness and mass matrices which are identical to those obtained by Davis. The performance of the Timoshenko element is then compared experimentally with that of more commonly used finite elements. The proposed method has the advantage of being simple, systematic and generalizable. Indeed, it allows new three-node beam elements to be constructed easily from high order interpolations. The resulting elements are more precise than the usual two-node Timoshenko element, provided that Guyan condensation remains valid in the frequency domain of interest.

0022–460X/97/130353 + 11 $25.00/0/sv960766

2. REVIEW OF THE TIMOSHENKO BEAM THEORY

2.1. Basic formulation

Consider a uniform prismatic straight beam of length ℓ. A Cartesian co-ordinate system (Ox, Oy, Oz) is defined on the beam, where (Ox) is the centroidal axis, and (Ox, Oy) is a symmetry plane. It is assumed, according to classical kinematics, that the cross-section remain plane (no warping) and that axis displacement is due only to the rotation angle $\psi(x, t)$ of cross-section. Let $v(x, t)$ be the time-dependent transverse displacement of the centroidal axis.

The dynamic equilibrium equations are written as (a list of main symbols is given in the Appendix).

$$\partial T/\partial x = m(\partial^2 v/\partial t^2), \qquad \partial M/\partial x + T = mr^2(\partial^2\psi/\partial t^2). \tag{1, 2}$$

According to Timoshenko's hypothesis, the shear force is expressed by the relation:

$$T = kAG(\partial v/\partial x - \psi) \tag{3}$$

Remark: There are several ways of obtaining the shear coefficient k. Timoshenko [5] presented a calculation based on the hypothesis of a parabolic distribution of the transverse stress σ_{xy} over the cross-section. The method developed by Cowper [6] consists of deducing k from the three-dimensional elasticity problem of a cantilever beam [7]. It proves to be more accurate because it accounts not only for the exact analytical expression of σ_{xy} but also that of σ_{xz}. In other studies, certain authors [8, 9] introduce the variation of the coefficient k as a function of the frequency. However, if one is to retain relatively simple results for arbitrary cross-sections and boundary conditions, the coefficient k obtained by Cowper is the most satisfactory one.

The displacement equation which governs free motion of the beam is

$$EI\,(\partial^4 v/\partial x^4) - mr^2(1 + (E/kG))\,(\partial^4 v/\partial x^2\partial t^2) + m(\partial^2 v/\partial t^2) + mr^2(m/kAG)\partial^4 v/\partial t^4 = 0. \tag{4}$$

Remark: In the static case, equations (1) and (4) lead to the following properties (independent of the boundary conditions): the shear force T is constant along the beam; the static deformation is a third order polynomial in x.

For slender beams (ℓ/r large), shear and rotary inertia effects are neglected. The well-known Euler–Bernoulli equation of motion is then

$$EI\,(\partial^4 v/\partial x^4) + m(\partial^2 v/\partial t^2) = 0. \tag{5}$$

The eigenfrequencies of the beam are expressed by

$$f_n = (\beta_n^2/2\pi\ell^2)\sqrt{EI/\rho A}. \tag{6}$$

2.2. Dynamic analysis

Let us consider an eigenmode having an angular frequency ω. By analogy with the Bernoulli case, natural frequencies of the beam are defined by

$$f_n = (\tau_n^2/2\pi\ell^2)\sqrt{EI/\rho A}. \tag{7}$$

The equation of motion can be transformed into

$$\ell^4\,(\mathrm{d}^4 v/\mathrm{d}x^4) + (\alpha + \eta)\Omega\ell^2\,(\mathrm{d}^2 v/\mathrm{d}x^2) + (\alpha\eta\Omega - 1)\Omega v = 0. \tag{8}$$

The study of the associated characteristic equation in λ;

$$\lambda^4 + (\alpha + \eta)\Omega\lambda^2 + (\alpha\eta\Omega - 1)\Omega = 0, \tag{9}$$

shows the existence of two families of solutions [2] depending on the position of the frequency parameter τ with respect to the critical value τ_c:

$$\tau_c = (\ell/r)\sqrt[4]{k/2(1+v)}. \tag{10}$$

Note that the value τ_c can appear in the analysis frequency band when the ratio ℓ/r decreases (short beams) or when k decreases (profiles, tubes, thin walled volumes).

In general, an application of the boundary conditions to obtain the frequency equation leads to rather fastidious calculations and the resulting equation is solvable analytically only in the simplest cases [2, 10]. For example, in the case of a pinned–pinned boundary for the first and second families, the preceeding equations lead to the solution

$$v(x) = v_0 \sin(n\pi\, x/\ell). \tag{11}$$

Equation (9) can be written as

$$\alpha\eta\Omega^2 - ((\alpha+\eta)n^2\pi^2 + 1)\Omega + n^4\pi^4 = 0, \qquad \text{with } \Omega = \tau^4. \tag{12}$$

Let Ω_{n1} and Ω_{n2} be the roots of this equation ($\Omega_{n1} \leqslant \Omega_{n2}$). It can be proved that $\Omega_{n2} \geqslant \Omega_c$, ($\Omega_c = \tau_c^4$). One can thus conclude that the frequency parameters corresponding to eigenmodes of the first family are given by Ω_{n1} and, in the second family, two types of solutions can be distinguished resulting, respectively, from Ω_{n1} and Ω_{n2}. Now, each value of n leads to two distinct values of Ω_n but to a single modal deformation. As a consequence, while the mode of number n in the first family possesses $n+1$ vibrational nodes, this is no longer the case in the second family. In other words, a new spectrum of eigenfrequencies appears in the second family, which superposes itself on the classical spectrum (see Figure 1). This kind of mode only appears when shear effects are present and they will thus be qualified as "shear modes". Indeed, it is shown that it is typically for these modes that the rotations of the cross-sections dominate over the transverse displacements (see Figure 2).

At the limit between the two families, when the solution exists, the integration of the equations lead to a very particular bending motion, where there is no transverse displacement. This can be called a "pure shear mode". It consists only of an alternative oscillation of the cross-sections about the z direction.

These important properties are taken into account by the Timoshenko finite element model studied thereafter.

3. FINITE ELEMENT FORMULATION

3.1. PRELIMINARIES

Consider a beam having the characteristics described above. This beam is discretized into n identical two-node finite elements of length $L = \ell/n$ with two degrees of freedom per node, a displacement and a rotation angle.

Let $q_n = [V_{i-1}\psi_{i-1}V_i\psi_i]^{\mathrm{T}}$ be the vector of generalized displacements for element i, and let $F_n = [T_{i-1}M_{i-1}T_iM_i]^{\mathrm{T}}$ be the vector of the corresponding generalized forces. The nodal approximation is written as

$$\begin{bmatrix} v(x) \\ \psi(x) \end{bmatrix} = \begin{bmatrix} N_V(x) \\ N_\psi(x) \end{bmatrix} q_n, \tag{13}$$

where $N_V(x)$ and $N_\psi(x)$ are, in $\mathbb{R}^{1,4}$, the polynomial interpolation shape functions. One obtains the following element stiffness and mass matrices:

$$K_T = \int_0^L EI\left(\frac{\mathrm{d}N_\psi}{\mathrm{d}x}\right)^{\mathrm{T}} \frac{\mathrm{d}N_\psi}{\mathrm{d}x}\,\mathrm{d}x + \int_0^L kAG\left(\frac{\mathrm{d}N_v}{\mathrm{d}x} - N_\psi\right)^{\mathrm{T}} \left(\frac{\mathrm{d}N_V}{\mathrm{d}x} - N_\psi\right)\mathrm{d}x, \qquad (14)$$

$$M_T = \int_0^L mN_V^T N_V\,\mathrm{d}x + \int_0^L mr^2 N_\psi^{\mathrm{T}} N_\psi\,\mathrm{d}x, \qquad K_T,\ M_T \in \mathbb{R}^{4,4}. \qquad (15)$$

Since the approximation for the displacement field depends only on the two nodal values, it is natural to choose a linear interpolation (isoparametric element). However, this choice makes the ‘‘shear locking’’ phenomenon appear, which leads to poor results for very thin beams.

In order to solve this problem, it is usual to construct a two-node beam element from a higher order polynomial interpolation. Shear locking is avoided by using an interpolation of order three (corresponding to the order of the exact static displacement field) or higher.

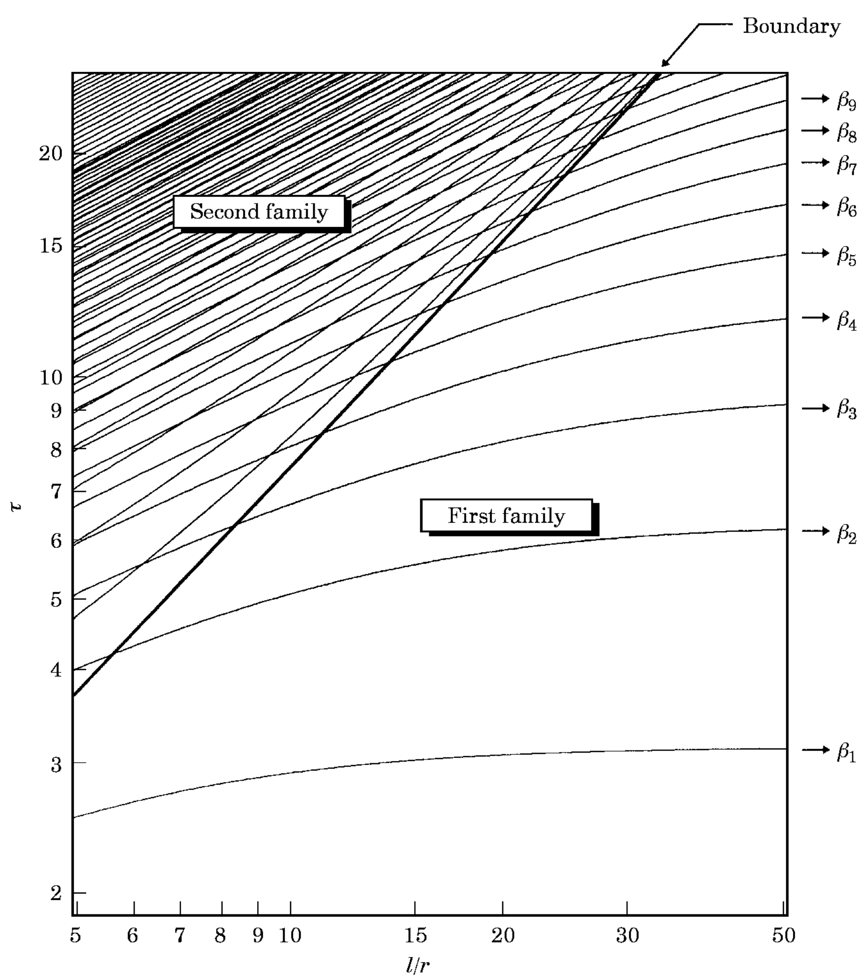

Figure 1. The frequency parameter as a function of the aspect ratio for a pinned–pinned beam of circular cross section.

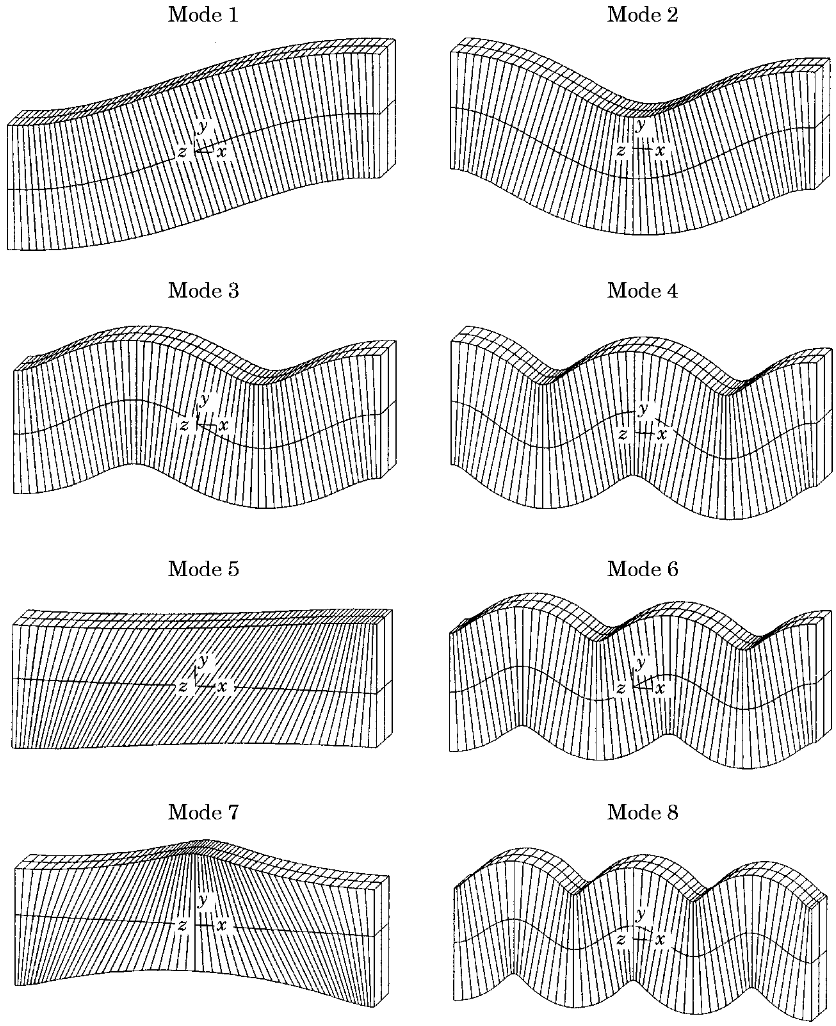

Figure 2. The first eigenmodes of a guided–guided beam with small aspect ratio, modelled with eight-node brick elements.

The method used by Davis [3] to obtain the stiffness matrix consists of interpolating the displacement $v(x)$ and the rotation $\psi(x)$ from the static equilibrium relations. The same interpolations are used for the mass matrix.

3.2. PROPOSED APPROACHES

3.2.1. *Two-node finite elements*

The method proposed here is based on Guyan static condensation. It allows the Timoshenko finite beam elements to be constructed in a simple and systematic manner.

Consider an isoparametric Timoshenko beam element with four equidistant nodes (see Figure 3). Given that the interpolation field is cubic for the independent variables v and ψ, the generalized displacement vector for the element is written as

$$q_n = [V_1 \quad \psi_1 \quad V_2 \quad \psi_2 \quad V_3 \quad \psi_3 \quad V_4 \quad \psi_4]^{\mathrm{T}}.$$

Let K and M be the elementary stiffness and mass matrices of the element ($K,\ M \in \mathbb{R}^{8,8}$). The shape functions N_V and N_ψ can be calculated immediately, and lead to K and M defined by the relations (14) and (15).

In order to construct the two-node finite beam elements, the d.o.f. are partitioned into two subsets: the master d.o.f. corresponding to junction nodes of the element (nodes 1 and 4) and the slave dof corresponding to the two internal nodes (nodes 2 and 3).

The vector of nodal unknowns, as well as the stiffness and mass matrices, can thus be partitioned in the following way:

$$q = \begin{bmatrix} q_m \\ q_s \end{bmatrix}, \qquad \text{with} \begin{Bmatrix} q_m = [V_1 \quad \psi_1 \quad V_4 \quad \psi_4]^{\mathrm{T}} \\ q_s = [V_2 \quad \psi_2 \quad V_3 \quad \psi_3]^{\mathrm{T}} \end{Bmatrix};$$

$$K = \begin{bmatrix} K_{mm} & K_{ms} \\ K_{ms}^T & K_{ss} \end{bmatrix}; \qquad M = \begin{bmatrix} M_{mm} & M_{ms} \\ M_{ms}^T & M_{ss} \end{bmatrix}; \qquad K,\ M \in \mathbb{R}^{8,8}.$$

The dynamic equilibrium of the element can be written as

$$(K - \omega^2 M)q = F, \tag{16}$$

with

$$\begin{bmatrix} F_m \\ 0 \end{bmatrix}$$

being the vector of junction forces between elements.

The use of Guyan [4] static condensation defines the following transformation at the element level:

$$q = \begin{bmatrix} I_m \\ -K_{ss}^{-1} K_{ms}^{\mathrm{T}} \end{bmatrix} q_m \triangleq T_G q_m, \tag{17}$$

I_m is the identity of matrix of order m. where Equation (16) can be expressed in condensed form as

$$(K_c - \omega^2 M_c) q_m = F_m, \tag{18}$$

with

$$K_c = T_G^{\mathrm{T}} K T_G, \qquad M_c = T_G^{\mathrm{T}} M T_G. \tag{19}$$

K_c and M_c are the elementary matrices condensed on the junction d.o.f. of the finite element. The compatibility relations between finite elements then allow elements to be assembled to obtain the model for the global beam.

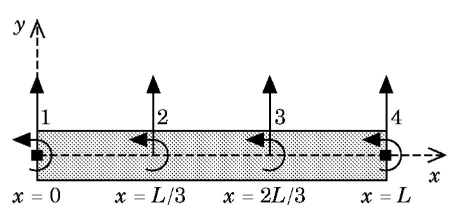

Figure 3. An isoparametric beam element with four equidistant nodes. ■, two master nodes.

Symbolic calculations allows the condensed elementary stiffness matrix to be obtained:

$$K_T = K_c = \frac{EI}{L^3(1+\phi)}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & L^2(4+\phi) & -6L & L^2(2-\phi) \\ -12 & -6L & 12 & -6L \\ 6L & L^2(2-\phi) & -6L & L^2(4+\phi) \end{bmatrix} \text{ with } \phi = \frac{12EI}{kAGL^2}. \quad (20)$$

Likewise, the condensed elementary mass matrix is

$$M_T = M_c = \frac{\rho AL}{(1+\phi)^2}\begin{bmatrix} m_1 & m_2 & m_3 & -m_4 \\ m_2 & m_5 & m_4 & -m_6 \\ m_3 & m_4 & m_1 & -m_2 \\ -m_4 & -m_6 & -m_2 & m_5 \end{bmatrix}, \quad (21)$$

$$\text{with} \left\{ \begin{aligned} m_1 &= \frac{13}{35} + \frac{7\phi}{10} + \frac{\phi^2}{3} + \frac{6}{5}\frac{r^2}{L^2} \\ m_2 &= \left(\frac{11}{210} + \frac{11\phi}{120} + \frac{\phi^2}{24} + \left(\frac{1}{10} - \frac{\phi}{2}\right)\frac{r^2}{L^2}\right)L \\ m_3 &= \frac{9}{70} + \frac{3\phi}{10} + \frac{\phi^2}{6} - \frac{6}{5}\frac{r^2}{L^2} \\ m_4 &= \left(\frac{13}{420} + \frac{3\phi}{40} + \frac{\phi^2}{24} - \left(\frac{1}{10} - \frac{\phi}{2}\right)\frac{r^2}{L^2}\right)L \\ m_5 &= \left(\frac{1}{105} + \frac{\phi}{60} + \frac{\phi^2}{120} + \left(\frac{2}{15} + \frac{\phi}{6} + \frac{\phi^2}{3}\right)\frac{r^2}{L^2}\right)L^2 \\ m_6 &= \left(\frac{1}{140} + \frac{\phi}{60} + \frac{\phi^2}{120} + \left(\frac{1}{30} + \frac{\phi}{6} - \frac{\phi^2}{6}\right)\frac{r^2}{L^2}\right)L^2 \end{aligned} \right\}$$

Note that when the aspect ratio of the element becomes large ($\phi \to 0$), K_T and M_T tend, respectively, towards matrices K_B and M_B of the two-node finite beam element derived from the Euler–Bernoulli formulation with rotary inertia effects.

Finally, the matrices K_T and M_T simply obtained by the proposed method are identical to those calculated by Davis [3] and other authors [11–13]. This is evident for the stiffness matrix K_T, since Guyan condensation is exact in the static case. However, for the mass matrix M_T, the Davis hypothesis, which consists of using a statically derived interpolation in the dynamic case, simply amounts to neglecting internal inertial forces in the condensation procedure.

The mass matrix M_T of the two-node Timoshenko finite element that is thus obtained is rarely presented in the literature. Many authors replace it by the Bernoulli matrix M_B, thereby creating a ‘‘mixed’’ finite element formulation in which the shear term ϕ intervenes in the stiffness matrix but not in the mass. This finite element is still commonly employed in certain structural calculation codes.

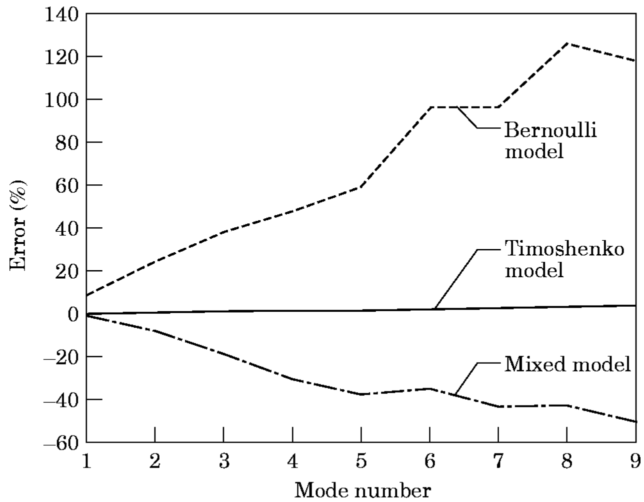

Figure 4. The frequency errors (in %) between the discrete and continuous models for three types of beam finite elements. Number of elements = 100.

3.2.2. *Numerical results*

To compare the behaviour of these finite elements, a guided–guided beam with a solid circular cross-section and an aspect ratio of $\ell/r = 12$ is considered. It is modelled separately by three types of beam elements: Timoshenko, Bernoulli and Mixed (respectively denoted Tbe, Bbe and Mbe). The accuracy of each model is shown in Figure 4. The Bernoulli model tends to overestimate eigenfrequencies. The inverse is true for the mixed approach. Indeed, the latter leads to error on the order of 30% and greater from the fourth mode, while the Timoshenko model yields very small errors (of the order of 0·04%).

It is also important to note that the Timoshenko beam element is the only one that obtains the "shear modes" (from the sixth mode) studied previously.

3.2.3 *Experimental test*

In order to validate the use of the two-node Timoshenko element, an experimental test is considered. The beam has a solid circular cross-section, with an aspect ratio of $\ell/r \approx 11{\cdot}47$. The boundary conditions are free–free. Four discretizations have been applied to this case: three beam element models (Tbe, Bbe and Mbe) $n = 100$ (202 d.o.f.); an eight-node solid element with three d.o.f. per node (denoted Sel); $n = 600$ (2520 d.o.f.).

TABLE 1

A comparison between the results obtained with different models $[e = (f_{cal} - f_{mes})/f_{mes}]$

		Finite elements models							
		Sel model		Tbe model		Bbe model		Mbe model	
Mode no.	Measure f(Hz)	f(Hz)	e(%)	f(Hz)	e(%)	f(Hz)	e(%)	f(Hz)	e(%)
1	4957	4867	− 1·81	4935	− 0·44	5296	6·83	4728	− 4·61
2	10 542	10 543	0·01	10 468	− 0·70	12 657	20·0	9020	− 14·4
3	16 476	16 722	1·49	16 382	− 0·57	21 657	31·4	12 251	− 25·6
4	20 514	21 315	3·90	20 690	0·85	31 494	53·5	14 323	− 30·1
5	24 439	25 350	3.72	25 019	2·37	41 782	70·9	15 694	− 35·8
6	24 679	25 723	4·23	25 186	2·05	52 270	112	16 600	− 32·7

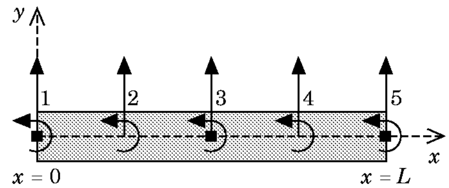

Figure 5. An isoparametric beam element with five equidistant nodes. ■, three master nodes.

The eigenfrequencies obtained for the first six bending modes, as well as the relative errors with respect to the measured, are reported in Table 1. This shows that the Timoshenko model gives results which are in complete agreement with experiments. Moreover, it performs as well as the model in which solid elements are used for a much reduced number of dof (12 times less in this case). However, the use of Bernoulli and Mixed elements leads to unacceptable errors despite a relatively fine mesh.

3.2.4. *Higher order finite element*

In order to enrich the model, it is interesting to use the proposed method to construct finite beam elements based on higher order interpolations.

Whatever the order of the interpolation, the resulting two-node element is necessarily the same. Indeed Guyan condensation comes down to performing an interpolation based on a static problem, and since the static deformation of a beam is a third order polynomial, all interpolations of order three or higher lead to the same condensed matrices.

In order to obtain an effectively higher order element, we propose a three-node beam finite element defined by two junction nodes and an internal node. Consider, for example the five-node isoparametric element in Figure 5, the master nodes of which are now nodes 1, 3 and 5. The proposed method allows the corresponding stiffness and mass matrices ($\in \mathbb{R}^{6,6}$) to be obtained.

3.2.5. *Numerical results*

The dynamic performance of this element is illustrated by the following test case. The structure is defined by a straight beam which is clamped at both ends and pinned at mid-length. Two discretized models of the same size are generated based, respectively, on two- and three-node Timoshenko beam element (denoted 2N and 3N). The eigenfrequencies of these models were evaluated and the results reported in Table 2. The

TABLE 2

A comparison of the results obtained by using the 2N *and* 3N *elements* [$e = (f_{fe} - f_{co})/f_{co}$]

Mode no.	Exact f(Hz)	2N model (29 d.o.f.) f(Hz)	e(%)	3N model (29 d.o.f.) f(Hz)	e(%)
1	3096·25	3097·22	0·03	3096·70	0·01
2	4408·22	4410·94	0·06	4409·47	0·03
3	9598·68	9626·23	0·30	9612·11	0·14
4	11 501·0	11 548·4	0·40	11 524·6	0·20
5	18 901·8	19 107·2	1·10	19 015·1	0·60
6	21 147·6	21 435·7	1·40	21 327·4	0·85
7	30 269·9	39 091·5	2·70	30 528·2	0·85
8	32 610·6	33 638·0	3·20	32 914·5	0·93

TABLE 3

Static results derived from interpolation of different orders for a single finite element

Order of the interpolation	Deflection, v_M	Deflection, v_M, for $\phi = 0$
4	$\frac{5}{3072}(3+11\phi)\frac{F\ell^3}{EI}$	$\frac{1}{204{\cdot}8}\frac{F\ell^3}{EI}$
6	$\frac{7}{4096}(3+11\phi)\frac{F\ell^3}{EI}$	$\frac{1}{195{\cdot}04}\frac{F\ell^3}{EI}$
8	$\frac{1}{262\,144}(1357+5053\phi)\frac{F\ell^3}{EI}$	$\frac{1}{193{\cdot}18}\frac{F\ell^3}{EI}$
10	$\frac{11}{3\,145\,728}(1485+5597\phi)\frac{F\ell^3}{EI}$	$\frac{1}{192{\cdot}57}\frac{F\ell^3}{EI}$
Continuous Timoshenko formulation	$\frac{1}{192}(1+4\phi)\frac{F\ell^3}{EI}$	$\frac{1}{192}\frac{F\ell^3}{EI}$

performance of the 3N elements is significantly better than that obtained with the 2N elements. The frequency error between the discrete model and the continuous one is less than 1% for the first eight modes with the 3N elements, while it is more than 3% for the eighth mode with the 2N elements.

Now, a question which remains is whether or not the interpolation order can still be increased. Indeed, the interpolation of the static deformation based on a three-node element is a third order polynomial which is continuous, but defined on two parts. In contrast to the two-node element, this deformation cannot be exactly represented by a single polynomial, regardless of its order. Consequently, the method for enriching the interpolation can be generalized *a priori*.

This property is illustrated by the static problem of a clamped–clamped beam of length ℓ, modelled by a single three-node finite element derived from a successively enriched interpolation, and subject to a transverse force F situated at the mid-length on the internal node. The corresponding maximal deflections are given in Table 3. The precision of the static results increases with the interpolation order.

However, the performance of the static condensation diminishes in the dynamic case. Indeed, the domain of validity of Guyan condensation is always defined between 0 and the cut-off frequency f_c [14] corresponding to the smallest eigenfrequency of the problem with the master d.o.f. grounded defined by $(k_{ss} - \omega^2 M_{ss})q_s = 0$. Thus, when the number of slave d.o.f. increases with the chosen order of interpolation, the frequency f_c decreases, resulting in a global degradation of the results. In general, the three-node element derived from a fourth order interpolation provides the best dynamic performance.

4. CONCLUSIONS

In this paper, the continuous dynamic model of the Timoshenko beam has been reviewed. In particular, the relation between the behavioural characteristics and the shear effect in short beams, as well as their specific eigenmodes, have been emphasized.

A new method based on Guyan condensation has been presented, which allows the Timoshenko beam element to be obtained. These isoparametric elements take into account rotary inertia and transverse shear, yielding results which are in agreement with the continuous model, especially in the case of short beams.

The technique proposed for constructing finite elements has the advantage of being simple and systematic. Moreover, it has been shown, for the two-node element, that all choices of polynomial interpolations of order three or higher lead necessarily to the same stiffness and mass element matrices. For the three-node element, the generalization of this method to higher order interpolations allows elements which perform better to be obtained (provided that Guyan condensation is still valid).

REFERENCES

1. S. P. Timoshenko 1922 *Philosophical Magazine* **43**, 125–131. On the transverse vibrations of bars of uniform cross sections.
2. D. Gay 1979 *Ph.D. Thesis*; *University of Toulouse*, *France*. Influence des effets secondaires sur les vibrations de flexion et de torsion des poutres.
3. R. Davis, R. D. Henshell and G. B. Warburton 1972 *Journal of Sound and Vibration* **22**, 475–487. A Timoshenko beam element.
4. R. J. Guyan 1965 *American Institute of Aeronautics and Astronautics Journal* **3**, 380. Reduction of stiffness and mass matrices.
5. S. P. Timoshenko 1921 *Philosophical Magazine* **41**, 744–746. On the correction for shear of the differential equation for transverse vibrations of prismatic bars.
6. G. R. Cowper 1966 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **33**, 335–340. The shear coefficient in Timoshenko's beam theory.
7. A. E. H. Love 1952 *A Treatise on the Mathematical Theory of Elasticity*. Cambridge: Cambridge University Press.
8. N .G. Stephen 1978 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **45**, 695–697. On the variation of Timoshenko's shear coefficient with frequency.
9. J. R. Hutchinson 1981 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **48**, 923–928. Transverse vibrations of beams, exact versus approximate solutions.
10. T. C. Huang 1961 *Transactions of the American Society of Mechanical Engineers*, *Journal of Applied Mechanics* **28**, 579–584. The effect or rotary inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions.
11. J. S. Prezemieneicki 1968 *Theory of Matrix Structural Analysis*. New York: McGraw-Hill.
12. D. L. Thomas, J. M. Wilson and R. R. Wilson 1973 *Journal of Sound and Vibration* **31**, 315–330. Timoshenko beam finite elements.
13. Z. Friedman and J. B. Kosmatka 1993 *Computers and Structures* **47**, 473–481. An improved two-node Timoshenko beam finite element.
14. N. Bouhaddi and R. Fillod 1992 *Computers and Structures* **45**, 941–946. A method for selecting master degrees of freedom in dynamic substructuring using the Guyan condensation menthod.

APPENDIX: MAIN SYMBOLS

A Cross-section area
I moment of inertia
r $=\sqrt{I/A}$, radius of gyration
ℓ length of beam
m $=\rho A$, mass per unit length
k shear coefficient
T shear force
M $=EI\,\partial\psi/\partial x$, bending moment
E Young's modulus
G shear modulus
v Poisson ratio
x distance along length of beam
v transverse displacement
Ψ rotation angle
β_n frequency parameter of the n^{th} mode (Bernoulli theory)
τ_n frequency parameter of the n^{th} mode (Timoshenko theory)
Ω $=\omega^2 m\ell^4/EI$, coefficient relative to angular frequency
α $=r^2/\ell^2$, coefficient relative to the rotary inertia
η $=(E/kG)(r^2/\ell^2)$, coefficient relative to shear