Journal of Sound and Vibration (1997) **201**(3), 385–387

CALCULATION OF THE NATURAL FREQUENCIES OF COMPOSITE PLATES BY THE RAYLEIGH–RITZ METHOD WITH ORTHOGONAL POLYNOMIALS

P. Cupiał

Cracow University of Technology, Al. Jana Pawła II 37, 31-864 Kraków, Poland

(*Received* 4 *June* 1996)

A summary is presented of an efficient approach to the analysis of the natural frequencies of composite rectangular plates based on the Rayleigh–Ritz method. Orthogonal polynomials are used as admissible functions and the approach has been found to be numerically stable and capable of treating in a uniform manner a wide class of boundary conditions for which the plate sides are either simply supported, free or clamped.

A composite plate with sides of length a and b is considered. The origin of the co-ordinate system is taken at the centre of the plate. By restricting analysis to symmetrically laminated plates and making use of the classical plate theory of composite plates [1], which is valid for sufficiently thin plates, the amplitudes of the potential and kinetic energies can be given by the following formulae (see also reference [2]):

$$V_{max} = \tfrac{1}{2}\frac{D_0 h^2}{r(a/2)^2}\int_{-1}^{1}\int_{-1}^{1}(\bar{D}_{11}\bar{W}^2_{,\xi\xi} + 2\bar{D}_{12}r^2\bar{W}_{,\xi\xi}\bar{W}_{,\eta\eta} + \bar{D}_{22}r^4\bar{W}^2_{,\eta\eta} + 4\bar{D}_{16}r\bar{W}_{,\xi\xi}\bar{W}_{,\xi\eta}$$

$$+ 4\bar{D}_{26}r^3\bar{W}_{,\eta\eta}\bar{W}_{,\xi\eta} + 4\bar{D}_{66}r^2\bar{W}^2_{,\xi\eta})\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

$$T_{max} = \tfrac{1}{2}\frac{D_0 h^2}{r(a/2)^2}\,\Omega^2\int_{-1}^{1}\int_{-1}^{1}\bar{W}^2\,\mathrm{d}\xi\,\mathrm{d}\eta. \qquad (1)$$

Here the following non-dimensional quantities are used:

$$\xi = x/(a/2), \qquad \eta = y/(b/2), \qquad r = a/b, \qquad \bar{W} = W/h, \qquad \bar{D}_{ij} = D_{ij}/D_0,$$

$$\Omega = \omega(a/2)^2(\rho/D_0)^{1/2}.$$

W is the amplitude of transverse vibration, h is the plate thickness, D_{ij} are the bending stiffnesses of a laminated plate (see reference [1]), ρ is the mass density per unit area of the plate, and D_0 is the reference stiffness, which can be taken arbitrarily.

According to the Ritz method the plate displacement is approximated by a series,

$$\bar{W}(\xi,\eta) = \sum_{m=1}^{M}\sum_{n=1}^{N}\gamma_{mn}\phi_m(\xi)\psi_n(\eta), \qquad (2)$$

where the admissible functions ϕ_m and ψ_n satisfy at least the geometric boundary conditions. The orthogonal polynomials in the interval [-1, 1] are generated from the following recursive formulae previously used in references [3] and [4]:

$$\phi_2(\xi) = (\xi - B_2)\phi_1(\xi), \qquad \phi_k(\xi) = (\xi - B_k)\phi_{k-1}(\xi) - C_k\phi_{k-2}(\xi) \qquad \text{for } k > 2. \qquad (3)$$

0022–460X/97/130385 + 03 $25.00/0/sv960802

Here

$$B_k = \int_{-1}^{1} \xi \phi_{k-1}^2(\xi)\,\mathrm{d}\xi \Big/ \int_{-1}^{1} \phi_{k-1}^2(\xi)\,\mathrm{d}\xi,$$

$$C_k = \int_{-1}^{1} \xi \phi_{k-1}(\xi)\phi_{k-2}(\xi)\,\mathrm{d}\xi \Big/ \int_{-1}^{1} \phi_{k-2}^2(\xi)\,\mathrm{d}\xi.$$

ϕ_1 is a properly chosen starting function, satisfying at least the geometric boundary conditions at $\xi = -1$ and $\xi = 1$. The orthogonal polynomials in the η direction are generated in a similar way.

The following starting functions have been used in the present analysis, which satisfy the geometric boundary conditions at $\xi = -1$ and $\xi = 1$:

$$\text{S–S} \quad \phi_1(\xi) = \text{const}\,(\xi^2 - 1), \qquad \text{S–F} \quad \phi_1(\xi) = \text{const}\,(\xi + 1),$$

$$\text{S–C} \quad \phi_1(\xi) = \text{const}\,(\xi + 1)(\xi - 1)^2, \qquad F\text{–}F \quad \phi_1(\xi) = \text{const},$$

$$\text{C–F} \quad \phi_1(\xi) = \text{const}\,(\xi + 1)^2, \qquad \text{C–C} \quad \phi_1(\xi) = \text{const}\,(\xi^2 - 1)^2, \tag{4}$$

where S denotes a simply supported edge, F a free edge and C a clamped one. From the formulae (3) it follows that if the starting function (4) satisfies one of the geometric boundary conditions (4), so do all generated polynomials. Also, in the numerical analysis all polynomials were normalized and orthonormal polynomials were used.

To show the accuracy of the present method, in Table 1 a comparison is made with the results obtained in reference [2]. The approach in reference [2] used ordinary polynomials and allowed the calculation of natural frequencies of a plate for which two adjacent sides were free. Even though the present approach is more general, only the results given in reference [2] are compared. Both analyses were done for the composite material properties: $E_1 = 138$ [GPa], $E_2 = 8{\cdot}96$ [GPa], $G_{12} = 7{\cdot}1$ [GPa], and $\nu_{12} = 0{\cdot}30$, and 8×8 polynomials were used.

TABLE 1

The non-dimensional frequency $\omega a^2(\rho/E_1h^3)^{1/2}$ *of an angle-ply* (30°, -30°, 30°) *square plate*

	Mode					
Boundary conditions	1	2	3	4	5	6
FFFF	1·620 (1·620)*	2·078 (2·078)	3·711 (3·712)	5·051 (5·052)	5·069 (5·071)	7·079 (7·080)
SFFF	0·916 (0·917)	2·536 (2·536)	3·275 (3·275)	4·517 (4·518)	5·692 (5·693)	7·505 (7·507)
CFFF	0·651 (0·651)	1·437 (1·437)	3·122 (3·123)	4·183 (4·184)	5·610 (5·611)	6·525 (6·526)
SSFF	0·465 (0·464)	1·842 (1·842)	3·926 (3·926)	4·654 (4·654)	6·751 (6·751)	8·077 (8·078)
CSFF	1·061 (1·060)	2·421 (2·421)	4·979 (4·979)	5·637 (5·634)	7·887 (7·877)	9·084 (9·077)
CCFF	1·291 (1·291)	3·051 (3·050)	5·552 (5·546)	6·270 (6·269)	8·564 (8·551)	10·232 (10·236)

*Reference [2] values are shown in parentheses.

TABLE 2

The non-dimensional frequency $\omega a^2[12(1-\nu_{12}\nu_{21})\rho/E_1h^3]^{1/2}$ *of a single-layer* (30°) *simply supported square plate*

Number of polynomials (*M*,*N*)	Mode 1	2	3	4	5	6
(6,6)	11·299	20·340	33·310	35·023	48·158	50·096
(8,8)	11·264	20·307	33·000	34·920	47·247	48·353
	(11·69)*	(21·00)	(33·91)	(35·62)	(48·72)	(49·78)
(10,10)	11·245	20·292	32·995	34·886	47·165	48·285
	(11·62)	(20·87)	(33·73)	(35·49)	(48·43)	(49·45)
(12,12)	11·233	20·282	32·995	34·866	47·133	48·283
	(11·56)	(20·78)	(33·61)	(35·40)	(48·24)	(49·25)
FEM	11·239	20·264	32·941	34·875	47·169	48·236

*Reference [5] values are shown in parentheses.

In Table 2 a comparison is made with the results given in reference [5] for a highly anisotropic simply supported plate. The material properties used are the same as above. In reference [5] a double-sine series was used with the Rayleigh–Ritz method. From the table it is observed that the present approach has a better convergence than the results obtained with harmonic functions. Also, the finite element results included in the table agree very well with the present calculations. It should also be emphasized that the results of the two approaches differ only in the case of a plate with high anisotropy. In the case of specially orthotropic simply supported plates the results given in reference [5] have been found practically to coincide with those of the present analysis. These conclusions agree with an earlier observation made in reference [6] that the convergence of the Rayleigh–Ritz method using beam functions may be slow for the free vibration frequencies of highly anisotropic plates with simply supported or free edges.

It is concluded that the Rayleigh–Ritz method with orthogonal polynomials is an efficient discretization method applicable to anisotropic problems. For all boundary conditions analyzed 8×8 polynomials showed good convergence for the first six frequencies. Due to the use of orthogonal functions, the mass and stiffness matrices are well-conditioned and the approach is very stable.

REFERENCES

1. R. M. JONES 1975 *Mechanics of Composite Materials*. Washington D.C: Scripta.
2. M. S. QATU 1991 *International Journal of Solids and Structures* **28**, 941–954. Free vibration of laminated composite rectangular plates.
3. R. B. BHAT 1985 *Journal of Sound and Vibration* **102**, 493–499. Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh–Ritz method.
4. S. M. DICKINSON and A. DI-BLASIO 1986 *Journal of Sound and Vibration* **108**, 51–62. On the use of orthogonal polynomials in the Rayleigh–Ritz method for the study of the flexural vibration and buckling of isotropic and orthotropic rectangular plates.
5. A. W. LEISSA, and Y. NARITA 1989 *Composite Structures* **12**, 113–132. Vibration studies for simply supported symmetrically laminated rectangular plates.
6. J. M. WHITNEY 1972 *Journal of Acoustical Society of America* **52**, 448–449. Free vibration of anisotropic rectangular plates.