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HIERARCHICAL FINITE ELEMENT ANALYSIS OF THE VIBRATION OF MEMBRANES

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A hierarchical finite element for the vibration of membranes is presented. The element has the ability to be joined to neighbouring elements and the numbers of hierarchical terms are allowed to vary in both directions of the element co-ordinate axes. The element transverse displacement is described by four linear shape functions plus a variable number of hierarchical functions which are forms of Legendre orthogonal polynomials. The four nodal displacements and the amplitudes of the hierarchical functions on the edges and in the interior of the element are used as generalized co-ordinates. Inter-element compatibility is achieved by matching the generalized co-ordinates at the nodes and edges shared by elements. Results are obtained for simply supported square and L-shaped membranes. Comparisons are made with exact solutions for the square membrane and with highly accurate approximate and linear finite element solutions for the L-shaped membrane. The results for the square membrane confirm that the solutions always converge from above to the exact values as the numbers of hierarchical terms are increased and highly accurate answers are obtained despite the use of a very few hierarchical terms. The results of the L-shaped membrane show that the hierarchical finite element solutions are largely more accurate than the linear finite element solutions despite the use of fewer system degrees of freedom.

1. INTRODUCTION

This paper deals with the Hierarchical Finite Element Method (HFEM) applied to membrane vibrations. There are a number of solutions [1–3] that are suggestive of this method. The HFEM has a few major features that make its use desirable for certain structural problems. The most important feature is that the gridwork of elements in a structure is kept unchanged and the number of hierarchical terms in each element is varied. The results can then be obtained to any desired degree of accuracy by simply increasing the number of hierarchical terms. The other important feature is a consequence of the inclusion principle [3], which guarantees that the solutions always converge from above to the exact values and are therefore upper bounds. Furthermore, a membrane is modelled as just one finite element and therefore the satisfaction of C_0 continuity at internal nodes is avoided.

The membrane hierarchical finite element presented in this paper has the ability to be joined to neighboring elements. Furthermore, the numbers of hierarchical terms are allowed to vary in both directions of the element co-ordinate axes. The first feature makes the element particularly useful for membranes with complex geometry, such as an L-shaped membrane, and the second feature makes the element particularly useful for elongated membranes discretized into one element in which more hierarchical terms are needed along the length than along the width in order to describe the membrane mode shapes accurately. The element transverse displacement is described by four linear shape

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functions plus a variable number of hierarchical functions which are basically forms of orthogonal Legendre polynomials. The linear shape functions are used to define the element four nodal displacements and the hierarchical functions are used to provide additional freedom to the edges and the interior of the element. The nodal displacements and the amplitudes of the hierarchical functions on the edges and in the interior of the element are used as generalized co-ordinates. Inter-element compatibilitiy is achieved by matching the generalized co-ordinates at the element's four nodes and four edges.

Results of frequency calculations by using the new hierarchical finite element are given for simply supported square and L-shaped membranes. These particular examples were chosen because known exact and highly accurate approximate solutions were available in the literature for comparisons. Comparisons were also made with linear finite element solutions for the L-shaped membrane.

2. FORMULATION

A rectangular membrane element is shown in Figure 1. Also shown in the figure are the dimensionless co-ordinates defined as follows (a list of notation is given in the Appendix):

$$\xi = x/a, \qquad \eta = y/b. \tag{1, 2}$$

The potential energy V and the kinetic energy T of the rectangular membrane element have the forms

$$V = \tfrac{1}{2}Sab \int_{-1}^{1}\int_{-1}^{1}\left[\left(\frac{1}{a}\frac{\partial w}{\partial \xi}\right)^2 + \left(\frac{1}{b}\frac{\partial w}{\partial \eta}\right)^2\right] \mathrm{d}\xi\, \mathrm{d}\eta, \qquad T = \frac{\omega^2}{2}\rho ab \int_{-1}^{1}\int_{-1}^{1} w^2\, \mathrm{d}\xi\, \mathrm{d}\eta. \tag{3, 4}$$

The displacement functions assumed for this element are written as

$$\begin{aligned} w = {} & w_1 f_1(\xi) f_1(\eta) + w_2 f_2(\xi) f_1(\eta) + w_3 f_2(\xi) f_2(\eta) + w_4 f_1(\xi) f_2(\eta) + w_{m1} f_m(\xi) f_1(\eta) \\ & + w_{m2} f_m(\xi) f_2(\eta) + w_{n1} f_1(\xi) f_n(\eta) + w_{n2} f_2(\xi) f_n(\eta) + w_{mn} f_m(\xi) f_n(\eta), \end{aligned} \tag{5}$$

where summation is implied on the indices m and n. The functions f_1 and f_2 are the following known linear shape functions:

$$f_1(\xi \text{ or } \eta) = \tfrac{1}{2}(1 - (\xi \text{ or } \eta)) \tag{6}$$

$$f_2(\xi \text{ or } \eta) = \tfrac{1}{2}(1 + (\xi \text{ or } \eta)) \tag{7}$$

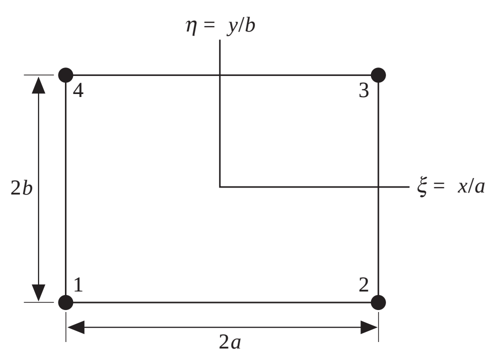
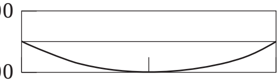
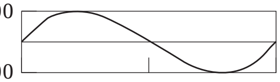
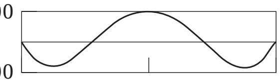
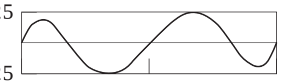
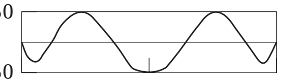
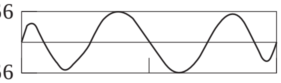
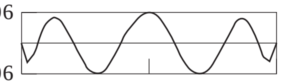
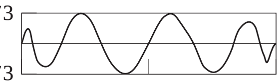


Figure 1. The element co-ordinates and dimensions

TABLE 1

The first eight hierarchical functions f_s ($\zeta = \xi$ or η) ($s = 3, 4, \ldots, 10$)

$f_3(\zeta) = \frac{1}{2}\zeta^2 - \frac{1}{2}$	
$f_4(\zeta) = \frac{1}{2}\zeta^3 - \frac{1}{2}\zeta$	
$f_5(\zeta) = \frac{5}{8}\zeta^4 - \frac{3}{4}\zeta^2 + \frac{1}{8}$	
$f_6(\zeta) = \frac{7}{8}\zeta^5 - \frac{5}{4}\zeta^3 + \frac{3}{8}\zeta$	
$f_7(\zeta) = \frac{63}{48}\zeta^6 - \frac{35}{16}\zeta^4 + \frac{15}{16}\zeta^2 - \frac{1}{16}$	
$f_8(\zeta) = \frac{99}{48}\zeta^7 - \frac{63}{16}\zeta^5 + \frac{35}{16}\zeta^3 - \frac{5}{16}\zeta$	
$f_9(\zeta) = \frac{429}{128}\zeta^8 - \frac{231}{32}\zeta^6 + \frac{315}{64}\zeta^4 - \frac{35}{32}\zeta^2 + \frac{5}{128}$	
$f_{10}(\zeta) = \frac{715}{128}\zeta^9 - \frac{429}{32}\zeta^7 + \frac{693}{64}\zeta^5 - \frac{105}{32}\zeta^3 + \frac{35}{128}\zeta$	

The functions f_m and f_n are hierarchical functions derived from Rodrigues form of the Legendre orthogonal polynomials and are given by

$$f_m(\xi) = \sum_{k=0}^{m/2} \frac{(-1)^k(2m-2k-5)!!}{2^k k!(m-2k-1)!}\,\xi^{m-2k-1}, \qquad m = 3, 4, \ldots, p+2, \tag{8}$$

$$f_n(\eta) = \sum_{k=0}^{n/2} \frac{(-1)^k(2n-2k-5)!!}{2^k k!(n-2k-1)!}\,\eta^{n-2k-1}, \qquad n = 3, 4, \ldots, q+2, \tag{9}$$

where $j!! = j(j-2)\ldots 2$ or 1 with $0!! = (-1)!! = 1$, $m/2$ and $n/2$ denote their own integer parts, and p and q are the numbers of hierarchical terms used in the ξ and η directions, respectively. The hierarchical functions f_m and f_n have zero values for arguments equal to ± 1. The first eight hierarchical functions are given in Table 1.

The assumed displacement functions are divided into three groups. The first group consists of the shape functions used to define the element four nodal displacements. The second group consists of hierarchical functions which give additional freedom to the element four edges. The third group consists of hierarchical functions which give additional freedom to the interior of the element.

The co-ordinates w_1, w_2, w_3, and w_4 are the element's four nodal displacements. The co-ordinates w_{m1}, w_{m2}, w_{n1}, and w_{n2} are the amplitudes of the hierarchical functions on the element's four edges. The co-ordinates w_{mn} are the amplitudes of the hierarchical functions in the interior of the element. The element's generalized co-ordinates are shown in Figure 2.

The quantities needed to form the element stiffness matrix and mass matrix can be written in matrix form as follows

$$w = \mathbf{Cu}, \qquad \frac{1}{a}\frac{\partial w}{\partial \xi} = \mathbf{Fu}, \qquad \frac{1}{b}\frac{\partial w}{\partial \eta} = \mathbf{Hu}, \tag{10–12}$$

where

$$\begin{aligned}\mathbf{C} = [&f_1(\xi)f_1(\eta), \quad f_2(\xi)f_1(\eta), \quad f_2(\xi)f_2(\eta), \quad f_1(\xi)f_2(\eta),\\ &f_m(\xi)f_1(\eta), \quad f_m(\xi)f_2(\eta), \quad f_1(\xi)f_n(\eta), \quad f_2(\xi)f_n(\eta),\\ &f_m(\xi)f_n(\eta)];\end{aligned} \tag{13}$$

$$\begin{aligned}\mathbf{F} = (1/a)[&f_1'(\xi)f_1(\eta), \quad f_2'(\xi)f_1(\eta), \quad f_2'(\xi)f_2(\eta), \quad f_1'(\xi)f_2(\eta),\\ &f_{\mathrm{m}}'(\xi)f_1(\eta), \quad f_{\mathrm{m}}'(\xi)f_2(\eta), \quad f_1'(\xi)f_n(\eta), \quad f_2'(\xi)f_n(\eta),\\ &f_{\mathrm{m}}'(\xi)f_n(\eta)];\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{H} = (1/b)[&f_1(\xi)f_1'(\eta), \quad f_2(\xi)f_1'(\eta), \quad f_2(\xi)f_2'(\eta), \quad f_1(\xi)f_2'(\eta),\\ &f_m(\xi)f_1'(\eta), \quad f_m(\xi)f_2'(\eta), \quad f_1(\xi)f_{\mathrm{n}}'(\eta), \quad f_2(\xi)f_{\mathrm{n}}'(\eta),\\ &f_m(\xi)f_{\mathrm{n}}'(\eta)].\end{aligned} \tag{15}$$

Here primes denote differentiation with respect to the argument. The vector of generalized co-ordinates $\mathbf{u}$ is as follows

$$\mathbf{u} = \{w_1, w_2, w_3, w_4, w_{m1}, w_{m2}, w_{n1}, w_{n2}, w_{mn}\}^{\mathrm{T}}. \tag{16}$$

The indices are defined as follows

$$m1 = 2 + m, \qquad m2 = 2 + p + m, \qquad n1 = 2 + 2p + n, \tag{17–19}$$

$$n2 = 2 + 2p + q + n, \qquad mn = 2 + 2(p + q) + (m - 3)q + n. \tag{20, 21}$$

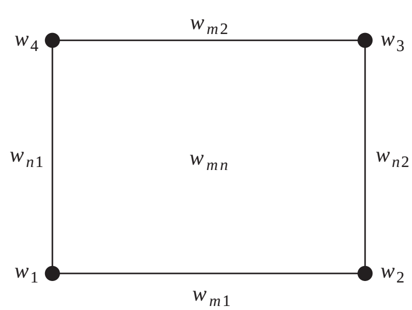


Figure 2. The element generalized co-ordinates.

Substituting equations (10)–(12) into equations (3) and (4) gives the following quadratic forms for the potential energy V and the kinetic energy T:

$$V = \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u}, \qquad T = (\omega^2/2)\mathbf{u}^{\mathrm{T}}\mathbf{M}\mathbf{u} \tag{22, 23}$$

Here $\mathbf{K}$ and $\mathbf{M}$ are, respectively, the element stiffness matrix and mass matrix, expressed as follows:

$$\mathbf{K} = Sab\int_{-1}^{1}\int_{-1}^{1}[\mathbf{F}^{\mathrm{T}}\mathbf{F} + \mathbf{H}^{\mathrm{T}}\mathbf{H}]\,\mathrm{d}\xi\,\mathrm{d}\eta, \qquad \mathbf{M} = \rho ab\int_{-1}^{1}\int_{-1}^{1}\mathbf{C}^{\mathrm{T}}\mathbf{C}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{24, 25}$$

The order N of the element stiffness matrix $\mathbf{K}$ and mass matrix $\mathbf{M}$ is as follows:

$$N = 4 + 2(p + q) + pq. \tag{26}$$

Any arbitrary elements $K_{\alpha\beta}$ of $\mathbf{K}$ and $M_{\alpha\beta}$ of $\mathbf{M}$ will have the forms

$$K_{\alpha\beta} = S\left(\frac{b}{a}I_{i,k}^{1,1}J_{j,l}^{0,0} + \frac{a}{b}I_{i,k}^{0,0}J_{j,l}^{1,1}\right), \qquad M_{\alpha\beta} = \rho ab I_{i,k}^{0,0}J_{j,l}^{0,0}, \tag{27, 28}$$

where the row number α and the column number β are related to the indices i, j, k, and l which denote the numbers of the functions used in the assumed displacement field. The integrals $I_{i,k}^{c,d}$ and $J_{j,l}^{c,d}$ are defined as

$$I_{i,k}^{c,d} = \int_{-1}^{1} f_i^c(\xi)f_k^d(\xi)\,\mathrm{d}\xi, \qquad J_{j,l}^{c,d} = \int_{-1}^{1} f_j^c(\eta)f_l^d(\eta)\,\mathrm{d}\eta, \tag{29, 30}$$

where the indices c and d ($c, d = 0, 1$) denote the order of the derivatives.

The values of the integrals in equations (29) and (30) are obtained by using Gaussian quadrature with an appropriate number of integration points for the polynomial in the integrand of each integral and are stocked in a file which is later used by the program that implements the membrane hierarchical finite element. This process greatly speeds up the generation of the element stiffness and mass matrices.

One may think of the new hierarchical membrane element as one with a variable number of fictitious nodes. The generalized co-ordinates w_{m1}, w_{m2}, w_{n1}, w_{n2}, and w_{mn} may therefore be considered as associated with fictitious nodes $m1$, $m2$, $n1$, $n2$, and mn respectively. The number of fictitious nodes in an element is equal to $2(p + q) + pq$. As a consequence of these considerations, the processes of assembly and application of boundary conditions will be identical to their counterparts in the finite element method and so the techniques used in the finite element method become applicable.

3. RESULTS

Results of the application of the membrane hierarchical finite element to the calculation of the frequency parameter Ω are found for a simply supported square membrane of side length equal to 2 and a simply supported L-shaped membrane shown in Figure 3.

Exact solutions are available in the literature for the square membrane [4]. By centering the co-ordinate system of the square membrane it is necessary to consider only one quarter. The solution for the entire membrane can be obtained from the solution for one quarter with three different sets of boundary conditions on the symmetry lines. The solutions for the one quarter will therefore fall into three groups. The first group consists of modes having even symmetry with respect to both co-ordinate axes, the second group consists

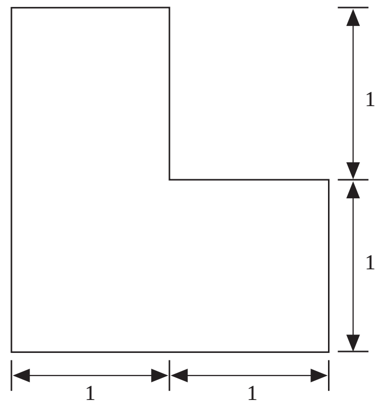


Figure 3. The simply supported L-shaped membrane.

of the modes having odd symmetry with respect to both co-ordinate axes, and the third group consists of the modes having even symmetry with respect to one co-ordinate axis and odd symmetry with respect to the other. In order to see the manner of convergence of the solutions, one quarter of the membrane is discretized into one element and the number of hierarchical terms $p\ (=q)$ is varied. An equal number of hierarchical terms is used in both directions because the element is a square. The results for the ten lowest modes are shown in Table 2 along with exact solutions. Modes 1, 4, 7 and 10 are even–even modes, modes 2, 5, 6 and 9 are even–odd modes, and modes 3 and 8 are odd–odd modes. In Table 2 it is clearly shown that rapid convergence from above to the exact values occurs as the number of hierarchical terms is increased from 2 to 6. It is interesting to note that highly accurate solutions are obtained for this problem despite the use of a very few hierarchical terms. In fact, the hierarchical finite element solutions for $p = q = 6$ agree up to five significant digits with the exact solutions for most of the modes.

The problem of the L-shaped membrane is one of the most troublesome to solve because of the re-entrant corner, which causes difficulty in estimating several of the lower modes. This problem has been used as a basis of comparison for other methods by some authors [4, 5]. There is no known analytical solution of this problem, but highly accurate approximate solutions are available in the literature [6]. The membrane consists of three identical squares. It is therefore discretized into three elements with one element in each square and an equal number of hierarchical terms is used in all three elements. The results for the ten lowest modes are shown in Table 3, along with the solutions of Fox *et al*. [6]

TABLE 2

Convergence of the ten lowest frequency parameters Ω of the simply supported square membrane as a function of the number of hierarchical terms $p\ (=q)$

$p\ (=q)$	1	2	3	4	5	6	7	8	9	10
2	2·22159	3·53097	4·47213	5·08517	5·77851	6·66841	6·83977	7·21110	8·08648	10·56450
3	2·22144	3·51243	4·44292	4·97887	5·67376	6·66839	6·68158	7·20206	8·02008	8·47772
4	2·22144	3·51243	4·44291	4·96802	5·66424	6·47835	6·66541	7·02647	7·85592	8·09290
5	2·22144	3·51241	4·44288	4·96732	5·66361	6·47834	6·66437	7·02646	7·85547	8·02059
6	2·22144	3·51241	4·44288	4·96729	5·66359	6·47656	6·66432	7·02482	7·85398	8·01056
Exact	2·22144	3·51241	4·44288	4·96729	5·66359	6·47656	6·66432	7·02481	7·85398	8·00952

TABLE 3

Comparison of the ten lowest frequency parameters Ω of the simply supported L-shaped membrane. Numbers in parentheses denote the numbers of system degrees of freedom

Method	1	2	3	4	5	6	7	8	9	10
HFEM (16)	3·1251	3·9303	4·4721	5·5459	5·7759	6·6242	6·9278	7·2111	7·2111	7·7435
FEM (16)	3·2463	4·0720	4·6466	5·8994	6·2412	7·4783	7·7333	8·0482	8·0482	8·6480
HFEM (33)	3·1114	3·8987	4·4429	5·4448	6·6819	6·5851	6·9098	7·2020	7·2020	7·6730
FEM (33)	3·1877	3·9974	4·5578	5·6989	5·9926	7·0486	7·3392	7·6411	7·6411	8·1852
HFEM (56)	3·1087	3·8985	4·4429	5·4340	5·6551	6·4450	6·7072	7·0264	7·0264	7·5361
FEM (56)	3·1600	3·9619	4·5162	5·6033	5·8730	6·8290	7·1166	7·4220	7·4220	7·9511
Fox *et al*. [6]	3·1048	3·8983	4·4428	5·4333	5·6492	6·4400	6·7043	7·0248	7·0248	7·5305

and the solutions obtained by using the degenerate case with no hierarchical terms to represent a linear finite element. The numbers of hierarchical terms p $(=q)$ used in each element are two, three and four, and the corresponding numbers of system degrees of freedom excluding the restrained ones are 16, 33 and 16, respectively. Comparisons of the HFE and the linear FE solutions favour the HFE solutions by a large margin. An interesting comparison is that of the HFE (16) and the FE (56) solutions. The former is much more accurate, although it has about 72% fewer system degrees of freedom. The HFE (56) solutions agree very well with the solutions of Fox *et al*. despite the use of only four hierarchical terms in each element.

4. CONCLUSIONS

A hierarchical finite element for membrane vibrations has been presented. The element has the ability to be joined to neighbouring elements. Thus, membranes of complex geometry such as an L-shaped membrane may be easily handled. Furthermore, the numbers of hierarchical terms in both directions of the element co-ordinate axes are allowed to vary. Thus, an elongated membrane may be discretized into only one rectangular element, but more hierarchical terms are needed along the membrane length than along its width in order to describe the membrane mode shapes accurately.

When compared with a linear finite element, the hierarchical finite element was found to yield a better accuracy with fewer system degrees of freedom. Although comparisons with quadratic and cubic finite elements were not made, one would expect the hierarchical finite element to give a better accuracy with fewer system degrees of freedom.

The results have shown that the hierarchical finite element solutions always converge from above to the exact values as the numbers of hierarchical terms are increased, and highly accurate answers are obtained with a very few hierarchical terms.

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APPENDIX: NOTATION

S	surface tension
ρ	surface density
a	element half-width
b	element half-height
x, y	element co-ordinates
ξ, η	element non-dimensional co-ordinates
w	membrane transverse displacement
p	number of hierarchical terms in the ξ direction
q	number of hierarchical terms in the η direction
V	element potential energy
T	element kinetic energy
$\mathbf{u}$	vector of generalized co-ordinates
$\mathbf{K}$	element stiffness matrix
$\mathbf{M}$	element mass matrix
N	dimension of the element stiffness and mass matrices
ω	natural frequency
Ω	$=\omega\sqrt{\rho/S}$, frequency parameter