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LETTERS TO THE EDITOR

FUNDAMENTAL FREQUENCY OF TENSIONED FREE–FREE BEAMS

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Recently, Liu *et al*. [1] investigated the frequencies and mode shapes of free-free beams under tensile axial loads. Of particular interest was the discovery of a previously unreported fundamental frequency which increased from zero as the tension increased from zero. Some numerical values were presented, as well as a graph for a range of small tension parameters and some approximate third-order polynomial expressions (obtained from a least squares fit) for larger tension parameters.

The main purpose of this paper is to derive, directly from the characteristic frequency equation itself, a simple explicit approximate analytical formula for the fundamental frequency parameter for small tension, thereby providing a much better estimate of the fundamental frequency than that obtainable from the graph of Figure 3 in reference [1]. Some indication of a possible reason why such a mode had hitherto escaped notice is also elicited. A similarly simple formula is derived for large tensions. The crossover occurs with less than 5% error in either formulae.

The characteristic equation for planar transverse vibrations of a free-free uniform Euler–Bernoulli beam of length L, cross-sectional area A, mass density ρ and flexural rigidity EI under constant axial tension T is given in non-dimensional form in [1] as

$$2\beta^6[1-\cosh(\alpha_1)\cos(\alpha_2)]-k^2(k^4+3\beta^4)\sinh(\alpha_1)\sin(\alpha_2)=0, \tag{1}$$

where

$$\alpha_{1,2}=(\pm k^2/2+(k^4/4+\beta^4)^{1/2})^{1/2}, \tag{2}$$

with

$$\beta^4=[\rho AL^4/(EI)]\omega^2, \qquad k^2=[L^2/(EI)]T. \tag{3}$$

Here ω is the radian frequency of vibration, and the tension magnitude is characterized by [1]

$$\gamma=k^2/\pi^2. \tag{4}$$

If tension $T=0$, $\beta=0$ is a solution, i.e., $\omega=0$. Now a solution for the lowest frequency parameter β_0 in terms of small tension parameter k may be sought by making series expansions of the terms in equation (1) under the assumption $\beta=\mathcal{O}(k)$ (i.e., $\beta^2=\mathcal{O}(\gamma)$ or $\omega=\mathcal{O}(T)$) which appears natural in view of the form of equations (2). However, this leads to the equation $k^4+4\beta^4\approx 0$ which is consistent as regards orders of magnitude but which implies $k=0=\beta$, i.e., no positive solution. This may be part of the reason why the positive fundamental frequency for positive T had eluded notice.

Numerical computations (see Table 1), and the form of the graph of β_0^2 versus γ in Figure 3 in reference [1], suggest that in fact β_0^2 may rather behave like the square root

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of γ (i.e., ω like $\sqrt{T}$). Thus in equation (1) the assumption is now made that $\beta = \mathcal{O}(\sqrt{k})$, so $k = \mathcal{O}(\beta^2)$, and expansions are made for small values of these parameters. Greater care must now be taken in collecting small terms of given order. In particular, it is found that

$$\alpha_{1,2} = \beta \pm (1/4)k^2/\beta + \mathcal{O}(\beta^5). \tag{5}$$

This procedure then yields $\beta^2 \approx (\sqrt{(12)})k$, yielding the following explicit analytical approximation for the fundamental free-free frequency parameter for small tensions (small γ):

$$\beta_0^2 = (2\sqrt{3}\pi)\sqrt{\gamma} \simeq 10{\cdot}882796\sqrt{\gamma}. \tag{6}$$

This is the main result of this paper, and replaces the graph of [1] with an explicit formula. Even up to $\gamma = 1$, the error in equation (6) is less than 2%.

For *large* tensions, the work of Bokaian [2] for some modes again suggests a square root type of increase, although there the fundamental free-free mode is missing (as stressed in [1]) and there is an additive constant within the square root since those modes are positive for all tensions. In any case, numerical computations on equation (1) here again imply a square root type of relationship. Accordingly, the *ansatz* $\beta^2 = \mathcal{O}(\sqrt{\gamma})$, i.e., $k = \mathcal{O}(\beta^2)$, $\omega = \mathcal{O}(\sqrt{T})$, is again made in equation (1), now for large parameters. Then $\alpha_1 \approx k$ and $\alpha_2 \approx \beta^2/k$. Neglect of the smaller order first term in equation (1) then gives $\sin(\beta^2/k) = 0$. Thus for large tensions, the approximation is

$$\beta_0^2 = \pi^2\sqrt{\gamma} \approx 9{\cdot}8696044\sqrt{\gamma}. \tag{7}$$

This is in fact the corresponding string frequency relation (c.f., [2]). The expressions (21), (22) in [1] do in general give more accurate results, but at the expense of considerably detailed coefficients and less direct motivation.

Table 1 shows some exact values of dimensionless frequency parameter β_0^2, for dimensionless tension parameter γ covering a whole range, from small to large, in powers of 10, together with both approximations (6) and (7) and their percentage errors. Also included are the results for $\gamma = 4$ which corresponds to the "crossover" error of 5%.

TABLE 1

Frequency parameter β_0^2 (c.f., equation (3)) for tension parameter γ (equation (4)) obtained from equation (1)

γ	Exact	Equation (6)	% error	Equation (7)	% error
0·000001	0·010882796	0·010882796	$2{\cdot}4 \times 10^{-6}$	0·009870	$-9{\cdot}3$
0·00001	0·034414415	0·034414423	$2{\cdot}4 \times 10^{-5}$	0·03121	$-9{\cdot}3$
0·0001	0·108827706	0·108827962	$2{\cdot}4 \times 10^{-4}$	0·09870	$-9{\cdot}3$
0·001	0·344136148	0·3441442	$2{\cdot}4 \times 10^{-3}$	0·3121	$-9{\cdot}3$
0·01	1·08802453	1·088280	$2{\cdot}3 \times 10^{-2}$	0·9870	$-9{\cdot}3$
0·1	3·43355582	3·44144	$2{\cdot}3 \times 10^{-1}$	3·121	$-9{\cdot}1$
1·0	10·6790062	10·8828	1·9	9·870	$-7{\cdot}6$
4·0	20·7539369	21·7656	4·9	19·739	$-4{\cdot}9$
10·0	32·1399000	34·414	7·1	31·210	$-2{\cdot}9$
100·0	99·1255883	108·828	9·8	98·6960	$-4{\cdot}3 \times 10^{-1}$
1000·0	312·254025	344·144	10·2	312·1043	$-4{\cdot}8 \times 10^{-2}$
10000·0	987·009159	1088·280	10·3	986·9604	$-4{\cdot}9 \times 10^{-3}$

These results may now be summarized as follows:
To within 5% (and much less for γ small, or large):

$$\text{if } \gamma \leqslant 4, \ \beta_0^2 \approx 2\sqrt{3}\pi\sqrt{\gamma} \approx 10{\cdot}8828\sqrt{\gamma}; \tag{8a}$$

$$\text{if } \gamma > 4, \ \beta_0^2 \approx \pi^2\sqrt{\gamma} \approx 9{\cdot}8696\sqrt{\gamma}. \tag{8b}$$

The true values lie between the two estimates, which are very good for small and for large γ respectively.

Insofar as this zeroth, fundamental, mode may be of considerable practical importance [1], the exceedingly simple formulae (8) allow a rapid estimate, especially accurate for very small (or very large) tensions, to be made of the lowest solution to the complicated frequency equation (1).

For rough estimates, either of (8a) or (8b) actually gives accuracy to within about 10% over the whole range. Evidently a more accurate formula for medium values of γ could be derived by taking expansions to the next order, but this would detract from the above simplicity.

REFERENCES

1. X. Q. Liu, R. C. Ertekin and H. R. Riggs 1996 *Journal of Sound and Vibration* **190,** 273–282. Vibration of a free-free beam under tensile axial loads.
2. A. Bokaian 1990 *Journal of Sound and Vibration* **142,** 481–498. †Natural frequencies of beams under tensile axial loads.

†(There appear to be errors in sign in the boundary conditions for the free–free case in Table 1 on p. 483 of [2].)