Journal of Sound and Vibration (1997) **201**(4), 520–527

EVALUATION OF NATURAL VIBRATION FREQUENCY OF A COMPRESSION BAR WITH VARYING CROSS-SECTION BY USING THE SHOOTING METHOD

R. S. Chen

P.O. Box 284, *Department of Computer Science*, *Nanjing University*, *Nanjing*, *Jiangsu* 210093, *People's Republic of China*

(*Received* 8 *January* 1996, *and in final form* 28 *June* 1996)

1. INTRODUCTION

To solve the boundary value problem of the ordinary differential equation, the shooting method was introduced in reference [1]. Recently, the shooting method has been extended and used for evaluating the eigenvalues in some problems of ordinary differential equations. For example, the buckling loading of a compression bar with a varying cross-section was analyzed in reference [2]. In addition, the natural vibration frequency of the varying cross section bar was also discussed in [3].

In this note, the natural vibration frequency of a compression bar with varying cross-section is studied. The shooting method is used to solve the differential equation of the problem. The merit of the shooting method is that one adjusts the parameter involved in the equation such that the boundary condition at the end point of the interval is satisfied.

In the following analysis, a bar with two hinged ends is taken as an example (Figure 1). In this case, evaluation of the natural vibration frequency of the bar is reduced to the eigenvalue problem of a particular ordinary differential equation. The governing equation takes the form

$$\frac{d^4w}{dx^4} = \omega^2 f_1(x)w + f_2(x)\frac{d^2w}{dx^2} + f_3(x)\frac{d^3w}{dx^3} \qquad (0 \leqslant x \leqslant l), \tag{1}$$

with the following boundary value conditions:

$$w(0)=0, \qquad w''(0)=0, \tag{2}$$

$$w(l)=0, \qquad w''(l)=0, \tag{3}$$

where $f_1(x)$, $f_2(x)$ and $f_3(x)$ are some given functions, and ω denotes the eigenvalue.

Since the vibration modes of the bar can differ from each other by a multiple, we can propose an alternative problem. The relevant solution of the problem consists of the following steps.

(a) To solve the following initial boundary value problem

$$w(0)=0, \qquad w''(0)=0, \qquad w'(0)=1, \qquad w'''(0)=a, \tag{4}$$

for the ordinary differential equation (1).

(b) To adjust two parameters, a in equation (4) and ω in equation (1), so that condition (3) is satisfied.

The problem is more difficult than the one cited in reference [3]. In fact, only one parameter needs to be adjusted and only one condition needs to be satisfied in the problem introduced in reference [3]. This means that the technique used in reference [3] cannot be directly applied to the present study.

After some manipulation, one can also convert the proposed problem to one in which only one parameter needs to be adjusted and only one condition at the end point of the

0022–460X/97/140520 + 08 $25.00/0/sv960807

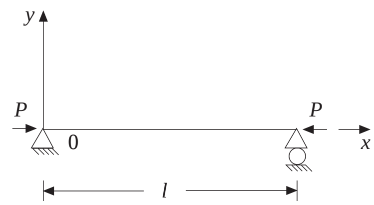

Figure 1. A bar with two hinged ends in compression.

bar needs to be satisfied. This is an essential step in the present study. The details of the relevant derivation will be given later.

2. ANALYSIS

In the following analysis, a compression bar with two hinged ends is considered (Figure 1). Clearly, for other support conditions, the problem can be solved in a similar manner.

2.1. *Buckling problem of a compression bar with varying cross-section*

Before discussing the vibration problem of a bar in compression, we first consider the buckling problem of the bar. It is well known that the buckling problem of a varying cross-section bar with two hinged ends can be reduced to a differential equation for the deflection $y(x)$:

$$EI(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Py = 0 \qquad (0 \leqslant x \leqslant l), \tag{5}$$

$$y(0)=0, \quad y'(0)=b, \tag{6}$$

$$y(l)=0, \tag{7}$$

where E denotes the elastic modulus of the bar, $I(x)$ is the moment of inertia and P is the compression loading. Since the buckling modes can differ from each other by an arbitrary multiple, the constant b in equation (6) can be assumed arbitrarily. For definiteness, we choose $b = 1$ in the analysis.

In the following analysis, we denote

$$I(x)=I_0 g(x), \qquad Q = P/EI_0 \tag{8}$$

Thus, equation (5) is reduced to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{Qy}{g(x)}. \tag{9}$$

In this case, the shooting method consists of the following main points.

(a) After choosing two parameters Q_1 and Q_2, and solving the equation (9) under condition (6), one obtains two solutions,

$$y_1(x, Q_1) \qquad \text{and} \qquad y_2(x, Q_2). \tag{10}$$

The numerical solution is obtained by using the Runge–Kutta method of fourth order accuracy [4]. Therefore from condition (7) the corresponding deviations can be obtained:

$$d_1 = y_1(l, Q_1) \qquad \text{and} \qquad d_2 = y_2(l, Q_2). \tag{11}$$

In general, the Q_1 and Q_2 values can be chosen by experience.

(b) We assume that the deviation d is a linear function of parameter Q. Thus, the expected solution for the Q value can be obtained from

$$Q_3 = Q_2 - d_2(Q_1 - Q_2)/(d_1 - d_2). \tag{12}$$

(c) From the pair Q_2, Q_3, we can perform the next round of iterative computation. Finally, when the deviation d_n at the nth step iteration becomes a very small value, the computation is ended and the approximate Q value is obtained with sufficient accuracy.

2.2. *Natural vibration frequency of a compression bar*

For the natural vibration frequency of a bar with compression loading P, the governing equation for the deflection $y(x, t)$ takes the form [3]

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) + P\frac{\partial^2 y}{\partial x^2} + D(x)\frac{\partial^2 y}{\partial t^2} = 0, \tag{13}$$

$$y(0, t) = 0, \qquad \frac{\partial^2}{\partial x^2}y(0, t) = 0, \qquad y(l, t) = 0, \qquad \frac{\partial^2}{\partial x^2}y(l, t) = 0, \tag{14}$$

where $D(x)$ denotes the mass density.

As before, we denote

$$I(x) = I_0 g(x), \qquad D(x) = D_0 h(x), \qquad Q = P/EI_0, \qquad S = D_0/EI_0 \tag{15}$$

and assume that

$$y(x, t) = w(x)\sin(\omega t). \tag{16}$$

After substituting equations (15) and (16) into equations (13) and (14), it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(g(x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) + Q\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \omega^2 S h(x) w = 0, \tag{17}$$

$$w(0) = 0, \quad w''(0) = 0, \tag{18}$$

$$w(l) = 0, \quad w''(l) = 0. \tag{19}$$

Note that in equation (17) Q is given beforehand, and ω denotes the eigenvalue, i.e., the investigated natural vibration frequency.

To obtain the vibration frequency by using the shooting method, the following main points are introduced.

(a) After choosing some particular value ω_1 for equation (17), we solve the following two initial problems:

$$w(0) = 0, \qquad w''(0) = 0, \qquad w'(0) = 1, \qquad w'''(0) = 0, \tag{20}$$

$$w(0) = 0, \qquad w''(0) = 0, \tag{21}$$

$$w'(0) = 0, \qquad w'''(0) = 1. \tag{22}$$

The obtained solutions are denoted by

$$w = p(x, \omega_1) \qquad \text{(for condition (20))}, \tag{22}$$

$$w = q(x, \omega_1) \qquad \text{(for condition (21))}, \tag{23}$$

respectively. Therefore, the general solution of equation (17) takes the form

$$w = c_1 p(x, \omega_1) + c_2 q(x, \omega_1), \tag{24}$$

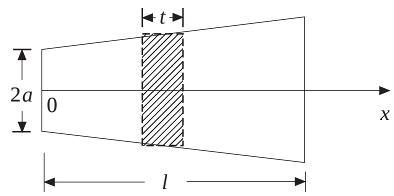

Figure 2. A tapered bar with constant thickness t.

where c_1 and c_2 are two undertermined constants. In addition, one has

$$w(l)=c_1p(l, \omega_1)+c_2q(l, \omega_1), \qquad w''(l)=c_1p''(l, \omega_1)+c_2q''(l, \omega_1). \tag{25}$$

Furthermore, we denote

$$\delta_1 = p(l, \omega_1)q''(l, \omega_1)-p''(l, \omega_1)q(l, \omega_1) \tag{26}$$

and δ_1 is called the deviation in the shooting process.

Comparing condition (19) with equation (25), we see that if $\delta_1 = 0$ we have a non-trivial solution for c_1 and c_2 such that condition (19) is satisfied. In this case, ω_1 will be the investigated eigenvalue. Obviously, for a given ω_1 value, the corresponding deviation δ_1 does not vanish in general.

(b) Similarly, we can choose a second trial value ω_2. After the initial boundary value problems (20) and (21) are solved, we can obtain the corresponding deviation

$$\delta_2 = p(l, \omega_2)q''(l, \omega_2)-p''(l, \omega_2)q(l, \omega_2). \tag{27}$$

In general, the ω_1 and ω_2 values used in the first round of computation can be chosen by experience.

(c) We assume that the deviation δ is a linear function of parameter ω. Thus, the expected solution for ω value can be obtained from

$$\omega_3 = \omega_2 - \delta_2(\omega_1 - \omega_2)/(\delta_1 - \delta_2). \tag{28}$$

(d) From the pair ω_2, ω_3, we can perform the next round of iterative computation. Finally, when the deviation δ_n at the nth step iteration becomes a very small value, for example, $\delta_n = 0{\cdot}0000000001$, the computation is ended and the approximate ω value is obtained with sufficient accuracy.

3. NUMERICAL EXAMPLES

As mentioned above, the buckling problem is governed by equation (9) and the vibration frequency problem is governed by equation (17). In the numerical solution of equations, the Runge–Kutta method with fourth order accuracy is used [4]. It is proven that, if $N = 80$, 120, 160 and 200 are taken, where N denotes the number of divisions used in integration, the final calculated results are actually the same. For compactness, only the smallest eigenvalues are investigated in the following examples.

3.1. *A tapered bar with constant thickness*

A tapered bar with constant thickness t is analyzed in the first example (Figure 2). In this case, we have

$$I(x)=I_0g(x), \qquad I_0 = at^3/6, \qquad g(x)=1 + mx/l,$$
$$D(x)=D_0h(x), \qquad D_0 = 2\rho at, \qquad h(x)=1 + mx/l, \tag{29}$$

TABLE 1

The normalized buckling loading $F(m)$ for a tapered bar with constant thickness (see Figure 2 and equation (31))

m	$F(m)$
0	1·000
1	1·470
2	1·908
3	2·329
4	2·741

where ρ is the mass density, and m represents the degree of the tapered configuration. If $m = 0$ the bar has a uniform section.

First, the buckling loading will be investigated. In the first round of iterative computation for equation (9) with condition (6), we can for example, let,

$$Q_1 = 1{\cdot}1(\pi/l)^2, \qquad Q_2 = 1{\cdot}2(\pi/l)^2. \tag{30}$$

It is found that the iterative process is convergent in general. Finally, the buckling loading is expressed by

$$P_c = F(m)EI_0(\pi/l)^2. \tag{31}$$

The calculated $F(m)$ values are listed in Table 1.

Second, we consider the natural vibration frequency of a compression bar. The applied compression force P is assumed as follows:

$$P = G(m)EI_0(\pi/l)^2 \quad \text{or} \quad Q = G(m)(\pi/l)^2 \qquad (0 \leqslant G(m) \leqslant F(m)). \tag{32}$$

Substituting the P value shown by equation (32) into equation (17), and performing the shooting method, one can obtain the natural vibration frequency, which is expressed by

$$\omega_c = H(m)(EI_0/D_0)^{1/2}(\pi/l)^2. \tag{33}$$

TABLE 2

The interaction between the normalized compression loading $G(m)$ and the normalized natural vibration frequency $H(m)$ for a tapered bar with two hinged ends (see Figure 2 and equations (32) and (33))

$m = 0$		$m = 1$		$m = 2$		$m = 3$		$m = 4$	
G	H	G	H	G	H	G	H	G	H
0·000	1·000	0·000	0·995	0·000	0·989	0·000	0·984	0·000	0·980
0·100	0·949	0·147	0·945	0·191	0·939	0·233	0·934	0·274	0·930
0·200	0·894	0·294	0·891	0·382	0·886	0·466	0·881	0·548	0·878
0·300	0·837	0·441	0·833	0·572	0·829	0·699	0·825	0·822	0·822
0·400	0·775	0·588	0·772	0·763	0·768	0·932	0·764	1·096	0·762
0·500	0·707	0·735	0·705	0·954	0·701	1·165	0·698	1·371	0·696
0·600	0·632	0·882	0·630	1·145	0·627	1·398	0·625	1·645	0·623
0·700	0·548	1·029	0·546	1·336	0·544	1·631	0·542	1·919	0·540
0·800	0·447	1·176	0·446	1·526	0·444	1·864	0·443	2·193	0·441
0·900	0·316	1·323	0·315	1·717	0·314	2·097	0·313	2·467	0·312
1·000	0·000	1·470	0·000	1·908	0·000	2·329	0·000	2·741	0·000

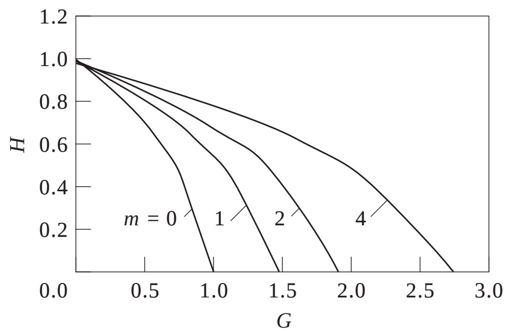

Figure 3. The interaction between the normalized compression loading $G(m)$ and the normalized natural vibration frequency $H(m)$ for a tapered bar with two hinged ends (see Figure 2 and equations (32) and (33)).

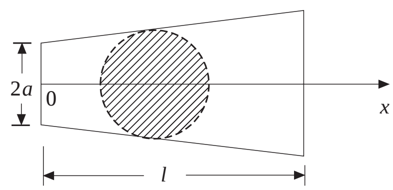

Figure 4. A truncated conical bar.

Note that in equation (32) $G(m)$ is given beforehand, and the $H(m)$ value in equation (33) is calculated in accordance with the given $G(m)$ value. If $G(m)=0$, the bar is in a pure vibration situation, and if $G(m)=F(m)$ the bar is in a purely static situation. The corresponding $G(m)$, $H(m)$ pairs for $m = 0$, 1, 2, 3, 4 are listed in Table 2 and Figure 3.

3.2. *A truncated conical bar in compression*

A truncated conical bar in compression is analyzed in the second example (Figure 4). In this case, we have

$$I(x)=I_0 g(x), \qquad I_0 = \pi a^4/4, \qquad g(x)=(1+mx/l)^4, \tag{34}$$

$$D(x)=D_0 h(x), \qquad D_0 = \rho\pi a^2, \qquad h(x)=(1+mx/l)^2. \tag{35}$$

where ρ is the mass density, and m represents the degree of conical configuration. If $m = 0$ the bar has a uniform section.

TABLE 3

The normalized buckling loading $A(m)$ for a truncated conical bar (see Figure 4 and equation (36))

m	$A(m)$
0	1·000
1	4·000
2	9·000
3	16·000
4	25·000

TABLE 4

The interaction between the normalized compression loading $B(m)$ and the normalized natural vibration frequency $C(m)$ for a truncated conical bar with two hinged ends (see Figure 4 and equations (37) and (38))

$m = 0$		$m = 1$		$m = 2$		$m = 3$		$m = 4$	
B	C	B	C	B	C	B	C	B	C
0·00	1·00	0·00	1·41	0·00	1·72	0·00	1·98	0·00	2·21
0·10	0·95	0·40	1·34	0·90	1·64	1·60	1·89	2·50	2·11
0·20	0·89	0·80	1·26	1·80	1·55	3·20	1·79	5·00	2·00
0·30	0·84	1·20	1·18	2·70	1·45	4·80	1·68	7·50	1·88
0·40	0·77	1·60	1·10	3·60	1·35	6·40	1·57	10·00	1·76
0·50	0·71	2·00	1·00	4·50	1·24	8·00	1·44	12·50	1·61
0·60	0·63	2·40	0·90	5·40	1·11	9·60	1·29	15·00	1·46
0·70	0·55	2·80	0·78	6·30	0·97	11·20	1·13	17·50	1·27
0·80	0·45	3·20	0·64	7·20	0·79	12·80	0·93	20·00	1·05
0·90	0·32	3·60	0·45	8·10	0·56	14·40	0·66	22·50	0·75
1·00	0·00	4·00	0·00	9·00	0·00	16·00	0·00	25·00	0·00

As before, in the purely static case, the buckling loading can be expressed by

$$P_c = A(m)EI_0(\pi/l)^2. \tag{36}$$

The calculated $A(m)$ values are listed in Table 3.

Second, in the interaction case, the applied compression force P is assumed as follows

$$P = B(m)EI_0(\pi/l)^2 \quad \text{or} \quad Q = B(m)(\pi/l)^2 \qquad (0 \leqslant B(m) \leqslant A(m)) \tag{37}$$

and the corresponding natural vibration frequency is expressed in the form

$$\omega_c = C(m)(EI_0/D_0)^{1/2}(\pi/l)^2. \tag{38}$$

The corresponding $B(m)$, $C(m)$ pairs for $m = 0$, 1, 2, 3, 4 are listed in Table 4.

4. REMARKS

Other methods can also be used to study the buckling or the natural vibration frequency problem for a compression bar. For example, one can use the energy method to solve the problem. However, in this method, one has to select an approximate deflection so as to make the compression loading a minimum [5]. For the second example, one can also use the finite element method to study this problem. Clearly, in this case, we have to assemble a matrix and to solve the eigenvalue problem of a set of linear algebraic equations. In general, these methods are more complicated than the suggested shooting method.

Due to the efficiency of the shooting method, this method has recently been used to solve the plastic limit loading of a circular plate [6], and the non-linear analysis of a shallow spherical shell. [7].

REFERENCES

1. X. X. SHEN 1985 *Handbook of Modern Mathematics in Science and Engineering*, *Volume* 2. Wuhan: Press of Huazhong University of Science and Tehcnology (in Chinese).
2. Y. Z. CHEN, Y. K. CHEUNG and J. R. XIE 1989 *American Society of Civil Engineers*, *Journal of Engineering Mechanics* **115,** 662–667. Buckling loads of columns with varying cross sections.
3. Y. Z. CHEN and J. R. XIE 1988 *Computers and Structures* **29,** 693–697. Evaluation of natural frequencies of non-uniform beams by numerical integration.

4. F. B. HILDEBAND 1974 *Introduction to Numerical Analysis*. New York: McGraw-Hill.
5. S. P. TIMOSHENKO and J. M. GERE 1961 *Theory of Elastic Stability*. New York: McGraw-Hill second edition.
6. Y. Z. CHEN 1986 *Computers and Structures* **24,** 821–822. A numerical procedure for evaluating the plastic limit loading of a circular plate using Mises yield condition.
7. J. R. XIE, Y. Z. CHEN and D. HO 1990 *Computers and Structures* **34,** 297–304. Axisymmetric buckling of truncated shallow spherical shells.