Journal of Sound and Vibration (1997) **201**(5), 567–576

INSTABILITY OF VIBRATIONS OF A MASS MOVING UNIFORMLY ALONG AN AXIALLY COMPRESSED BEAM ON A VISCOELASTIC FOUNDATION

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(*Received 2 July 1996, and in final form 10 October 1996*)

The uniform motion of a mass along an axially compressed Euler–Bernoulli beam on a viscoelastic foundation is investigated. It is assumed that the mass is subjected to a constant vertical load and that the beam and mass are in continuous contact. The velocity of the mass after which the vibrations of the system are unstable is found. The instability implies that the amplitude of the mass vibrations is growing exponentially and that the problem does not have a steady state solution. It is shown that the instability starts at lower velocities as the compresional force increases. The instability occurs even for over-critical viscosities of the foundation when there is no dynamical amplification of the steady state vibrations due to resonance.

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1. INTRODUCTION

The instability of vibrations of a mass moving uniformly along a continuously supported beam was found first by Denisov *et al*. [1]. The results obtained indicated that there is a velocity after which the amplitude of the mass vibrations grows exponentially even if there is a non-zero viscosity of the beam foundation. Mathematically, the exponential growth implies that the problem does not have a steady state solution. The physical explanation of the phenomenon was given in reference [2], where it was shown that the instability is caused by anomalous Doppler waves [3], radiated by the moving object.

In earlier investigations of the instability of a moving mechanical object interacting with an elastic system, the object was assumed to be distributed [4, 5]. The instability of a point object can be considered as the basic phenomenon for this situation.

This paper is mainly concerned with the effect of compressional axial stresses in the beam on the instability phenomenon. It is of practical importance for continuously welded tracks, where a temperature increase can cause considerable axial compressional forces. One can expect that the instability takes place for lower velocities as the compressional force increases. Indeed, in references [1, 2] it was shown that the instability can occur if the velocity of the mass is larger than the minimum phase velocity V_{ph}^{min} of waves in the beam. Furthermore, Kerr [6] has shown that V_{ph}^{min} decreases as the compressional force

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increases. Therefore the compressioal stresses can probably reduce the velocity at which the instability starts. It is shown in this paper that this is indeed the case.

Furthermore, the effect of the viscosity of the foundation on the instability phenomenon is investigated. It is shown that due to the viscosity the instability domain moves to an area of larger velocities and masses. It is important to underline that, even with viscosity in the system, the amplitude of vibrations can grow exponentially in time, resulting in an infinite displacement when time goes to infinity (in the frame of the linear model). This is quite different from the effect of viscosity on the amplitude of beam vibrations at a resonance caused by a load. In this case the amplitude of vibrations decreases when the viscosity increases.

2. MODEL AND GENERAL SOLUTION

Consider a uniform motion of a mass along an axially compressed Euler–Bernoulli beam on a viscoelastic foundation. It is assumed that the mass and the beam are in continuous contact and a vertical constant force acts on the moving mass. The model is depicted in Figure 1.

The equations of motion for the model are

$$\rho \frac{\partial^2 \tilde{U}}{\partial t^2} + EI \frac{\partial^4 \tilde{U}}{\partial x^4} + N \frac{\partial^2 \tilde{U}}{\partial x^2} + \mu \frac{\partial \tilde{U}}{\partial t} + \chi \tilde{U} = -\left(m \frac{\mathrm{d}^2 \tilde{U}^0}{\mathrm{d}t^2} + P \right) \delta(x - Vt),$$

$$\tilde{U}^0(t) = \tilde{U}(Vt, t), \qquad (1)$$

where $\tilde{U}(x, t)$ and $\tilde{U}^0(t)$ are the vertical deflections of the beam and the mass respectively, ρ and EI are the mass per unit length and the bending stiffness of the beam, N is the compressional force μ and χ are the viscosity and the stiffness of the foundation per unit length, m is the mass of the body, V is the velocity of the body, P is the vertical force and $\delta(\cdots)$ is the Dirac delta function. The units of the parameters are $[\rho] = \mathrm{kg/m}$, $[EI] = \mathrm{N\ m^2}$, $[N] = \mathrm{N}$, $[\mu] = \mathrm{kg/(m\ s)}$, $[\chi] = \mathrm{kg/(m\ s^2)}$, $[m] = \mathrm{kg}$, $[V] = \mathrm{m/s}$, $[P] = \mathrm{N}$ and $[\delta(x)] = 1/\mathrm{m}$.

Introducing dimensionless variables and parameters by the definitions

$$\tau = t\sqrt{\chi/\rho}, \qquad y = x(4EI/\chi)^{-1/4}, \qquad \{U, U^0\} = \{\tilde{U}, \tilde{U}^0\}(4EI/\chi)^{-1/4},$$

$$\alpha = V(4\chi EI/\rho^2)^{-1/4}, \qquad T = N(4EI\chi)^{-1/2}, \qquad v = \mu(\rho\chi)^{-1/2}, \qquad M = m(4EI/\chi)^{-1/4}/\rho,$$

and

$$F = P(4EI/\chi)^{-1/2}/\chi,$$

one can rewrite equation (1) as

$$\frac{\partial^2 U}{\partial \tau^2} + \frac{1}{4}\frac{\partial^4 U}{\partial y^4} + T\frac{\partial^2 U}{\partial y^2} + v\frac{\partial U}{\partial \tau} + U = -\left(M \frac{\mathrm{d}^2 U^0}{\mathrm{d}t^2} + F \right)\delta(y - \alpha\tau), \qquad U^0(\tau) = U(\alpha\tau, \tau). \quad (2)$$

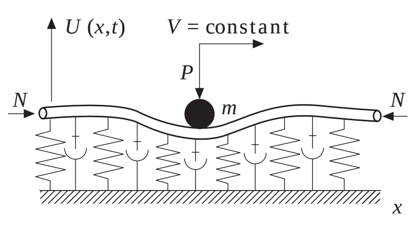

Figure 1. The uniform motion of a mass (subjected to a constant vertical force) along a beam on a viscoelastic foundation.

For the analysis it is convenient to introduce a moving reference system $\{\xi = y - \alpha\tau, \tau = \tau\}$. In this system, the first of equations (2) takes the form (see reference [7])

$$\frac{\partial^2 U}{\partial \tau^2} + \frac{1}{4}\frac{\partial^4 U}{\partial \xi^4} + (T+\alpha^2)\frac{\partial^2 U}{\partial \xi^2} - 2\alpha\frac{\partial^2 U}{\partial \tau \partial \xi} + v\frac{\partial U}{\partial \tau} - v\alpha\frac{\partial U}{\partial \xi} + U = -\left(M\frac{\partial^2 U}{\partial \tau^2} + F\right)\delta(\xi). \quad (3)$$

The advantage of equation (3) is that the Dirac delta function is independent of time and on the right side of the equation one has $\partial^2 U/\partial\tau^2$ instead of $\mathrm{d}^2U^0/\mathrm{d}\tau^2$ (this is due to the equality $\mathrm{d}^2U^0/\mathrm{d}\tau^2 = \partial^2 U/\partial\tau^2 + 2\alpha\partial^2 U/\partial\tau\partial y + \alpha^2\partial^2 U/\partial y^2|_{y=\alpha\tau}$, which has to be rewritten in the moving reference system).

As boundary conditions one requires that the solution must vanish as $\xi \to \pm\infty$. The trivial initial conditions $U(\xi, 0) = U_\tau(\xi, 0) = 0$ can be taken, since the initial shape of the beam does not effect either the stability of the system or the steady state solution (in the frame of the linear model).

Equation (3) can be solved by using the Fourier transform with respect to ξ and the Laplace transform with respect to τ. These transforms are

$$W_{k,s}(k,s) = \int_{-\infty}^{\infty} V_s(\xi, s)\exp(-\mathrm{i}k\xi)\,\mathrm{d}\xi, \qquad V_s(\xi, s) = \int_0^{\infty} U(\xi,\tau)\exp(-s\tau)\,\mathrm{d}\tau,$$

and applying them to equation (3) results in

$$D(k,s)W_{k,s}(k,s) = -(Ms^2V_s(0,s) + F/s),$$
$$D(k,s) = s^2 + (1/4)k^4 - (T+\alpha^2)k^2 - s\mathrm{i}\alpha sk + vs - \mathrm{i}v\alpha k + 1. \quad (4)$$

Inverting the Fourier transform to obtain the solution in the Laplace domain yields

$$V_s(\xi, s) = -\left(Ms^2V_s(0,s) + \frac{F}{s}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k\xi)\,\mathrm{d}k}{D(k,s)}. \quad (5)$$

To determine $V_s(\xi, s)$ one has to know $V_s(0, s)$, which is the Laplace image of $U^0(\tau)$. Assuming $\xi = 0$ in equation (5), one finds that

$$V_s(0,s) = -\frac{F}{s(Ms^2 + \chi_{eq}(s))}, \qquad \chi_{eq}(s) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{D(k,s)}\right)^{-1}. \quad (6)$$

The expression for χ_{eq} determines the equivalent stiffness of the system (the beam on foundation) under the moving body.

Substituting equations (6) into equation (5) yields

$$V_s(\xi,s) = -\frac{F}{2\pi s}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k\xi)}{D(k,s)}\,\mathrm{d}k + \frac{Ms^2}{Ms^2 + \chi_{eq}(s)}\frac{F}{2\pi s}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k\xi)}{D(k,s)}\,\mathrm{d}k. \quad (7)$$

The first member of equation (7) describes the beam displacement under the constant load ($M = 0$). The second member is related to the beam deflection caused by the mass.

3. INSTABILITY OF VIBRATIONS AND THE EFFECT OF THE COMPRESSIONAL AXIAL FORCE

Now it can be shown that the second member of equation (7) can have a pole $s^* = a + \mathrm{i}b$, with $a > 0$. This case physically implies that the beam displacement is growing exponentially in time ($(s - s^*)^{-1}/\exp(t(a + \mathrm{i}b))$), so the beam vibrations are unstable. Note, that it is assumed that $N < 2\sqrt{\chi EI}$, so the instability is not related to an over-critical axial compressional force (see reference [6]).

Consider the roots of the equation

$$Ms^2 + \chi_{eq}(s) = 0. \tag{8}$$

These roots determine the eigenfrequencies of vibrations of the moving mass as it interacts with the beam. In fact, the goal is to determine whether or not equation (8) has a root with a positive real part. To determine this, it is appropriate to use the D-decomposition method [1, 7]. The idea of the method is to map the imaginary axis of the complex (s)-plane on to the plane of a complex parameter M (here, initially the mass parameter M is regarded as a complex parameter, without regard to its physical meaning). The mapped line will divide the M-plane into domains with different numbers of roots with a positive real part.

Substituting $s = \mathrm{i}\Omega$ (the imaginary axes of the (s)-plane) into equation (8) and expressing M explicitly one obtains the following rule for the mapping,

$$M = \chi_{eq}(\mathrm{i}\Omega)/\Omega^2, \tag{9}$$

in which Ω is a real value which has to be varied from minus infinity to plus infinity. The expression for $\chi_{eq}(\mathrm{i}\Omega)$ according to equations (4) and (6) is

$$\chi_{eq}(\mathrm{i}\Omega) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{D(k, \mathrm{i}\Omega)}\right)^{-1},$$

$$D(k, \mathrm{i}\Omega) = -\Omega^2 + (1/4)k^4 - (T + \alpha^2)k^2 + 2\alpha\Omega k + \mathrm{i}v\Omega - \mathrm{i}v\alpha k + 1. \tag{10}$$

By using the contour integration method the integral (10) can be developed and this results in the expression (the contour of integration is closed along an infinite semicircle in the upper half-plane of the complex variable k)

$$\chi_{eq}(\mathrm{i}\Omega) = \left(4\mathrm{i}\sum_n \frac{k - k_n}{(k - k_1)(k - k_2)(k - k_3)(k - k_4)}\bigg|_{k = k_n}\right)^{-1}, \tag{11}$$

where k_n are the roots of the equation $D(k, \mathrm{i}\Omega) = 0$ which possess a positive imaginary part. All roots have an imaginary part and all of them are simple due to the viscosity of the foundation of the beam.

Therefore, the final expression for the mapping is

$$M = \frac{1}{4\mathrm{i}\Omega^2}\left(\sum_n \frac{k - k_n}{(k - k_1)(k - k_2)(k - k_3)(k - k_4)}\bigg|_{k = k_n}\right)^{-1}, \tag{12}$$

Now, according to the D-decomposition method, one has to plot a curve Re(M) versus Im(M) (according to equation (12)), using Ω as the parameter for this curve. This can easily be done numerically with the help of any standard program for finding the complex roots

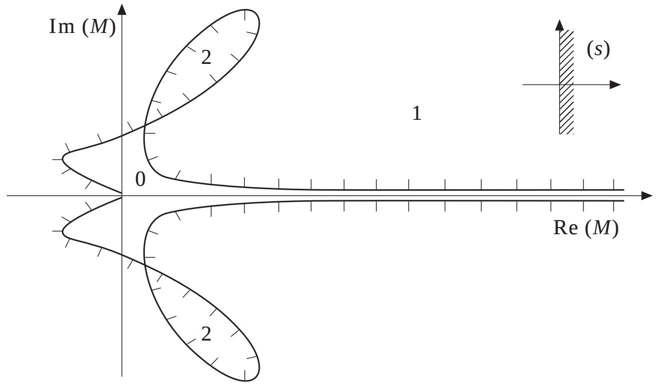

Figure 2. Separation of the complex (M)-plane into domains with different numbers of roots with a positive real part (the number of roots is shown in each domain). The parameters are $v = 0{\cdot}001$, $T = 0{\cdot}5$ and $\alpha = 0{\cdot}5$.

of polynomials. This curve is depicted in Figure 2 and Figure 3 for $v = 0{\cdot}001$ (very small viscosity) and $T = 0{\cdot}5$ ($N = 0{\cdot}5N_{cr}$). For Figure 2 the value of α has been taken as $\alpha = 0{\cdot}5$ ($V = 0{\cdot}5V_{cf}^{N=0}$, $V_{cr}^{N=0}$ being the critical velocity of the constant load along an axially uncompressed beam) and, for Figure 3, $\alpha = 0{\cdot}9$. One side of the lines in the figures is shaded (this side is related to the right side of the imaginary axes in the (s)-plane). Crossing of the lines in the direction of the shading implies that one has an additional root with a positive real part.

It is seen from the figures that for a "small" velocity of the mass (Figure 2) the mapped line does not cross the real axes of the (M)-plane as $\mathrm{Re}(M)>0$. This implies that for all physically relevant values of the mass (real and positive) the number of "unstable" roots of the equation (8) is the same. Another situation takes place for "large" velocities of the mass (Figure 3). In this case the mapped line crosses the real axis of the (M)-plane twice at the point $M^* > 0$. Therefore, one has two more "unstable" roots for $M > M^*$ than for $0 < M < M^*$ (see the direction of the shading).

Now one has to derive the number of "unstable" roots for some particular value of M because so far one knows only the relative number of "unstable" roots in the different domains of the (M)-plane, but not the number itself. This can be done

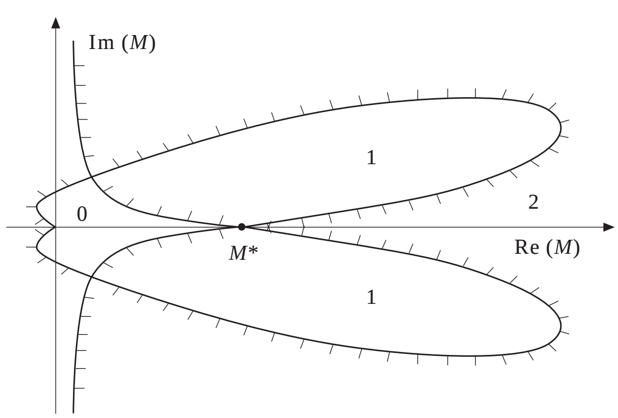

Figure 3. Separation of the complex (M)-plane for $v = 0{\cdot}001$, $T = 0{\cdot}5$ and $\alpha = 0{\cdot}9$.

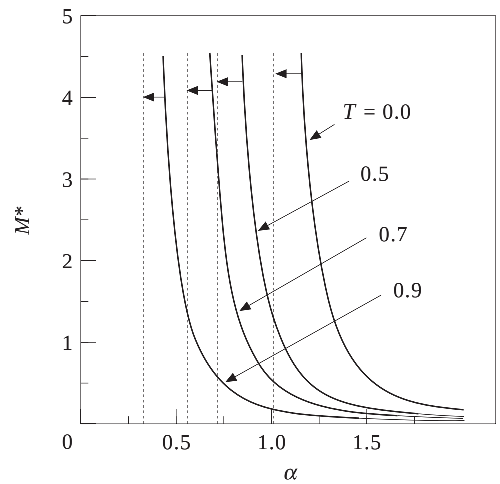

Figure 4. $M^*(\alpha)$ for different compressional forces. Each curve separates the plane into two domains. The domain located above a curve is associated with unstable vibrations of the beam.

numerically; for example, for $M = 0$. Calculations show that in this case equation (8) does not possess roots with a positive real part. Physically, this result is evident since the instability is the result of an interaction between the moving body and the beam (see reference [2]). Therefore, if the body is absent ($M = 0$), instability cannot occur.

Using this result and starting in the domains which include the point $M \to +0$ (M is real), one can determine the number of "unstable roots" for arbitrary M. These numbers are depicted in Figures 2 and 3.

One can now easily see from the figures that for "small" velocities of the mass the beam vibrations are stable for all M. However, if the velocity is "large", there exists a critical value of the mass (M^*), after which the vibrations become unstable.

In Figure 4 the dependency of this critical mass versus the velocity is depicted on the plane (M, α) for different values of the compressional axial force in the beam ($\nu = 0{\cdot}001$). The vertical lines in the figure are asymptots for the curves as $M^* \to \infty$. The following conclusions can be drawn from the figure.

1. If the beam is not axially compressed, the instability can take place only for $\alpha > 1$ ($\alpha = 1 \Leftrightarrow V = (4\chi EI/\rho^2)^{1/4}$). This result has been obtained in reference [1].

2. The instability starts at smaller velocities (for a fixed value of the mass) when the compressional force increases.

3. The larger the mass, the smaller the velocity that can cause the instability.

Thus, vibrations of the beam interacting with the uniformly moving mass can be unstable starting from some velocity V_{cr}^{inst}. This velocity decreases if an axial compressional force acts on the beam. As shown in reference [2], the energy to increase the amplitude of the vibrations is delivered by the energy source, maintaining the uniform motion of the mass.

Mathematically the existence of roots with a positive real part of equation (8) implies that the problem (1) does not possess a steady state solution.

If one applies the model to describe a train-wheel motion along a rail, the contact between the wheel and the rail is not necessarily continuous. Therefore, the instability should be considered as one of the reasons for the loss of contact.

4. EFFECT OF VISCOSITY ON INSTABILITY AND RESONANCE

As shown in the previous section, there is a critical velocity V_{cr}^{inst} after which instability can occur. If one neglects the viscosity ($\nu \rightarrow 0$) this velocity can be found analytically to give the expression $V_{cr}^{inst} = (4\chi EI/\rho^2 - N/\rho)^{1/4}$. This expression is exactly the same as the one which has been obtained by Kerr [6] for the critical (resonance) velocity V_{cr}^{res} of a constant load moving uniformly along an axially compressed Euler–Bernoulli beam on an elastic foundation. If the load moves with V_{cr}^{res}, the steady state solution of the problem is infinite. Now the following question can arise: Why is it of interest to analyze the instability phenomenon if it can occur only for velocities larger than the critical velocity V_{cr}^{res} leading to resonance? To answer this question, one can now analyze the effect of the viscosity of the foundation on the instability phenomenon (V_{cr}^{inst}) and on the amplitude of the steady state beam vibrations.

First assume that the steady state vibrations of the beam exist. Then the expression for these vibrations can be obtained from equation (7). To this end one can formally apply the inverse Laplace transform to equation (7) taking into account the pole $s = 0$ only (for a proof of this statement see the Appendix). This yields

$$U^{st}(\xi) = -\frac{F}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k\xi)}{D(k,0)}\,\mathrm{d}k. \tag{13}$$

By using the contour integration method equation (13) can be rewritten in the form

$$U^{st}(\xi) = 4\mathrm{i}F\sum_n \left.\frac{k-k_n}{(k-k_1)(k-k_2)(k-k_3)(k-k_4)}\right|_{k=k_n}, \tag{14}$$

where k_n are the roots of the equation $D(k,0) = (1/4)k^4 - (T+\alpha^2)k^2 - \mathrm{i}\nu\alpha k + 1 = 0$ which have a positive imaginary part.

With the aid of equation (14) the steady state displacement of the beam can be easily found by using a standard program for finding the complex roots of polynomials. In Figure 5 the maximum steady state beam deflection (relative to the static deflection)

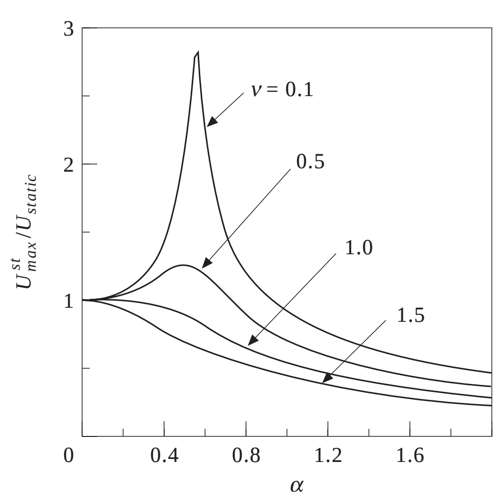

Figure 5. The ratio of the maximum steady state displacement of the beam and the static beam displacement versus the load velocity for different values of the foundation viscosity.

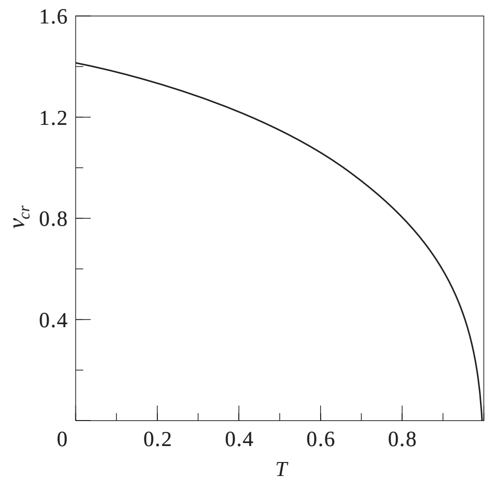

Figure 6. The critical viscosity of the foundation versus the compressional axial force.

as a function of the load velocity is depicted for different values of the foundation viscosity ($T = 0{\cdot}7$). It is shown in the figure that for small viscosities ($v = 0{\cdot}1$) there is a large dynamical amplification of the beam deflection for the critical (resonance) velocity V_{cr}^{res}. When the viscosity grows, the dynamical amplification becomes smaller and there is a critical viscosity of the foundation v_{cr} after which the static displacement is maximal.

The dependency of v_{cr} (critical viscosity) upon the compressional force in the beam is depicted in Figure 6. As a consequence of these results it is interesting to note that the critical viscosity decreases when the axial compressional force increases. This implies that the resonance vibrations of the axially compressed beam will be damped by a smaller viscosity than vibrations of the beam without compression. However, the static displacement of the beam grows when the compressional force increases.

Coming back to the main problem, one can now ask: Can the instability occur for an overcritical viscosity? If so, this phenomenon is of practical importance, since it can take

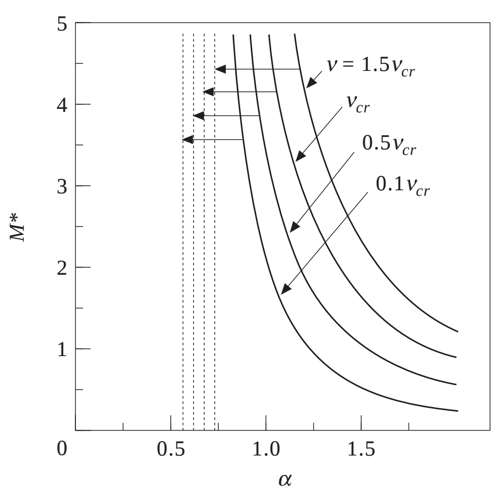

Figure 7. The dependency $M^*(\alpha)$ for different viscosities of the foundation ($T = 0{\cdot}7$).

place for parameters of the system when there is no dynamical amplification due to the load (the instability occurs due to the presence of the mass).

By analyzing the roots of the equation (8) one can give a positive answer to this question. In Figure 7 the lines are depicted (dependencies of M^* versus α) which separate the instability domain from the domain where the beam vibrations are stable. Each line is related to a different value of the foundation viscosity. The domain located above a line is the instability domain. The compressional force is taken as $T = 0{\cdot}7$. It is shown in the figure that the instability still occurs for overcritical viscosities, although the instability starts at larger velocities (for a fixed mass) as the viscosity increases.

Thus, with increase of the foundation viscosity it becomes more and more important to take into account the instability phenomenon. The reason is that the resonance vibration due to the moving load can be effectively damped (even totally for an overcritical viscosity); however, the instability domain is then only moved to the higher (M, α) domain.

5. CONCLUSIONS

The instability of vibrations of a mass moving uniformly along an axially compressed beam on a viscoelastic foundation has been investigated. Instability has been found, implying that the amplitude of the system vibrations grows exponentially in time. It has beem confirmed that the velocity V_{cr}^{inst} at which the instability starts, decreases with an increasing compressional force.

Furthermore, the effect of the foundation viscosity upon V_{cr}^{inst} has been analyzed. It has been shown that when the viscosity increases the instability domain moves towards the region of larger velocities and masses. The instability even occurs for overcritical viscosities of the foundation when there is no dynamical amplification of the steady state vibrations due to resonance.

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APPENDIX

Assume that the problem (1) possesses a steady state solution. Then, evidently, the beam displacement under the constant force P has to be constant in steady state. This implies that $\mathrm{d}^2\tilde{U}^0/\mathrm{d}t^2 = 0$ and the first of equations (1) takes the form (after introduction of the dimensionless variables and parameters)

$$\frac{\partial^2 U^{st}}{\partial \tau^2} + \frac{1}{4}\frac{\partial^4 U^{st}}{\partial y^4} + T\frac{\partial^2 U^{st}}{\partial y^2} + v\frac{\partial U^{st}}{\partial \tau} + U^{st} = -F\delta(y - \alpha\tau). \tag{A1}$$

If the beam motion approaches the steady state (from the point of view of an observer moving with the load) the deflection of the beam will appear static (see, for example, reference [6]). Thus one can reduce the partial differential equation (A1) into an ordinary one by introducing the moving reference system $\xi = y - \alpha\tau$. This gives

$$\frac{1}{4}\frac{\partial^4 U^{st}}{\partial \xi^4} + (T + \alpha^2)\frac{\partial^2 U^{st}}{\partial \xi^2} - \alpha v\frac{\partial U^{st}}{\partial \xi} + U^{st} = -F\delta(\xi) \tag{A2}$$

Now applying the Fourier transform

$$\tilde{W}_k(k) = \int_{-\infty}^{\infty} U^{st}(\xi)\exp(-\mathrm{i}k\xi)\,\mathrm{d}\xi$$

to equation (A2), one obtains

$$\tilde{W}_k = -\frac{F}{k^4/4 - k^2(T + \alpha^2) - \mathrm{i}k\alpha v + 1}. \tag{A3}$$

Finally, inverting equation (A3) yields

$$U^{st}(\xi) = -\frac{F}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}k\xi)\,\mathrm{d}k}{k^4/4 - k^2(T + \alpha^2) - \mathrm{i}k\alpha v + 1},$$

which is identical to equation (13).