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A NOTE ON THE FUNDAMENTAL SHAPE FUNCTION AND FREQUENCY FOR BEAMS UNDER OFF-CENTER LOAD

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1. Introduction

As an example of the application of Rayleigh's method to find the natural frequency of vibrating systems, Timoshenko *et al*. [1, p. 38, 39] consider a simply-supported beam of uniform cross-section loaded at position $x = a$, with a block of weight W (see Figure 1). The static deflection for a light beam carrying a concentrated mass was used in reference [1], rather than the static deflection for a beam with a uniformly distributed load and a concentrated mass. If the distributed load is greater than the concentrated mass, it is advisable to assume the static deflection curve for the beam with a uniformly distributed load for the fundamental mode shape [2, section 3.2]. In fact, Timoshenko's example [1] uses the strength-of-materials closed solution, $y = Wb/(6lEI)[x^3 - (l^2 - b^2)x]$, to which the sine series converges [3, section 2.7.]. Chai and Low [4] confirm that a 100 term series is equivalent to the strength-of-materials expression.

Rayleigh's principle states that a reasonable mode shape satisfies at least the slope and deflection conditions at the ends; it leads to a good approximation for the natural frequency [1, 2]. The accuracy of Rayleigh's method depends on how closely one can predict the dynamic deflection curve. The static deflection curve is often used to approximate the dynamic deflection for the fundamental mode. James *et al*. [5, example 2-9] apply Rayleigh's energy method to determine the frequency of clamped-clamped centrally loaded beam by using the static-deflection curve, $y = P(3lx^2 - 4x^3)/(48EI)$, and a trigonometric function, $y = A[1 - \cos(2\pi x/l)]$. A more detailed comparison was summarized in Table 2.2 of reference [3], in which six deflection functions were used for $w(x)$ for a fixed-ended beam under a load at the center. Also given was the deflection coefficient K in the formula $w_{max} = Pl^3/(KEI)$ [3].

In the present work, four assumed deflection curves are used, one at a time, to obtain the kinetic and potential energies of a simply-supported beam. Their effect on the natural frequency of the vibrating beams under an *off-center* load is investigated.

2. Energy method using different shape functions

Since the potential energy U equals the work done during beam bending, one has

$$U = \tfrac{1}{2}\int M_b \, d\theta = \tfrac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx, \tag{1}$$

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in which M_b is the bending moment, θ the bending angle, EI and y the flexural rigidity and lateral deflection of the beam, respectively. Equation (1) gives U_{max} if the deflection y is the amplitude of the assumed deflection curve.

The maximum kinetic energy of the system, which includes that of the beam (m) and the rigid mass (M), is

$$T = \tfrac{1}{2}\int \dot{y}^2 \, \mathrm{d}m + \tfrac{1}{2}M\dot{y}^2|_{x=a}. \tag{2}$$

After equating T_{max} and U_{max}, the square of the fundamental natural frequency of the beam is given by

$$\omega^2 = EI\int\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \bigg/ \left[\int \dot{y}^2\,\mathrm{d}m + M\dot{y}^2|_{x=a}\right]. \tag{3}$$

Note that $\dot{y} = \omega y$.

As mentioned in the introduction, the accuracy of Rayleigh's method depends on how closely one can predict the deflection curve, $y(x)$. In this work, each of the following four shape functions is incorporated into the energy terms, equations (1) and (2), to obtain the system frequency in equation (3):

$$y_{W1} = Wb\xi/(6lEI)[a(l+b) - \xi^2], \quad \text{for } 0 \leqslant \xi \leqslant a; \tag{4a}$$

$$y_{W2} = Wa\eta/(6lEI)[b(l+a) - \eta^2], \quad \text{for } 0 \leqslant \eta \leqslant b; \tag{4b}$$

$$y_m = wx/(24EI)(x^3 - 2lx^2 + l^3), \quad \text{for } 0 \leqslant x \leqslant l; \tag{5}$$

$$y_t = y_m + y_W, \quad \text{for } 0 \leqslant x \leqslant l; \qquad y_s = A\sin\pi(x/l), \quad \text{for } 0 \leqslant x \leqslant l. \tag{6, 7}$$

Note that y_{W1} and y_{W2} are the deflection curves defined from the left and right ends, respectively. They are static-deflection curves considering the load (W) only. On the other hand, the deflection curve y_m is defined in terms of the distributed beam mass (m) only, while y_t considers both the distributed mass and the load. Note that y_W represents the terms of y_{W1} or y_{W2} in its respective range. A sine function (y_s) is defined in equation (7), where parameter A refers to both the displacement of the beam at $x = a$ and the displacement owing to the load.

It is worth mentioning that the deflection curve (for example, y_W) may not account for both the beam and the load, even though the terms of M and m are always included in equation (2) for the kinetic energy.

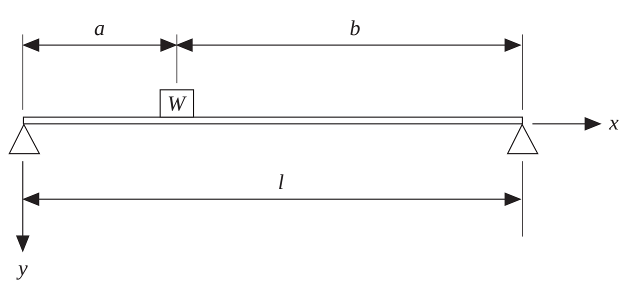

Figure 1. A loaded beam with simply-supported ends.

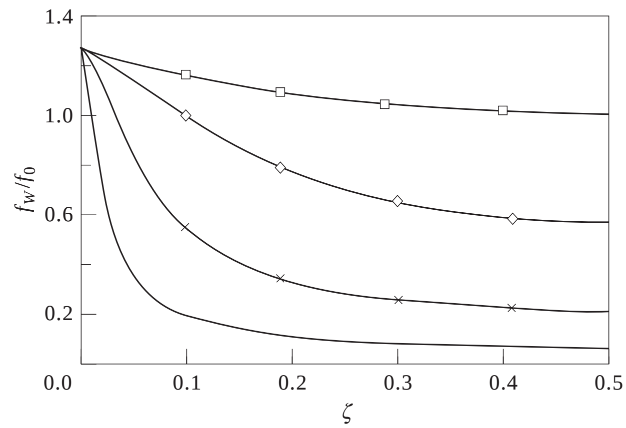

Figure 2. Frequency ratio obtained by using y_W: —□—, $M_r = 0{\cdot}001$; —◇—, $M_r = 1$; —×—, $M_r = 10$; —, $M_r = 100$.

3. RESULTS AND REMARKS

To study the effect of different shape functions on the system natural frequency of the loaded beam shown in Figure 1, let f_W, f_m, f_t and f_s be defined as the associated frequency obtained from equation (3) by using y_W, y_m, y_t and y_s, respectively. In fact, the frequency f_W is equivalent to the expression given in reference [1], $f = (1/2\pi)\sqrt{3lEIg/\{[W + w(\alpha a + \beta b)]a^2b^2\}}$.

Figure 2 gives the frequency ratio (f_W/f_0) as the load is located along the beam's position, $\zeta = x/l$, where f_0 is the frequency of the unloaded beam obtained by using y_m. Only results for a half of the beam ($0 \leqslant \zeta \leqslant 0{\cdot}5$) need to be plotted by virtue of the symmetrical boundary ends. One would expect that the frequency ratio (f/f_0) is less than one as a mass is placed on the beam. Also, the frequency ratio should be unchanged (i.e., $f/f_0 = 1$) if the mass is placed at the beam's ends.

Four curves with different mass ratios ($M_r = M/m$) are shown in Figure 2 to illustrate the effect of the load's mass on the system frequency. Another set of curves with y_m, y_s and y_t is shown in Figures 3 and 4, respectively. The effect of heavy loads is investigated. The frequency ratio (f/f_∞) is defined and plotted in Figure 5 as a function of the load's position. Note that f_∞ is the frequency with $M_r = 100$ for $y = y_W$.

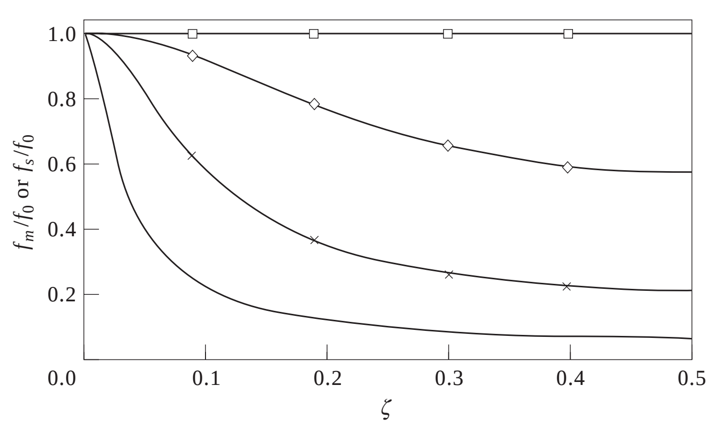

Figure 3. Frequency ratio obtained by using y_m and y_s. Key as in Figure 2.

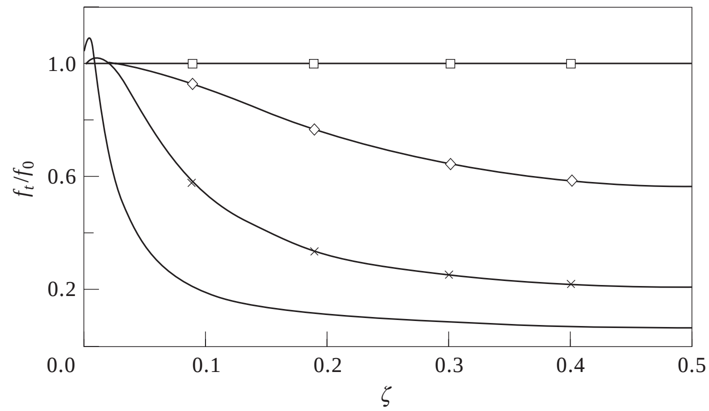

Figure 4. Frequency ratio obtained by using y_t. Key as in Figure 2.

In view of Figures 2–5, the following points can be made:

(1) The system frequency decreases as the mass of the load increases, while it increases as the load is moved away from the beam's center. (2) As shown in Figure 2, the expression by Timoshenko *et al*. [1] is only applicable to a beam system with heavy loads. The frequency ratio (f_W/f_0) is greater than one with a zero or large mass placed at the ends. Results using other shape functions show a factor of one for such a trivial case. In fact, the frequency ratio with y_W is greater than one if the load is placed near the ends. (3) The curves using trigonometric functions (y_s) are almost identical to those using a distributed function (y_m), as can be seen in Figures 3 and 5. (4) The curves obtained by the total deflection y_t are similar to those obtained by y_m, except for cases with a load near the ends. As can be seen in Figure 4, the frequency ratio (f_t/f_0) is greater than one for a large mass near the ends. This is due to the predominant contribution from the load W on y_t. (5) As shown in Figure 5, the frequency ratio (f_t/f_∞) quickly converges to unity, whereas the other two curves (f_m/f_∞ and f_s/f_∞) approach one at the center after they rise above one. (6) In all the cases, the frequency ratios using various shape functions are identical if the load is placed at the beam's center, $\zeta = 0{\cdot}5$. (7) It is apparent that the curves (y_t) that consider both the beam and the load give the best results, except when the load is placed very near the ends.

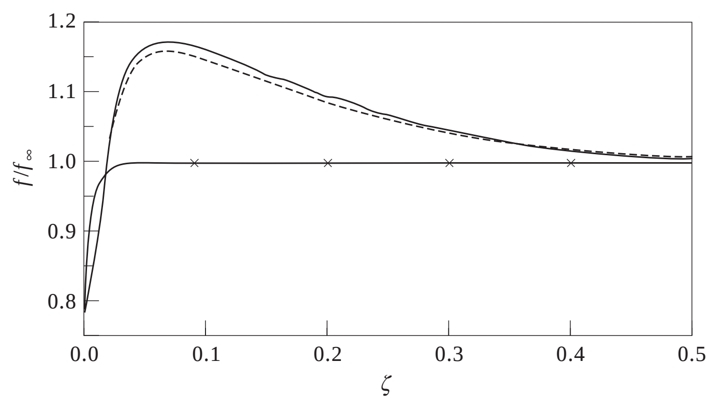

Figure 5. Frequency ratio for heavy mass ($M_r = 100$): —, f_s/f_∞; – – –, f_m/f_∞; —×—, f_t/f_∞.

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