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OSCILLATIONS OF A TIMOSHENKO SANDWICH CANTILEVER BEAM

I. K. Silverman
1155 *Ask Street*, *Denver*, *CO* 80220, *U.S.A.*

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1. INTRODUCTION

An inverse approach, based on Hamilton's Principle, to the study of the dynamical behavior of a beam type body in plane stress [1] leads to the Euler governing equations for two unknown displacement functions $\alpha(x)$, $\beta(x)$. Simultaneously the associated natural boundary conditions are defined. Except for the simply supported beam a formal solution of the Euler equations appears intractable. This difficulty leads to the use of a Galerkin type of approximate solution for the practically important eigenvalues. The Galerkin method deals with an integral which has a zero value when the solutions of the Euler equations are known. When non-solutions are used to evaluate the integral a weighted error over the range $(0, l)$ is equated to zero to meet the requirement of a minimum.

The Galerkin method consists of choosing a class of admissible co-ordinate functions so that the forced and natural conditions are satisfied or that the terms at the limits arising from the variational process vanish. Since the chosen functions are not solutions of the Euler equations they are considered to be pseudo-eigenfunctions. It is possible to choose the co-ordinate functions on the basis of knowledge of the eigenfunctions of a system with slightly different characteristics but which satisfy the same boundary conditions. They preserve the inherent peaks and nodes of known eigenfunctions. The eigenfunctions of the classical Bernoulli–Euler (B–E) model serve as a fruitful source. This is especially true for end supported beams. However for the cantilever beam (Figure 1) the present anlaysis furnishes natural boundary conditions at the free end which are considerably different from those of the B–E theory. Compare equations (11) of reference [1].

2. THE PSEUDO-EIGENFUNCTIONS FOR α(X)AND β(X)

The pseudo-eigenfunctions $f_{\alpha m}(x)$ for the dominant $\alpha(x)$ which represents the pure bending configuration is taken as identical with that of the B–E solution with identical peaks and nodes. It satisfies the forced boundary conditions at $x = 0$. For the free end $x = l$, it yields the values

$$f''_{\alpha m}(l) = f'''_{\alpha m}(l) = 0.$$

The pseudo-eigenfunctions $f_{\alpha m}(x)$ for the cantilever are taken [2] as

$$f_{\alpha m}(x) = (\mathrm{ch}\,\beta_m x - \cos\beta_m x) - \alpha_m(\mathrm{sh}\,\beta_m x - \sin\beta_m x), \tag{1}$$

where $\beta_m l$ are eigenvalues and

$$\alpha_m = (\mathrm{sh}\,\beta_m l - \sin\beta_m l)/(\mathrm{ch}\,\beta_m l + \cos\beta_m l).$$

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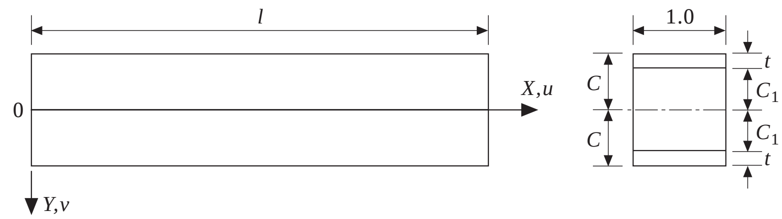

Figure 1. Co-ordinate system and dimensions.

The pseudo-eigenfunction for $\beta_m(x)$ denoted by $f_{\beta m}(x)$ must satisfy the forced boundary conditions at $x = 0$. It must also furnish derivative values at $x = l$ so that the limit terms vanish. Accordingly it is required that

$$f_{\beta m}(0) = f'_{\beta m}(0) = f'_{\beta m}(l) = f''_{\beta m}(l) = f'''_{\beta m}(l) = 0.$$

To act as a correction to the dominant $\alpha(x)$ function it must additionally have the same number and location of nodes. A proposed pseudofunction for $\beta(x)$ is the trigonometrical polynomial

$$f_{\beta m}(x) = c_0 + \sum_{i=1}^{m+1} c_i \cos(i\pi x/l), \tag{2}$$

where m is the mode number and $i = 1, 2, 3, \ldots$. The conditions $\beta'(0) = \beta'(l) = \beta'''(l) = 0$ are identically satisfied for all values of c_i. If a node exists at $x = x_0$, a supplementary forced condition is $\beta(x_0) = 0$. Table 1 gives the values of the c_0 and c_i for the first three modes.

3. THE TIMOSHENKO SANDWICH CANTILEVER BEAM

The pseudo-eigenfunctions of the Galerkin method for the cantilever beam are given by equations (1) and (2) of section 2. With the vanishing of the terms at the limits, equation (11) of reference [1] furnishes

$$\int\int \{L_1(x) - c^2 L_2(\beta)\}\delta\alpha \, \mathrm{d}x \, \mathrm{d}t = 0, \qquad \int\int \{L_3(\alpha) - c^2 L_4(\beta)\}\delta\beta \, \mathrm{d}x \, \mathrm{d}t = 0,$$

where $\alpha = C_m f_{\alpha m}(x) \cos p_m t$; $\beta = D_m f_{\beta m}(x) \cos p_m t$.

The condition for non-zero values for C_m and D_m is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = 0, \tag{3}$$

TABLE 1

Trigonometric polynomial coefficients c_0, c_i *of equation* (2)

m	c_0	c_1	c_2	c_3	c_4
1	−1·25	1·0	0·25	—	—
2	1·5056291	1·0	−1·6577433	−0·8478859	—
3	−1·2356298	1·0	−0·728263	0·4603693	0·5033524

where

$$a_{11} = \int_0^l [\lambda_3 I f_{\alpha m} f_{\alpha m}^{IV} + \mu_3 I p_m^2 f_{\alpha m} f''_{\alpha m} - A\mu_1 p_m^2 f_{\alpha m}^2]\,\mathrm{d}x,$$

$$a_{12} = c^2\left(\int_0^l \left[A\mu_1 p_m^2 f_{\alpha m} f_{\beta m} - \frac{\mu_5}{5} I\left[p_m^2 f_{\alpha m} f''_{\beta m} - \frac{\lambda_5}{5} I f_{\alpha m} f_{\beta m}^{IV}\right] \mathrm{d}x\right),$$

$$a_{21} = \int_0^l [\lambda_5 I f_{\alpha m}^{IV} f_{\beta m} + \mu_5 I p_m^2 f''_{\alpha m} f_{\beta m} - 5A\mu_1 p_m^2 f_{\alpha m} f_{\beta m}]\,\mathrm{d}x,$$

$$a_{22} = c^2\left(\int_0^l \left[-\tfrac{5}{21}\lambda_7 f_{\beta m}^{IV} f_{\beta m} - (\tfrac{5}{21}\mu_1 I p_m^2 - \tfrac{8}{3}AG)\right) f_{\beta m} f''_{\beta m} + 5A\mu_1 p_m^2 f_{\beta m}^2\right] \mathrm{d}x\right),$$

and

$$I = 2c^3/3, \quad A = 2c, \quad \gamma = c_1/c, \quad I/A = r^2 = c^2/3 \quad \text{(see Figure 1)},$$

$$\lambda_i = E_f(1 - \gamma^i + \eta\gamma^i), \quad \mu_i = \rho_f(1 - \gamma^i + \varepsilon\gamma^i), \qquad \eta = E_c/E_f, \qquad \varepsilon = \rho_c/\rho_f,$$

$$G = G_f[1 + \tfrac{15}{8}(G_c/G_f - 1)(\gamma - \tfrac{2}{3}\gamma^3 + \gamma^5/5)].$$

E_c, E_f, G_c, G_f are Young's and shear moduli of core and face, respectively; ρ_c and ρ_f are mass densities.

4. APPLICATION TO STEEL ALUMINUM BEAM

The procedure for determining the natural frequencies from equation (3) is illustrated in the following for the first mode. The pseudofunction $f_{\alpha m}(x)$ is given by equation (1) with $m = 1$. Numerical values are taken from Table 2.

Since the function $f_{\alpha m}(x)$ is orthogonal

$$\int_0^l f_{\alpha m}^2\,\mathrm{d}x = l; \qquad \int_0^l f_{\alpha m} f''_{\alpha m}\,\mathrm{d}x = \alpha_m\beta_m(2 - \alpha_m\beta_m l) = \phi_m = 0{\cdot}8580652/l$$

$$\int_0^l f_{\alpha m} f_{\alpha m}^{IV}\,\mathrm{d}x = \beta_m^4 l = 12{\cdot}362364/l^3.$$

The pseudofunction $f_{\beta m}(x)$ for the first mode is

$$f_{\beta m}(x) = -1{\cdot}25 + \cos(\pi x/l) + 0{\cdot}25\cos(2\pi x/l)$$

TABLE 2

Values of $\phi_m l$ and $(\beta_m l)^4$ for first three modes

m	1	2	3
$\phi_m l$	0·8580652	−13·294271	−45·904225
$(\beta_m l)$	12·362364	485·5188	3806·5462

and the integrals involving $f_{\beta m}(x)$ and its derivatives lead to $\int_0^l \cos^2(i\pi x/l)\,\mathrm{d}x = l/2$ since the trigonometric functions are also orthogonal. Integrals involving the products of trigonometric and hyperbolic functions are of the following type.

$$\beta_m(1 + i^2\pi^2/\beta_m^2 l^2)\int_0^l \cos\frac{i\pi x}{l}\,\mathrm{sh}\,\beta_m x\,\mathrm{d}x = (-1)^i(\mathrm{ch}\,\beta_m l + (-1)^{i+1})$$

$$\beta_m(1 + i^2\pi^2/\beta_m^2 l^2)\int_0^l \cos\frac{i\pi x}{l}\,\mathrm{ch}\,\beta_m x\,\mathrm{d}x = (-1)^i\,\mathrm{sh}\,\beta_m l.$$

Computation yields the following:

$$\int_0^l f_\beta f_\alpha\,\mathrm{d}x = 2\cdot2880l; \qquad \int_0^l f_\alpha f_\beta''\,\mathrm{d}x = -12\cdot4493/l; \qquad \int_0^l f_\alpha f_\beta^{IV}\,\mathrm{d}x = 104\cdot11851/l^3$$

$$\int_0^l f_\beta f_\alpha^{IV}\,\mathrm{d}x = 2\cdot2880\beta_m^4/l^3; \qquad \int_0^l f_\beta f_\alpha''\,\mathrm{d}x = 6\cdot4734/l; \qquad \int_0^l f_\beta f_\beta^{IV}\,\mathrm{d}x = \pi^4/l^3$$

$$\int_0^l f_\beta f_\beta''\,\mathrm{d}x = -\tfrac{5}{8}\pi^2/l; \qquad \int_0^l f_\beta^2\,\mathrm{d}x = \tfrac{67}{32}l.$$

It follows that

$$a_{11} = [\lambda_3 I\beta_m^4 l + p^2(0\cdot8580\mu_3 I/l - A\mu_1 l)],$$

$$a_{12} = c^2[p^2(2\cdot2880A\mu_1 l + 2\cdot4899\mu_5 I/l) - 20\cdot8237\lambda_3 I/l^3],$$

$$a_{21} = [p^2(6\cdot4731\mu_5 l/l - 11\cdot4398A\mu_1 l) + 2\cdot2880\beta_m^4 l\lambda_5 I/l],$$

$$a_{22} = c^2\left[-\tfrac{5}{21}\lambda_7 I\frac{\pi^4}{l^3} + p^2\left(\tfrac{335}{32}Al\mu_1 + \tfrac{25}{168}\frac{\pi^2}{l}\mu_7 I\right) - \tfrac{5}{3}AG\pi^2/l\right].$$

A meaningful value for p_m is obtained from

$$a_{11}a_{22} - a_{12}a_{21} = 0 = ap_m^4 + bp_m^2 + c = 0, \tag{4}$$

or

$$p_m = [(-b/a \pm \sqrt{(b/a)^2 - 4(c/a)})/2]^{1/2},$$

where $b/a = (V_f/l)^2(N_1/D)$, $c/a = (V_f/l)^4(r/l)^2(\beta_m l)^4(\lambda_3/E_f)N_2/D$, $V_f\sqrt{E_f/\rho_f}$. The following values are taken for a steel-aluminum beam: $G_f/E_f = 3/8$, $\eta = E_c/E_f = 1/3$, $\varepsilon = \rho_c/\rho_f = 1/3$, $G_c/C_f = 1/3$. For $\gamma = 0\cdot95$ i.e., for a face thickness of $0\cdot05c$ the values of N_1, N_2 and D are

$$N_1 = 14\cdot6887(r/l)^4 - 30\cdot5135(r/l)^2 + 0\cdot7544,$$

$$N_2 = 13\cdot6723(r/l)^2 - 2\cdot0575, \; D = -3\cdot4893(r/l)^4 + 3\cdot5504(r/l)^2 + 2\cdot1115.$$

Table 3 gives the values of pl/V_f for various r/l. It also contains the values of pl/V_f for the second and third modes which have been computed in the manner indicated for the first mode.

TABLE 3

Values of pl/V_f and p/p_0 for a cantilever steel-aluminum beam $\gamma = 0{\cdot}95$

	First mode		Second mode		Third mode	
r/l	pl/V_f	p/p_0	pl/V_f	p/p_0	pl/V_f	p/p_0
0·025	0·0948	0·9979	0·5722	0·9670	1·5150	0·9807
0·05	0·1886	0·9924	1·0448	0·8773	2·4447	0·7331
0·075	0·2800	0·9824	1·4191	0·7946	2·9812	0·5960
0·10	0·3697	0·9728	1·7340	0·7304	3·3478	0·5020

TABLE 4

Values of pl/V_f and p/p_0 for a cantilever steel-aluminium beam $\gamma = 0{\cdot}80$

	First mode		Second mode		Third mode	
r/l	pl/V_f	p/p_0	pl/V_f	p/p_0	pl/V_f	p/p_0
0·025	0·1042	0·9981	0·6114	0·9342	1·4137	0·7715
0·05	0·2075	0·9933	1·0626	0·8119	1·9398	0·5282
0·075	0·3098	0·9859	1·3934	0·7097	2·2315	0·4059
0·10	0·4078	0·9763	1·7310	0·6613	2·5021	0·3414

Table 4 gives the corresponding values for the case of $\gamma = 0{\cdot}8$.

In both Tables 3 and 4, $p_0 = k^2\sqrt{\lambda_3 I/\mu_1 A}$, where kl are the classical values (1·8751, 4·6940, 7·8548).

5. THE HOMOGENEOUS CASE

For the homogeneous case $\eta = \varepsilon = \gamma =$ unity, $E_f = E_c = E$, $G_f = G_c = G$, $V_f = V_1 = \sqrt{E/\rho}$, $\lambda_i = E$, $\mu_i = \rho$. For the first mode the values of $a_{11}, a_{12}, a_{21}, a_{22}$ are obtained by substituting E for all λ_i and ρ for all μ_1. The values of N_1, N_2, and D are

$$N_1 = -62{\cdot}698(r/l)^4 + 155{\cdot}6273(r/l)^2 - 6{\cdot}1685, \qquad N_2 = 24{\cdot}4559(r/l)^2 - 6{\cdot}1684,$$

$$D = 14{\cdot}8569(r/l)^4 - 26{\cdot}1874(r/l)^2 - 15{\cdot}7051.$$

Table 5 gives the numerical values for the first three modes, where $p_0 = (kl)^2(r/l)(V_1/l)$ and (k/l) are the classical values (1·8751, 4·6940, 7·8548).

TABLE 5

Values of pl/V_f and p/p_0 for a cantilever homogeneous beam

	First mode		Second mode		Third mode	
r/l	pl/V_f	p/p_0	pl/V_f	p/p_0	pl/V_f	p/p_0
0·025	0·0881	0·996	0·5313	0·9645	1·4375	0·9320
0·05	0·1763	0·989	0·9739	0·8839	2·3645	0·7740
0·075	0·2570	0·976	1·3179	0·7975	2·8932	0·6256
0·10	0·3382	0·960	1·5943	0·7235	3·2492	0·5258

6. REMARKS

As in previous results [1], the increase in face thickness led to an increase in natural frequencies for the first mode but not universally in the second or third mode.

REFERENCES

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