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NON-LINEAR DYNAMIC ANALYSIS OF ORTHOTROPIC OPEN CYLINDRICAL SHELLS SUBJECTED TO A FLOWING FLUID

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A theory is presented to predict the influence of non-linearities associated with the wall of the shell and with the fluid flow on the dynamics of elastic, thin, orthotropic and non-uniform open cylindrical shells submerged and subjected simultaneously to an internal and external fluid. The open shells are assumed to be freely simply supported along their curved edges and to have arbitrary straight edge boundary conditions. The method developed is a hybrid of thin shell theory, fluid theory and the finite element method. The solution is divided into four parts. In part one, the displacement functions are obtained from Sander's linear shell theory and the mass and linear stiffness matrices for the empty shell are obtained by the finite element procedure. In part two, the modal coefficients derived from the Sanders–Koiter non-linear theory of thin shells are obtained for these displacement functions. Expressions for the second and third order non-linear stiffness matrices of the empty shell are then determined through the finite element method. In part three, a fluid finite element is developed; the model requires the use of a linear operator for the velocity potential and a linear boundary condition of impermeability. With the non-linear dynamic pressure, we develop in the fourth part three non-linear matrices for the fluid. The non-linear equation of motion is then solved by the fourth-order Runge–Kutta numerical method. The linear and non-linear natural frequency variations are determined as a function of shell amplitudes for different cases.

1. INTRODUCTION

The analysis of thin shells containing flowing fluid has been the focus of many investigations. Most of these studies have involved linear analysis of thin shells both with and without interaction between the structure and the surrounding fluid medium. Results proved to be satisfactory where wall deflections of the shell are very small compared to the wall thickness [1–11]. In several practical reports, however, the linear analysis was not sufficiently accurate for satisfactory design. In those cases, a non-linear analysis was required.

Several methods have been developed for the analysis of dynamic, non-linear thin cylindrical shells. Among these were Galerkin's method [12–14], the small perturbation method [15, 16], the Rayleigh–Ritz method [17], the modal expansion method [18, 19], the finite element method [20–22] and the hybrid finite element method [23].

The finite element method appears to be ideally suited to the analysis of complex shell structures. Numerous general computer programs are available for industrial use for the linear and non-linear analysis, where the displacement functions of the finite elements used are assumed to be polynomial. Precise prediction of both the high and the low frequencies requires the use of a great many elements in the classical finite element method.

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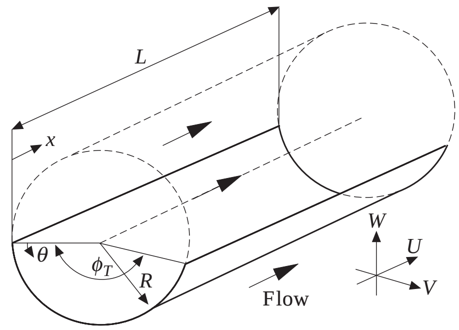

Figure 1. The open cylindrical shell geometry.

In order to achieve this, the present study presents a general approach to the non-linear analysis of elastic, thin, orthotropic and circumferentially non-uniform open cylindrical shells submerged in liquid under flow or no-flow condition (Figure 1). We investigate the effect of non-linearities associated with the wall of the shell and with fluid flow on the natural frequencies of an interactive fluid–shell system. The shells are assumed to be freely simply supported along their curved edges and to have arbitrary straight edge boundary conditions. The finite element method is employed, but it is a hybrid, a combination of the finite element method, shell theory and fluid theory. This choice allows us to derive the displacement functions from the shell's equations of equilibrium and, furthermore, the mass, stiffness and damping matrices for the shell and the fluid element.

The analytical solution involves four steps.

(1) Using the linear strain–displacement and stress–strain relationships which are inserted into Sanders' equations of equilibrium [24], we determine the displacement functions by solving the linear equation system. We then determine the mass and linear stiffness matrices for an empty finite element and assemble the matrices for the complete shell.

(2) Using strain–displacement relationships from the Sanders–Koiter non-linear theory [25, 26], the modal coefficients are obtained from the displacement functions. The second and third order non-linear stiffness matrices for an empty finite element are then calculated by precise analytical integration with respect to modal coefficients.

(3) To account for the effect of the fluid on the structure, a panel finite fluid element bounded by two nodal lines is considered. By solving the linear equations of motion for the fluid element, an expression for linear fluid pressure as a function of the displacement of the element is obtained. Analytical integration for the pressure distribution along the element yields three components: the mass, linear stiffness and linear damping matrices for a fluid element.

(4) With the non-linear dynamic pressure, we develop in the fourth part three non-linear matrices for the fluid: stiffness, damping and combination of the two.

The linear and non-linear natural vibration frequency ratio is then obtained by solving the non-linear equations of motion.

2. DISPLACEMENT FUNCTIONS

Sanders' [24] linear equations for thin cylindrical shells, in terms of axial, tangential and radial displacements (U, V, W) of the mean surface of the shell (Figure 1) and in terms of elements p_{ij} of the orthotropic matrix of elasticity $[P]$ are

$$L_i(U, V, W, p_{ij}) = 0, \tag{1}$$

where $L_i(i = 1, 2, 3)$ are three linear differential operators. These equations, where inertia terms are not included, are given in Appendix A.

The shell is subdivided into several finite elements defined by two nodes i and j and by components U, V, W and $\mathrm{d}W/\mathrm{d}\theta$, representing axial, tangential and radial displacements and the rotation, respectively (Figure 2).

The displacement functions are assumed to be

$$U(x, \theta) = A\,\mathrm{e}^{\eta\theta}\cos(m\pi x/L), \qquad V(x, \theta) = B\,\mathrm{e}^{\eta\theta}\sin(m\pi x/L),$$
$$W(x, \theta) = C\,\mathrm{e}^{\eta\theta}\sin(m\pi x/L), \tag{2}$$

where m is the axial mode and η is a complex number.

Substituting equation (2) into equations of motion (1), a system of three homogeneous linear functions of constants A, B and C are obtained. For the solution to be non-trivial, the determinant of this system must be equal to zero. This brings us to the following characteristic equation in η:

$$h_8\eta^8 + h_6\eta^6 + h_4\eta^4 + h_2\eta^2 + h_0 = 0, \tag{3}$$

where h_0, h_2, h_4, h_6 and h_8 are listed in Appendix B.

Each root η of this equation yields a solution to the linear equations of motion (1). The complete solution is obtained by adding the eight solutions independently with the constants A_p, B_p and C_p ($p = 1, \ldots, 8$). The constants A_p, B_p and C_p are not independent. We can therefore express A_p and B_p as a function of C_p, for example,

$$A_p = \alpha_p C_p, \quad B_p = \beta_p C_p, \qquad p = 1, \ldots, 8. \tag{4}$$

The values of α_p and β_p can be obtained from linear system (1) by introducing

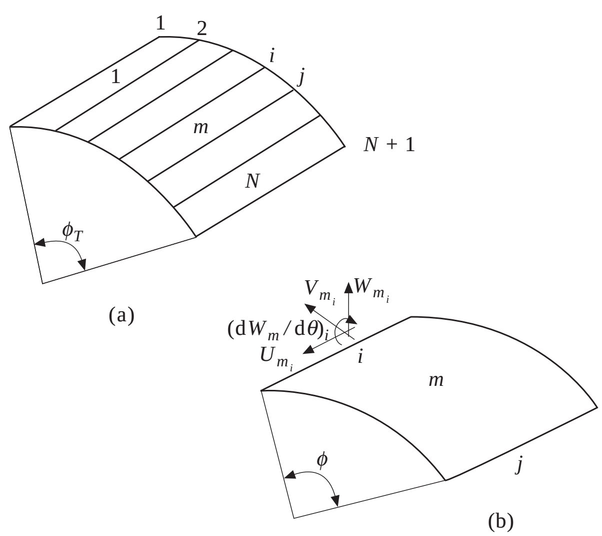

Figure 2. (a) Finite element idealization. (b) Nodal displacements at node i, where U_{mi}, V_{mi} and W_{mi} are, respectively, the axial, tangential and radial displacements; $(\mathrm{d}W_m/d\theta)_i$ is the rotation and ϕ is the opening angle for one finite element.

relations (4). Substituting expressions (4) into equations (2), the displacements $U(x, \theta)$, $V(x, \theta)$ and $W(x, \theta)$ can then be expressed in conjunction with the eight C_p constants. We then have

$$\{U(x, \theta) W(x, \theta) V(x, \theta)\}^{\mathrm{T}} = [T_m][R]\{C\}, \tag{5}$$

where $[T_m]$ and $[R]$ are matrices given in Appendix C and $\{C\}$ is an eighth order vector of the C_p constants.

To determine the eight C_p constants, it is necessary to formulate eight boundary conditions for the finite elements. The axial, tangential and radial displacements, as well as rotation, will be specified for each node. The elements which have two nodes and eight degrees of freedom will have $i(\theta = 0)$ and $j(\theta = \phi)$ as nodal displacements at the boundaries:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \left\{ U_i W_i \left(\frac{\mathrm{d}W}{\mathrm{d}\theta}\right)_i V_i U_j W_j \left(\frac{\mathrm{d}W}{\mathrm{d}\theta}\right)_j V_j \right\}^{\mathrm{T}} = [A]\{C\}, \tag{6}$$

where the terms of matrix $[A]$, given in Appendix C, are obtained from matrix $[R]$ by successively setting $\theta = 0$ and $\theta = \phi$.

Multiplying equation (6) by $[A^{-1}]$ and substituting into equations (5) we obtain

$$\begin{Bmatrix} U(x, \theta) \\ W(x, \theta) \\ V(x, \theta) \end{Bmatrix} = [T_m][R][A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}, \tag{7}$$

where the matrices $[T_m]$, $[R]$ and $[A]$ are given in Appendix C. $[N]$ represents the displacement functions matrix.

3. MASS AND LINEAR STIFFNESS MATRICES FOR AN EMPTY ELEMENT

Introducing the displacement functions (equation (7)) into the linear deformation vector $\{\varepsilon_L\}$ (Sanders [24]), we obtain

$$\{\varepsilon_L\} = \begin{bmatrix} [T_m] & [O] \\ [O] & [T_m] \end{bmatrix} [Q][A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}, \tag{8}$$

where the matrices $[A]$ and $[Q]$ are given in Appendix C.

For an orthotropic laminated material, the stress resultants may be expressed as

$$\{\sigma\} = \{N_{xx} \;\; N_{\theta\theta} \;\; \overline{N}_{x\theta} \;\; M_{xx} \;\; M_{\theta\theta} \;\; \overline{M}_{x\theta}\}^{\mathrm{T}} = [P]\{\epsilon_L\}, \tag{9}$$

where $[P]$ is the elasticity matrix, in which the general term is designated by p_{ij}. It may be written as

$$[P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix}. \tag{10}$$

Referring to equation (8), the stress vector (9) may be rewritten as

$$\{\sigma\} = [P][B]\begin{Bmatrix}\delta_i \\ \delta_j\end{Bmatrix}. \tag{11}$$

The mass and linear stiffness matrices can then be expressed as

$$[m_s] = \rho_s t \iint [N^T][N]\,\mathrm{d}A, \qquad [k_s^{(L)}] = \iint [B^T][P][B]\,\mathrm{d}A, \tag{12}$$

where $\mathrm{d}A = R\,\mathrm{d}x\,\mathrm{d}\theta$, ρ_s is the density of the shell, t is its thickness, $[P]$ is the elasticity matrix given by equation (10), and the matrices $[N]$ and $[B]$ are derived from equations (7) and (8), respectively. The matrices $[m_s]$ and $[k_s^{(L)}]$ were obtained analytically by carrying out the necessary matrix operations and integration over x and θ in equation (12). These matrices are also given in Appendix C.

4. NON-LINEAR STIFFNESS MATRICES FOR AN EMPTY ELEMENT

The non-linear Sanders–Koiter [25, 26] theory for thin shells describes the behaviour of open cylindrical shells. This theory is derived by approximation from the three-dimensional elasticity equation. In common with linear theory, it is based on Love's "first approximation" but the assumption concerning the order of magnitude of the bending has been modified. The displacement gradients are small and the squares of the rotation do not exceed the reference surface deformation in order of magnitude.

The following approach, developed by Radwan and Genin [18], is used with particular attention to geometric non-linearities. The coefficients of the modal equations are obtained through the Lagrange method. Thus, the non-linear stiffness matrices, once calculated, are overlaid onto the linear system. Before we embark on formulation, however, a brief summary of the method is in order.

(1) Shell displacements are expressed as generalized product coordinate sums and spatial functions:

$$u = \sum_i q_i(t)U(x,\theta), \qquad v = \sum_i q_i(t)V(x,\theta), \qquad w = \sum_i q_i(t)W(x,\theta), \tag{13}$$

where the functions $q_i(t)$ are the generalized co-ordinates and the spatial functions U, V and W are given by equation (2).

(2) The non-linear Sanders–Koiter [25, 26] theory for thin shells postulated differences in the first and second fundamental forms between the reference surfaces, deformed and non deformed, as deformation measures in elongation and bending, respectively. The deformation vector is written as a function of the generalized coordinates by separating the linear portion from the non-linear:

$$\{\varepsilon\} = \{\varepsilon_L\} + \{\varepsilon_{NL}\} = \{\varepsilon_{xx}\;\; \varepsilon_{\theta\theta}\;\; 2\varepsilon_{x\theta}\;\; \kappa_{xx}\;\; \kappa_{\theta\theta}\;\; 2\kappa_{x\theta}\}^{\mathrm{T}}, \tag{14}$$

where subscripts "L" and "NL" mean "linear" and "non-linear", respectively; ε_{xx}, $\varepsilon_{\theta\theta}$, and $\varepsilon_{x\theta}$ represent the deformations of reference surface and κ_{xx}, $\kappa_{\theta\theta}$ and $\kappa_{x\theta}$

are the changes in curvature and torsion of the reference surface. In general, these terms can be expressed as

$$\varepsilon_{xx} = \sum_j a_j q_j + \sum_j \sum_k A_{jk} q_j q_k, \quad \varepsilon_{\theta\theta} = \sum_j b_j q_j + \sum_j \sum_k B_{jk} q_j q_k, \quad \varepsilon_{x\theta} = \sum_j c_j q_j + \sum_j \sum_k C_{jk} q_j q_k, \tag{14a}$$

$$\kappa_{xx} = \sum_j p_j q_j, \qquad \kappa_{\theta\theta} = \sum_j s_j q_j, \qquad \kappa_{x\theta} = \sum_j t_j q_j. \tag{14b}$$

For a cylindrical shell, the expressions for $\{\varepsilon_L\}$ and $\{\varepsilon_{NL}\}$ are given in Appendix D.

(3) In the usual way, using equations (13) and Hamilton's principle leads to Lagrange's equations of motion in the generalized co-ordinates $q_i(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i, \tag{14c}$$

where T is the total kinetic energy, V is the total elastic strain energy of deformation and the Q_i's are the generalized forces.

(4) After developing the total kinetic and elastic strain energy and substituting equation (14) into Lagrange equations (14c), we obtain the dynamic behaviour of an empty cylindrical shell, in the absence of external loads, in terms of the following non-linear modal equations:

$$\sum_r m_{pr} \ddot{\delta}_r + \sum_r k_{pr}^{(L)} \delta_r + \sum_r \sum_s k_{prs}^{(NL2)} \delta_r \delta_s + \sum_r \sum_s \sum_q k_{prsq}^{(NL3)} \delta_r \delta_s \delta_q = 0, \qquad p = 1, 2, \ldots, \tag{15}$$

where m_{pr} and $k_{pr}^{(L)}$ are the terms of mass and linear stiffness matrices given by equation (12); the terms $k_{prs}^{(NL2)}$ and $k_{prsq}^{(NL3)}$ which represent the second and third non-linear stiffness may be obtained by the following integrals in the case of the laminated orthotropic open cylindrical shell:

$$k_{prs}^{(NL2)} = \iint \{p_{11} A_{prs} + p_{22} B_{prs} + p_{12}(D_{prs} + E_{prs}) + p_{33} C_{prs}\} \, \mathrm{d}A, \tag{16}$$

$$k_{prsq}^{(NL3)} = \iint \{p_{11} A_{prsq} + p_{22} B_{prsq} + p_{12}(D_{prsq} + E_{prsq}) + p_{33} C_{prsq}\} \, \mathrm{d}A, \tag{17}$$

where $\mathrm{d}A = R \, \mathrm{d}x \, \mathrm{d}\theta$, p_{ij} are the terms of the elasticity matrix $[P]$, and the terms A_{prs}, B_{prs}, C_{prs}, D_{prs}, E_{prs} and A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq}, E_{prsq} represent the coefficients of the modal equations mentioned in step (4). These coefficients are given by

$$\begin{aligned}
A_{prs} &= a_p A_{rs} + a_r A_{sp} + a_s A_{pr}, & A_{prsq} &= 2A_{pq} A_{rs}, \\
B_{prs} &= b_p B_{rs} + b_r B_{sp} + b_s B_{pr}, & B_{prsq} &= 2B_{pq} B_{rs}, \\
C_{prs} &= c_p C_{rs} + c_r C_{sp} + c_s C_{pr}, & C_{prsq} &= 2C_{pq} C_{rs}, \\
D_{prs} &= a_r B_{sp} + a_s B_{pr} + b_p A_{rs}, & D_{prsq} &= 2A_{pq} B_{rs}, \\
E_{prs} &= b_r A_{sp} + b_s A_{pr} + a_s B_{rs}, & E_{prsq} &= 2B_{pq} A_{rs},
\end{aligned} \tag{18}$$

with

$$a_p = U_{p,x}, \qquad b_p = \frac{1}{R}(V_{p,\theta} + W_p), \qquad c_p = \tfrac{1}{2}\left(\frac{U_{p,\theta}}{R} + V_{p,x}\right), \tag{19}$$

$$A_{pq} = \frac{1}{8R^2}(RV_{p,x} - U_{p,\theta}) \cdot (RV_{q,x} - U_{q,\theta}) + \tfrac{1}{2} W_{p,x} W_{q,x}, \tag{20}$$

$$B_{pq} = \frac{1}{8R^2}(RV_{p,x} - U_{p,\theta}) \cdot (RV_{q,x} - U_{q,\theta}) + \frac{1}{2R^2}(W_{p,\theta} - V_p) \cdot (W_{q,\theta} - V_q), \tag{21}$$

$$C_{pq} = \frac{1}{4R}(W_{p,x} W_{q,\theta} - W_{q,x} W_{p,\theta}) - \frac{1}{4R}(V_p W_{q,x} + V_q W_{p,x}), \tag{22}$$

where U, V and W are spatial functions determined by equation (5):

In equations (18)–(22), the subscripts "p, q", "p, r, s" and "p, r, s, q" represent the coupling between two, three and four modes, respectively.

Introducing equation (5) into equation (19), we obtain

$$a_p = C_p a'_p \, e^{\eta_p \theta}, \qquad a'_p = a_p^{(1)} \sin \bar{m}x, \qquad a_p^{(1)} = -\bar{m}\alpha_p, \tag{23}$$

$$b_p = C_p b'_p \, e^{\eta_p \theta}, \qquad b'_p = b_p^{(1)} \sin \bar{m}x, \qquad b_p^{(1)} = (\eta_p \beta_p + 1)/R, \tag{24}$$

$$c_p = C_p c'_p \, e^{\eta_p \theta}, \qquad c'_p = c_p^{(1)} \cos \bar{m}x, \qquad c_p^{(1)} = \eta_p \alpha_p / 2R + \bar{m}\beta_p / 2. \tag{25}$$

Furthermore, introducing equation (5) into equations (20)–(22), we obtain

$$A_{pq} = C_p a'_{pq} \, e^{(\eta_p + \eta_q)\theta} C_q, \qquad a'_{pq} = a_{pq}^{(1)} \cos^2 \bar{m}x,$$
$$a_{pq}^{(1)} = \frac{1}{8R^2}[R\bar{m}\beta_p - \alpha_p \eta_p][R\bar{m}\beta_q - \alpha_q \eta_q] + \tfrac{1}{2}\bar{m}^2, \tag{26}$$

$$B_{pq} = C_p b'_{pq} \, e^{(\eta_p + \eta_q)\theta} C_q, \qquad b'_{pq} = b_{pq}^{(1)} \cos^2 \bar{m}x + b_{pq}^{(2)} \sin^2 \bar{m}x,$$
$$b_{pq}^{(1)} = \frac{1}{8R^2}[R\bar{m}\beta_p - \alpha_p \eta_p][R\bar{m}\beta_q - \alpha_q \eta_q], \qquad b_{pq}^{(2)} = \frac{1}{2R^2}[\eta_p - \beta_p][\eta_q - \beta_q], \tag{27}$$

$$C_{pq} = C_p c'_{pq} \, e^{(\eta_p + \eta_q)\theta} C_q, \qquad c'_{pq} = c_{pq}^{(1)} \cos \bar{m}x \sin \bar{m}x,$$
$$c_{pq}^{(1)} = \frac{\bar{m}}{4R}[\eta_p + \eta_q - \beta_p - \beta_q]; \tag{28}$$

η_p ($p = 1, \ldots, 8$) are the roots of characteristic equation (3); α_p and β_q are given by relation (4); R is the mean radius of the shell; $\bar{m} = m\pi/L$, where m is the axial wavenumber and L is the length of the shell.

The constants C_p ($p = 1, \ldots, 8$) and C_q ($q = 1, \ldots, 8$) may be obtained from equation (6) as

$$\{C\} = [A^{-1}]\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}. \tag{29}$$

The matrix $[A^{-1}]$ is the inverse of $[A]$, where $[A]$ is given by equation (6) and listed in Appendix C.

The present theory, expressed by equation (15), constitutes a general approach to the dynamic study of non-linear cylindrical shells. These equations of motion will be solved numerically only in the cases in which the coupling between different modes is ignored. Future numerical development will tackle the cases of coupling modes.

Assuming $r = s$ in equation (16), replacing the terms of A_{prs}, B_{prs}, C_{prs}, D_{prs} and E_{prs} by their expressions (equation (18)), using relations (23)–(28) and then integrating over x and θ, we obtain for the second order non-linear matrix for an empty element

$$[k_s^{(NL2)}] = [A^{-1}]^{\mathrm{T}}[J^{(NL2)}][A^{-1}], \tag{30}$$

where the (p, q) term in matrix $[J^{(NL2)}]$ is written as

$$J^{(NL2)}(p, q) = \begin{cases} \sum_{k=1}^{8} \dfrac{RGG(p, q)}{(\eta_p + \eta_q + \eta_k)} [\mathrm{e}^{(\eta_p + \eta_q + \eta_k)\phi} - 1] & \text{if } \eta_p + \eta_q + \eta_k \neq 0, \\ \sum_{k=1}^{8} RGG(p, q)\phi, & \text{if } \eta_p + \eta_q + \eta_k = 0. \end{cases} \tag{31}$$

$GG(p, q)$ is a coefficient in conjunction with α, β, η and element p_{ij} in matrix $[P]$. The general expression of $GG(p, q)$ is

$$\begin{aligned} GG(p, q) = {} & p_{11} I_1 [a_p^{(1)} A_{pq}^{-1} a_{qk}^{(1)} + a_q^{(1)} A_{qk}^{-1} a_{kp}^{(1)} + a_k^{(1)} A_{kp}^{-1} a_{pq}^{(1)}] \\ & + p_{22} I_1 [b_p^{(1)} A_{pq}^{-1} b_{qk}^{(1)} + b_q^{(1)} A_{qk}^{-1} b_{kp}^{(1)} + b_k^{(1)} A_{kp}^{-1} b_{pq}^{(1)}] \\ & + p_{22} I_2 [b_p^{(1)} A_{pq}^{-1} b_{qk}^{(2)} + b_q^{(1)} A_{qk}^{-1} b_{kp}^{(2)} + b_k^{(1)} A_{kp}^{-1} b_{pq}^{(2)}] \\ & + p_{33} I_1 [c_p^{(1)} A_{pq}^{-1} c_{qk}^{(1)} + c_q^{(1)} A_{qk}^{-1} c_{kq}^{(1)} + c_k^{(1)} A_{kp}^{-1} c_{pq}^{(1)}] \\ & + p_{12} I_1 [a_q^{(1)} A_{qk}^{-1} b_{kp}^{(1)} + a_k^{(1)} A_{kp}^{-1} b_{pq}^{(1)} + b_p^{(1)} A_{pq}^{-1} a_{qk}^{(1)} \\ & + b_q^{(1)} A_{qk}^{-1} a_{kp}^{(1)} + b_k^{(1)} A_{kp}^{-1} a_{pq}^{(1)} + a_p^{(1)} A_{pq}^{-1} b_{qk}^{(1)}] \\ & + p_{12} I_2 [a_q^{(1)} A_{qk}^{-1} b_{kp}^{(2)} + a_k^{(1)} A_{kp}^{-1} b_{pq}^{(2)} + a_p^{(1)} A_{pq}^{-1} b_{qk}^{(2)}], \end{aligned} \tag{32}$$

where

$$I_1 = \frac{1}{3\bar{m}} [1 - (-1)^m], \qquad I_2 = 2I_1, \qquad \bar{m} = m\pi/L, \tag{33}$$

The terms $a_p^{(1)}$, $b_p^{(1)}$, $c_p^{(1)}$, $a_{pq}^{(1)}$, $b_{pq}^{(1)}$, $c_{pq}^{(1)}$ and $b_{pq}^{(2)}$ are terms appearing in expressions of coefficients a_p, b_p, c_p, A_{pq}, B_{pq} and C_{pq} (relations (23)–(28)) and A_{pq}^{-1} is the term (p, q) of matrix $[A^{-1}]$, where $[A]$ is the matrix defined by relation (6).

Assuming $r = s = q$ in equation (17), replacing the terms of A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq} and E_{prsq} by their expressions (equation 18), using relations (26)–(28) and then integrating over x and θ, we obtain for the third order non-linear matrix for an empty element

$$[k_s^{(NL3)}] = [A^{-1}]^{\mathrm{T}}[J^{(NL3)}][A^{-1}], \tag{34}$$

where the (p, q) term in matrix $J^{(NL3)}$ is written as

$$J^{(NL3)}(p, q) = \begin{cases} \sum_{k=1}^{8} \sum_{l=1}^{8} \dfrac{RLE(l, k) SS(p, q)}{8(\eta_p + \eta_q + \eta_k + \eta_l)} [\mathrm{e}^{(\eta_p + \eta_q + \eta_k + \eta_l)\phi} - 1], & \text{if } \eta_p + \eta_q + \eta_k + \eta_l \neq 0, \\ \sum_{k=1}^{8} \sum_{l=1}^{8} \tfrac{1}{8} RLE(l, k) SS(p, q)\phi, & \text{if } \eta_p + \eta_q + \eta_k + \eta_l = 0. \end{cases} \tag{35}$$

$E(l, k)$ is the term (l, k) of matrix $[E]$, where $[E]$ represents a matrix of constants defined by $[E] = [A^{-1}]^{\mathrm{T}}[A^{-1}]$, $SS(p, q)$ is a coefficient in conjunction with α, β, η and element p_{ij} in matrix $[P]$. The general expression of $SS(p, q)$ is

$$SS(p, q) = 3p_{11}a_{pl}^{(1)}a_{kq}^{(1)} + p_{22}(3b_{pl}^{(1)}b_{kq}^{(1)} + 3b_{pl}^{(2)}b_{kq}^{(2)} + b_{pl}^{(1)}b_{kq}^{(2)} + b_{pl}^{(2)}b_{kq}^{(1)}) + p_{33}c_{pl}^{(1)}c_{kq}^{(1)} + p_{12}(3a_{pl}^{(1)}b_{kq}^{(1)} + a_{pl}^{(1)}b_{kq}^{(2)} + 3b_{pl}^{(1)}a_{kq}^{(1)} + b_{pl}^{(2)}a_{kq}^{(1)}), \tag{36}$$

where the terms $a_{pq}^{(1)}$, $b_{pq}^{(1)}$, $c_{pq}^{(1)}$ and $b_{pq}^{(2)}$ are coefficients given in relations (26)–(28).

5. DYNAMIC BEHAVIOUR OF THE FLUID–SHELL INTERACTION

The pressure exerted by the fluid is given by using a non-linear development of the Bernoulli equation. From the solution of the potential equation we derive an expression of non-linear pressure as a function of (1) the nodal displacements of the fluid element, (2) the inertial, centrifugal and Coriolis forces and (3) a combination of non-linear effects. Through the usual finite element procedure, we obtain the linear mass, damping and stiffness matrices for the fluid as well as the non-linear matrices for damping and stiffness and a combination of the two.

The mathematical model which is developed is based on the following hypothesis: (1) the fluid flow is potential; (2) vibration is non-linear; (3) pressure on the wall is purely lateral; (4) the fluid mean velocity distribution is assumed to be constant across a shell section; and (5) the fluid is incompressible and non-viscous.

5.1. DYNAMIC PRESSURE

With the previous hypothesis, the potential function must satisfy the Laplace equation. This relation is expressed in the cylindrical co-ordinate system as

$$\nabla^2\varphi = \frac{1}{r}(r\varphi_{,r})_{,r} + \frac{\varphi_{,\theta\theta}}{r^2} + \varphi_{,xx} = 0, \tag{37}$$

where φ is the potential function that represents the velocity potential.
Therefore,

$$V_x = U_{xu} + \varphi_{,x}, \qquad V_\theta = \varphi_{,\theta}/R, \qquad V_r = \varphi_{,r}, \tag{38}$$

where V_x, V_θ and V_r are, respectively, the axial, tangential and radial components of the fluid velocity; U_{xu} is the velocity of the liquid through the shell section.

The Bernoulli equation is given by

$$\left.\varphi_{,t} + \tfrac{1}{2}V^2 + \frac{P_u}{\rho_{fu}}\right|_{r=\xi} = 0. \tag{39}$$

Introducting equation (38) into equation (39) and taking into account the linear and non-linear terms, we find the dynamic pressure P_u:

$$P_u = -\rho_{fu}\{\varphi_{,t} + U_{xu}\varphi_{,x} + \tfrac{1}{2}[(\varphi_{,x})^2 + (\varphi_{,\theta})^2/r^2 + (\varphi_{,r})^2]\}|_{r=\xi}, \tag{40}$$

where the subscript u represents "i, internal" or "e, external" fluid as the case may be:

$$\text{if } u = i, \qquad \text{then } \xi = R_i = R - t/2, \tag{41}$$

$$\text{if } u = e, \qquad \text{then } \xi = R_e = R + t/2. \tag{42}$$

A full definition of the flow requires that a condition be applied to the structure–fluid interface. The impermeability condition ensures contact between the shell and the fluid. This should be

$$V_r|_{r=R} = \varphi_{,r}|_{r=R} = W_{,t} + U_{xu} W_{,x} + \frac{U_{xu}^2}{2} W_{,xx}|_{r=R}. \tag{43}$$

From the theory of shells (equation (5)), we have

$$W(x, \theta, t) = \sum_{j=1}^{8} C_j \, e^{\eta_j \theta} \sin(m\pi x/L) \, e^{i\omega t}. \tag{44}$$

Assuming then

$$\varphi(x, \theta, r, t) = \sum_{j=1}^{8} R_j(r) S_j(x, \theta, t) \tag{45}$$

and applying the impermeability condition (equation (43)) with the radial displacement given by relation (44), we determine the function $S_j(x, \theta, t)$ explicitly. Using equation (37), we find the following differential Bessel equation:

$$r^2 \frac{d^2 R_j(r)}{dr^2} + r \frac{d R_j(r)}{dr} + R_j(r)\left[\left(\frac{im\pi}{L}\right)^2 r^2 - (i\eta_j)^2\right] = 0, \tag{46}$$

where i is the complex number, $i^2 = -1$, and η_j is the complex solution of the characteristic equation for the empty shell (relation (5)).

The general solution of equation (46) is given by

$$R_j(r) = A \, J_{i\eta_j}\left(\frac{im\pi}{L} r\right) + B \, Y_{i\eta_j}\left(\frac{im\pi}{L} r\right), \tag{47}$$

where $J_{i\eta_j}$ and $Y_{i\eta_j}$ are, respectively, the Bessel functions of the first and second kind of complex order "$i\eta_j$".

For inside flow, the solution (47) must be finite on the axis of the shell ($r = 0$); this means we have to set the constant B equal to zero. For outside flow ($r \to \infty$), this means that the constant A is equal to zero. When the shell is simultaneously subjected to internal and external flow, we have to take the complete solution (47).

We carry the Bessel equation solution back into equation (45) to obtain the final expression of velocity potential evaluated at the shell wall:

$$\varphi_u(r, \theta, x, t)_j = Z_{uj}(im\pi R_u/L)\left[W_{j,t} + U_{xu} W_{j,x} + \frac{U_{xu}^2}{2} W_{j,xx}\right], \tag{48}$$

where

$$Z_{uj}(im\pi R_u/L) = R_u \Bigg/ \left[i\eta_j - \frac{im\pi R_u}{L} \frac{J_{i\eta_j+1}(im\pi R_u/L)}{J_{i\eta_j}(im\pi R_u/L)}\right], \qquad \text{if } u = i, \tag{49}$$

$$Z_{uj}(im\pi R_u/L) = R_u \Bigg/ \left[i\eta_j - \frac{im\pi R_u}{L} \frac{Y_{i\eta_j+1}(im\pi R_u/L)}{Y_{i\eta_j}(im\pi R_u/L)}\right], \qquad \text{if } u = e, \tag{50}$$

where η_j ($j = 1, \ldots, 8$) are the roots of the characteristic equation of the empty shell; $\mathrm{J}_{\mathrm{i}\eta_j}$ and $\mathrm{Y}_{\mathrm{i}\eta_j}$ are, respectively, the Bessel functions of the first and second kind of order "$\mathrm{i}\eta_j$"; m is the axial mode number; R is the mean radius of the shell; L is its length; the subscript "u" is equal to "i" for internal flow and is equal to "e" for external flow.

Substituting relation (48) into the non-linear condition (40), we obtain the equation for the pressure on the shell wall. It is useful to separate the total pressure into its linear and non-linear terms:

$$P_u = P_{uL} + P_{uNL}, \tag{51}$$

where

$$P_{uL} = -\rho_{fu} \sum_{j=1}^{8} Z_{uj} \left[W_{j,tt} + 2U_{xu} W_{j,tx} + \frac{U_{xu}^2}{2} W_{j,txx} + U_{xu}^2 W_{j,xx} + \frac{U_{xu}^3}{2} W_{j,xxx} \right] \tag{52}$$

and

$$\begin{aligned} P_{uNL} = -\frac{\rho_{fu}}{2} \sum_{j=1}^{8} \sum_{k=1}^{8} Z_{uj} Z_{uk} \Bigg[& W_{j,tx} W_{k,tx} + U_{xu}^2 W_{j,xx} W_{k,xx} + \frac{U_{xu}^4}{4} W_{j,xxx} W_{k,xxx} \\ & + 2U_{xu} W_{j,tx} W_{k,xx} + U_{xu}^3 W_{j,xx} W_{k,xxx} + U_{xu}^2 W_{j,tx} W_{k,xxx} \Bigg] \\ & + \left(\frac{\eta_j \eta_k}{R^2} Z_{uj} Z_{uk} + 1 \right) \Bigg[W_{j,t} W_{k,t} + U_{xu}^2 W_{j,x} W_{k,x} + \frac{U_{xu}^4}{4} W_{j,xx} W_{k,xx} \\ & + 2U_{xu} W_{j,t} W_{k,x} + U_{xu}^3 W_{j,x} W_{k,xx} + U_{xu}^2 W_{j,t} W_{k,xx} \Bigg]. \end{aligned} \tag{53}$$

5.2. LINEAR MATRICES FOR THE MOVING FLUID

By introducing the displacement function (44) into the dynamic pressure expression (52) and performing the matrix operation required by the finite element method, the mass, damping and stiffness matrices for fluid are obtained by evaluating the integral

$$\int [N]^{\mathrm{T}} \{P_{uL}\} \, \mathrm{d}A. \tag{54}$$

We obtain

$$[m_f^{(L)}] = [A^{-1}]^{\mathrm{T}} [S_f^{(L)}] [A^{-1}], \qquad [c_f^{(L)}] = [A^{-1}]^{\mathrm{T}} [D_f^{(L)}] [A^{-1}], \qquad [k_f^{(L)}] = [A^{-1}]^{\mathrm{T}} [G_f^{(L)}] [A^{-1}]. \tag{55}$$

The matrix $[A]$ is given by equation (6) and the elements of $[S_f^{(L)}]$, $[D_f^{(L)}]$ and $[G_f^{(L)}]$ are given as

$$S_f^{(L)}(r, s) = -\frac{RL}{2} I_{rs} \rho_{fu} Z_{us}, \qquad D_f^{(L)}(r, s) = \frac{Rm^2\pi^2}{4L} I_{rs} \rho_{fu} U_{xu}^2 Z_{us},$$

$$G_f^{(L)}(r, s) = \frac{Rm^2\pi^2}{2L} I_{rs} \rho_{fu} U_{xu}^2 Z_{us}, \tag{56–58}$$

where $r, s = 1, \ldots, 8$; ρ_{fu} is the density of the fluid; U_{xu} is the velocity of the fluid; Z_{us} is defined by relations (49) and (50); the subscript "u" is equal to "i" for internal flow and is equal to "e" for external flow, and I_{rs} is defined by

$$\begin{cases} I_{rs} = \dfrac{1}{(\eta_r + \eta_s)} [\mathrm{e}^{(\eta_r + \eta_s)\phi} - 1], & \text{for } \eta_r + \eta_s \neq 0, \\ I_{rs} = \phi, & \text{for } \eta_r + \eta_s = 0, \end{cases} \tag{59}$$

where $r, s = 1, \ldots, 8$; η_r are the roots of the characteristic equation of the empty shell and ϕ is the angle for one finite element.

Finally, the global matrices $[M_f^{(L)}]$, $[C_f^{(L)}]$ and $[K_f^{(L)}]$ may be obtained, respectively, by superimposing the mass $[m_f^{(L)}]$, damping $[c_f^{(L)}]$ and stiffness $[k_f^{(L)}]$ matrices for each individual fluid finite element.

5.3. NON-LINEAR MATRICES FOR THE MOVING FLUID

We use the procedure outlined in the previous section, ignoring the cross products in the non-linear dynamic procedure expression (53). We obtain the following matrices for the non-linear effects:

$$[c_f^{(NL)}] = [A^{-1}]^{\mathrm{T}}[D_f^{(NL)}][A^{-1}], \qquad [kc_f^{(NL)}] = [A^{-1}]^{\mathrm{T}}[GD_f^{(NL)}][A^{-1}],$$
$$[k_f^{(NL)}] = [A^{-1}]^{\mathrm{T}}[G_f^{(NL)}][A^{-1}]. \tag{60}$$

The matrix $[A]$ is given by equation (6) and the elements of $[D_f^{(NL)}]$, $[GD_f^{(NL)}]$ and $[G_f^{(NL)}]$ are given as

$$D_f^{(NL)}(r, s) = -\frac{\rho_{fu}}{2} \Pi_{rs}\left[\left(\frac{m\pi}{L}\right)^2 Z_{us}^2 I_{SC2} + \frac{\eta_s^2}{R^2} Z_{us}^2 I_{S3} + I_{S3}\right], \tag{61}$$

$$G_f^{(NL)}(r, s) = -\frac{\rho_{fu}}{2} \Pi_{rs}\left\{U_{xu}^2\left[\left(\frac{m\pi}{L}\right)^4 Z_{us}^2 I_{S3} + \frac{\eta_s^2}{R^2}\left(\frac{m\pi}{L}\right)^2 Z_{us}^2 I_{SC2} + \left(\frac{m\pi}{L}\right)^2 I_{SC2}\right]\right.$$
$$\left. + \frac{U_{xu}^4}{4}\left[\left(\frac{m\pi}{L}\right)^6 Z_{us}^2 I_{SC2} + \frac{\eta_s^2}{R^2}\left(\frac{m\pi}{L}\right)^4 Z_{us}^2 I_{S3} + \left(\frac{m\pi}{L}\right)^4 I_{S3}\right]\right\}, \tag{62}$$

$$GD_f^{(NL)}(r, s) = -\frac{\rho_{fu}}{2} \Pi_{rs} U_{xu}^2\left[-\left(\frac{m\pi}{L}\right)^4 Z_{us}^2 I_{SC2} - \frac{\eta_s^2}{R^2}\left(\frac{m\pi}{L}\right)^2 Z_{us}^2 I_{S3} - \left(\frac{m\pi}{L}\right)^2 I_{S3}\right], \tag{63}$$

where $r, s = 1, \ldots, 8$; ρ is the density of the fluid; U_x is the velocity of the fluid; Z_{us} is defined by relations (49) and (50); the subscript "u" is equal to "i" for inside flow and is equal to "o" for outside flow; Π_{rs} is defined by

$$\begin{cases} \Pi_{rs} = \dfrac{1}{(\eta_r + 2\eta_s)} [\mathrm{e}^{(\eta_r + 2\eta_s)\phi} - 1], & \text{for } \eta_r + 2\eta_s \neq 0, \\ \Pi_{rs} = \phi, & \text{for } \eta_r + 2\eta_s = 0, \end{cases} \tag{64}$$

where $r, s = 1, \ldots, 8$; η_r are the roots of the characteristic equation of the empty shell and ϕ is the angle for one finite element; I_{SC2} and I_{S3} are defined by

$$I_{SC2} = \frac{L}{3m\pi}\,[1 - (-1)^{3m}], \qquad I_{S3} = \frac{L}{3m\pi}\,[(-1)^{3m} - 3(-1)^{m}], \tag{65}$$

where m is the axial mode number and L is the length of the shell.

Finally, the global matrices $[C_f^{(NL)}]$, $[KC_f^{(NL)}]$ and $[K_f^{(NL)}]$ may be obtained, respectively, by superimposing the non-linear damping $[c_f^{(NL)}]$, non-linear combination of damping and stiffness $[kc_f^{(NL)}]$ and non-linear stiffness $[k_f^{(NL)}]$ matrices for each individual fluid finite element.

6. INFLUENCE OF THE NON-LINEARITIES ON THE NATURAL FREQUENCIES

Taking into account the linear and non-linear matrices of the shell and of the fluid in the case in which the coupling between different modes is ignored, the dynamic behaviour of the open or closed cylindrical shell containing flowing fluid can be represented by the following system of equations

$$\begin{aligned}
&[M_t^{(L)}]\{\ddot{\delta}\} - [C_t^{(L)}]\{\dot{\delta}\} + [K_t^{(L)}]\{\delta\} + [K_s^{(NL2)}]\{\delta^2\} + [K_s^{(NL3)}]\{\delta^3\}\\
&\quad -[C_f^{(NL)}]\{\dot{\delta}^2\} - [KC_f^{(NL)}]\{\delta\dot{\delta}\} - [K_f^{(NL)}]\{\delta^2\} = \{0\},
\end{aligned} \tag{66}$$

where $[M_t^{(L)}] = [M_s] - [M_f^{(L)}]$; $[K_t^{(L)}] = [K_s^{(L)}] - [K_f^{(L)}]$; $\{\delta\}$ is the displacement vector; $[M_s]$ and $[K_s^{(L)}]$ are the global mass and linear stiffness matrices for the shell *in vacuo*; $[K_s^{(NL2)}]$ and $[K_s^{(NL3)}]$ are the global second and the third order non-linear stiffness matrices of the shell *in vacuo*; $[M_f^{(L)}]$, $[C_f^{(L)}]$ and $[K_f^{(L)}]$ are the global linear mass, damping and stiffness matrices for the fluid; $[C_f^{(NL)}]$, $[KC_f^{(NL)}]$ and $[K_f^{(NL)}]$ are the global non-linear matrices for the fluid.

These matrices are square matrices of order $4(N+1)$, where N represents the number of finite elements. In practice, very specific conditions are applied to the shell boundaries. Thus, matrices are reduced to square matrices of order NREDUC $= 4(N+1) - J$, where J represents the number of constraints applied.

Setting

$$\{\delta\} = [\Phi]\{q\}, \tag{67}$$

where $[\Phi]$ represents the square matrix for the eigenvectors of the linear system and $\{q\}$ is a time-related vector; and substituting equation (67) into system (66) and multiplying by $[\Phi]^{\mathrm{T}}$, we obtain

$$\begin{aligned}
&[M_t^{(L)}]^{\mathrm{D}}\{\ddot{q}\} - [C_t^{(L)}]^{\mathrm{D}}\{\dot{q}\} + [K_t^{(L)}]^{\mathrm{D}}\{q\}\\
&\quad +[\Phi^{\mathrm{T}}][K_s^{(NL2)}]([\Phi]\{q\})^2 + [\Phi^{\mathrm{T}}][K_s^{(NL3)}]([\Phi]\{q\})^3\\
&\quad -[\Phi^{\mathrm{T}}][C_f^{(NL)}]([\Phi]\{\dot{q}\})^2 - [\Phi^{\mathrm{T}}][KC_f^{(NL)}][\Phi]\{q\}[\Phi]\{\dot{q}\}\\
&\quad -[\Phi^{\mathrm{T}}][K_f^{(NL)}]([\Phi]\{q\})^2 = \{0\},
\end{aligned} \tag{68}$$

where

$$[M_t^{(L)}]^{\mathrm{D}} = [\Phi^{\mathrm{T}}][M_t^{(L)}][\Phi], \qquad [C_t^{(L)}]^{\mathrm{D}} = [\Phi^{\mathrm{T}}][C_t^{(L)}][\Phi], \qquad [K_t^{(L)}]^{\mathrm{D}} = [\Phi^{\mathrm{T}}][K_t^{(L)}][\Phi], \tag{69}$$

and where D denotes diagonal, the matrices quantifying the fluid contribution to the matrix equations of motions are non-symmetric. To facilitate the analysis, therefore, we consider only the symmetric portion of the matrices. This simplifying hypothesis is valid, since the original and simplified systems have comparable dynamic behaviour.

Tables 1–4 of Lakis and Laveau [27] show the variance between the eigenvalues in the original and simplified systems, corresponding to cases of damped and undamped free vibration. A trend was observed toward minimum variance of 1% at the extreme modes ($m = 1, \ldots, 6$ and $m > 11$) and maximum variances of 20% at the median modes ($m = 6, \ldots, 10$).

We saw how matrices contained in the linear part of system (66) could be reduced to diagonal matrices. On the other hand, by neglecting the cross-product in $([\Phi]\{q\}^2, \ldots$ of equation (68) we obtain

$$m_{ii}\ddot{q}_i - c_{ii}^{(L)}\dot{q}_i + k_{ii}^{(L)}q_i + \sum_{j=1}^{\mathrm{NREDUC}} (k_{ij}^{(NL2)}q_j^2 + k_{ij}^{(NL3)}q_j^3$$
$$- c_{ij}^{(NL)}\dot{q}_j^2 - KC_{ij}^{(NL)}q_j\dot{q}_j - K_{ij}^{(NL)}q_j^2) = 0, \tag{70}$$

where coefficients m_{ii}, $c_{ii}^{(L)}$ and $k_{ii}^{(L)}$represent the ith diagonal terms of linear matrices $[M_t^{(L)}]^{\mathrm{D}}$, $[C_t^{(L)}]^{\mathrm{D}}$ and $[K_t^{(L)}]^{\mathrm{D}}$, respectively; $k_{ij}^{(NL2)}$ and $k_{ij}^{(NL3)}$ are the (i, j) terms of the products $([\Phi]^{\mathrm{T}}[K_s^{NL2}][\Phi]^2)$ and $([\Phi]^{\mathrm{T}}[K_s^{NL3}][\Phi]^3$; $C_{ij}^{(NL)}$, $KC_{ij}^{(NL)}$ and $K_{ij}^{(NL)}$ are the (i, j) terms of the products $([\Phi]^{\mathrm{T}}[C_f^{NL}][\Phi]^2)$, $([\Phi]^{\mathrm{T}}[KC_f^{NL}][\Phi]^2)$ and $([\Phi]^{\mathrm{T}}[K_f^{NL}][\Phi]^2)$, respectively.

Here we have "NREDUC" simultaneous equations of the form of equation (70). Numerical solution of such a system is difficult and costly. At first, we limit ourselves to solving equation (70) by taking into account only the diagonal terms of the products $([\Phi]^{\mathrm{T}}[K^{NL2}][\Phi]^2, \ldots$ and therefore equation (70) is written

$$m_{ii}\ddot{q}_i - c_{ii}^{(L)}\dot{q}_i + k_{ii}^{(L)}q_i + k_{ii}^{(NL2)}q_i^2 + k_{ii}^{(NL3)}q_i^3 - C_{ii}^{(NL)}\dot{q}_i^2 - KC_{ii}^{(NL)}q_i\dot{q}_i - K_{ii}^{(NL)}q_i^2 = 0. \tag{71}$$

Setting

$$q_i(\tau) = A_i f_i(\tau), \qquad \text{with } f_i(0) = 1 \quad \text{and} \quad \dot{f}_i(0) = 0, \tag{72}$$

equation (71) becomes, after the A_p simplification and dividing by m_{ii},

$$\ddot{f}_i - \kappa_i\dot{f}_i + \omega_i^2 f_i + \lambda_i(A_i/t)f_i^2 + \sigma_i(A_i/t)^2 f_i^3$$
$$-(A_i/t)[\zeta_i\dot{f}_i^2 + \xi_i f_i\dot{f}_i + \gamma_i f_i^2] = 0, \tag{73}$$

where

$$\omega_i^2 = k_{ii}^{(L)}/m_{ii}, \qquad \kappa_i = c_{ii}^{(L)}/m_{ii}, \qquad \lambda_i = (k_{ii}^{(NL2)}/m_{ii})t, \qquad \sigma_i = (k_{ii}^{(NL3)}/m_{ii})t^2, \tag{74, 75}$$

$$\zeta_i = (c_{ii}^{(NL)}/m_{ii})t, \qquad \xi_i = (KC_{ii}^{(NL)}/m_{ii})t, \qquad \gamma_i = (K_{ii}^{(NL)}/m_{ii})t, \tag{76}$$

and where t represents shell thickness; the coefficient $[k_{ii}^{(L)}/m_{ii}]$ represents the ith linear vibration frequency of the system.

The solution $f_i(\tau)$ of the non-linear differential equation (73) which satisfies the conditions of equation (72) is calculated by a fourth order Runge–Kutta numerical method. The linear and non-linear natural frequencies are evaluated by a systematic search for the $f_i(\tau)$ roots as a function of time. The ω_{NL}/ω_L ratio of linear and non-linear frequency is expressed as a function of non-dimensional ratio (A_i/t), where A_i is the vibration amplitude.

7. CALCULATIONS AND DISCUSSION

The influence of non-linearities associated with the wall of the shell and with the fluid on the open or closed cylindrical shell's free vibrations is expressed by equation (73). For a shell of given physical characteristics, we first present the results for the convergence of the model and, second, those obtained by the present method in the case of linear vibration. Then the ratio ω_{NL}/ω_L of linear and non-linear frequency is graphically represented in Figures 5–10 with respect to the non-dimensional ratio, A_p/t. The straight horizontal line represents the linear vibration cases, where the frequency is independent of the motion's amplitude.

7.1. CONVERGENCE OF THE METHOD

A first set of calculations was undertaken to determine the required number of finite elements for a precise determination of natural frequencies. Calculations were made for the same closed cylindrical shell completely filled with fluid for a number of finite elements, $N = 2$, 4, 6, 8, 10, 15 and 20. This steel shell is simply supported at both ends and has the following data: $R = 37{\cdot}7$ mm, $t = 0{\cdot}229$ mm, $L = 234$ mm, $\nu = 0{\cdot}3$, $\rho_{fi}/\rho_s = 0{\cdot}128$.

The results for $m = 1$ and for $n = 2$, 3, 4, 5 and 6 are shown in Figure 3. We conclude that the convergence of the shell–fluid system demands ten elements for both the low and the high modes.

7.2. LINEAR FREE VIBRATION OF CLOSED CYLINDRICAL SHELL

We present a calculation to test the method incorporating linear analysis, which is developed in this paper. The closed cylindrical shell is simply supported at both ends and has the same physical properties as those given in the previous section. This shell was studied by Goncalves and Batista [3], who used the Rayleigh–Ritz technique to obtain the natural frequencies of the shell–fluid system. In Figure 4 are shown the linear natural frequencies as a function of the circumferential mode number n for the axial mode $m = 1$.

As may be seen, the results obtained by the present method are in good agreement with those of Goncalves and Batista [3]. For the case of empty or liquid-filled shells, there is the well-known dip in the frequency curve as the shell makes a transition through the lower values of n. This phenomenon can be explained by the interchange in the relative contributions of the bending and stretching strain energies of the shell.

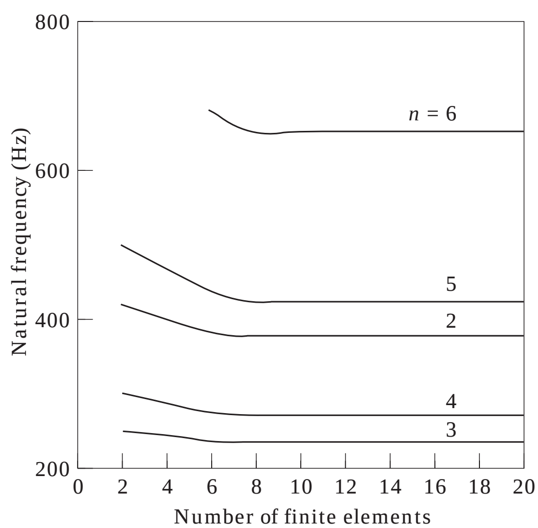

Figure 3. The linear natural frequency for a simply supported closed cylindrical shell completely filled with internal fluid as a function of the number of finite elements; n is the number of circumferential mode; the number of axial mode is $m = 1$; $R = 37{\cdot}7$ mm, $t = 0{\cdot}229$ mm, $L = 234$ mm, $\nu = 0{\cdot}3$, $\rho_{fi}/\rho_s = 0{\cdot}128$.

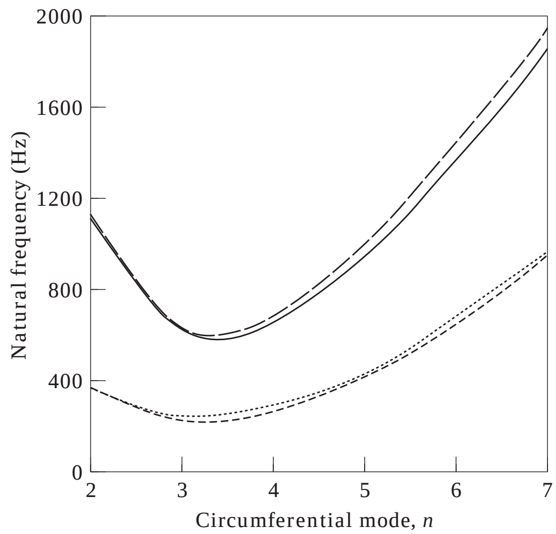

Figure 4. The linear natural frequency for an empty and liquid-filled closed simply supported cylindrical shell as a function of the circumferential mode n ($m = 1$); $R = 37{\cdot}7$ mm, $t = 0{\cdot}229$ mm, $L = 234$ mm, $\nu = 0{\cdot}3$, $\rho_{fi}/\rho_s = 0{\cdot}128$. ——, Empty, present method; — —, empty, ref [3]; ----, liquid-filled, present method; ····, liquid-filled, ref [3].

7.3 NON-LINEAR FREE VIBRATION OF CLOSED CYLINDRICAL SHELL

7.3.1. EMPTY SHELL

This set of calculations is designed to determine the influence of geometric non-linearities in strain–displacement relations on the free vibrations of an empty isotropic cylindrical shell, simply supported at both ends. The shell has the properties: $\zeta = \pi Rm/nL = 2$, $\chi = (n^2t/R)^2 = 1$ and $\nu = 0{\cdot}3$. The variations in frequency ratio as a function of A_p/t for this shell (Figure 5) were calculated using the present method, and compared to the results of Evensen [13] and Atluri [15]. Evensen's analysis involved a two modes approximation and his equation was obtained using the Galerkin procedure. The work of Atluri is based on Donnell's equations, a modal expansion was used for displacements and the Galerkin technique was used to reduce the problem to a non-linear ordinary differential equation for the modal amplitudes.

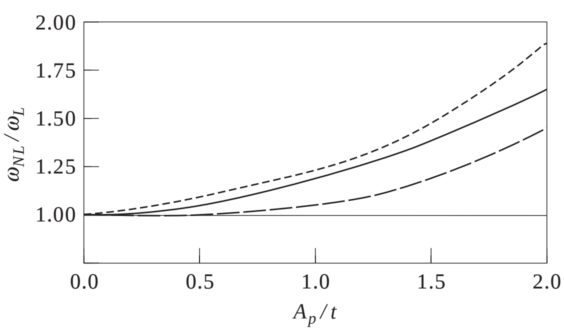

Figure 5. A comparison of the effects of amplitude upon frequency for an empty simply supported closed cylindrical shell; $\zeta = \pi Rm/nL = 2$, $\chi = (n^2t/R)^2 = 1$, $\nu = 0{\cdot}3$. ——, Present method; — —, Evensen [13]; ----, Atluri [15].

As may be seen, the results obtained by the present method are in satisfactory agreement with those of other authors.

7.3.2. *Submerged shell*

The second comparative example is shown in Figure 6: the closed cylindrical shell is simply supported at both ends and completely submerged in liquid. The pertinent data are $E = 21{\cdot}981 \times 10^{11}$ N/m^2, $\nu = 0{\cdot}3$, $\rho_f/\rho_s = 0{\cdot}128$, $R = 0{\cdot}235$ m, $t = 0{\cdot}00235$ m, $\phi_T = 360°$.

This case was previously analyzed by Ramachandran [17] who used the Rayleigh–Ritz procedure. In his study, he took into account only the influence of non-linearities associated with the shell and neglected the effect of non-linearities associated with the fluid. In addition, only lateral displacements were considered for the non-linear analysis.

In Figure 6, we present a comparison between the present work and that of Ramachandran [17], and show results for different modes and geometry. For ratio $L/R = 4$ and the mode ($n = 4$, $m = 1$), we observe that the ratio between linear and non-linear natural frequency decreases as ratio A/t increases. The variations are small for values A/t below 1·0. For the value of $A/t = 2$, the variation calculated by the present method is more pronounced than that of Ramanchadran [17]; the results obtained are in agreement within a range of 5%. For ratio $L/R = 2$ and the mode ($n = 8$, $m = 1$), we observe that the ratio between linear and non-linear natural frequency decreases and is more pronounced than the previous results. For the value $A/t = 2$, the variation calculated by the present method is less pronounced than that of Ramanchadran [17], the difference between the two results is of the order of 25%.

7.4. NON-LINEAR FREE VIBRATION OF AN OPEN CYLINDRICAL SHELL TOTALLY SUBMERGED IN LIQUID AND SUBJECTED SIMULTANEOUSLY TO AN INTERNAL AND EXTERNAL FLUID

One of the great advantages of the finite element method is the ease with which it can be applied to any geometry and any boundary condition. Thus, this step of calculation is to study the non-linear dynamic characteristics of an open cylindrical shell totally submerged in liquid as a function of flow velocity, circumferential and axial modes, boundary conditions, material properties, etc.

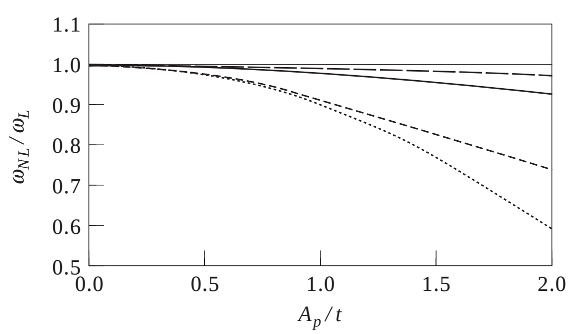

Figure 6. A comparison of the effects of amplitude upon frequency for a submerged simply supported closed cylindrical shell; $m = 1$; $E = 21{\cdot}981 \times 10^{11}$ N/m^2, $\nu = 0{\cdot}3$, $\rho_f/\rho_s = 0{\cdot}128$, $R = 235$ mm, $t = 2{\cdot}35$ mm, $\phi_T = 360°$. ——, Present method, $L/R = 4$, $n = 4$; — —, Ramachandran [17], $L/R = 4$, $n = 4$; ----, present method, $L/R = 2$, $n = 8$; ····, Ramachandran [17], $L/R = 2$, $n = 8$.

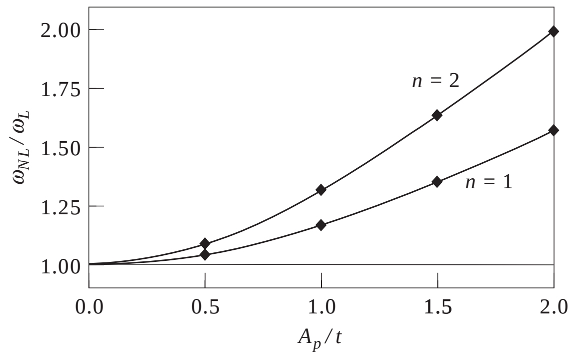

Figure 7. The influence of non-linearities associated with the wall of the shell versus non-linearities associated with the fluid at rest for a simply supported open cylindrical shell; $m = 1$: $R = 450$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 100^{\circ}$, $\rho_f/\rho_s = 0{\cdot}128$. ——, NL shell, NL fluid; ◆, NL shell, L fluid.

7.4.1. *Influence of non-linearities associated with the wall of the shell versus non-linearities associated with the fluid*

In this first calculation, we analyze the influence of non-linearities associated with the wall of the shell versus the non-linearities associated with the fluid at rest. The study is made on an open cylindrical shell totally submerged in fluid. Calculations have been made by solving equation (73), both when the non-linearity associated with the fluid is taken into account and when it is not taken into account.

The steel open shell is simply supported at the four edges and has the following data: $R = 450$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 100^{\circ}$, $\rho_f/\rho_s = 0{\cdot}128$. The results of this analysis are presented in Figure 7. We observe that the influence of non-linearities associated with fluid on the dynamic behaviour of the shell–fluid structure is negligible.

7.4.2. *Effects of flow velocity*

In order to establish the effect of the flow velocity on the non-linear free vibration, we turn to Figure 8. The parameters of the investigation are as follows: $m = 1$, $n = 9$ and 10; Reynolds number, $R_N = 0{\cdot}0$ and $1{\cdot}0 \times 10^6$, with $R_N = 2U_x R\rho_f/\nu_f$, where U_x is the mean

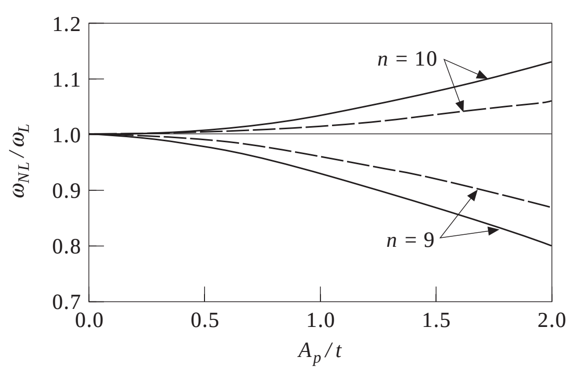

Figure 8. The influence of large amplitude on the natural frequency of a submerged clamped-clamped open cylindrical shell with axial flow for different Reynolds numbers; $m = 1$: $R = 225$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 120^{\circ}$, $\rho_f/\rho_s = 0{\cdot}128$. ——, $R_N = 0{\cdot}0$; — —, $R_N = 1{\cdot}0 \times 10^6$.

velocity of the flow, R is the average radius of the open cylindrical shell and ρ_f and v_f are, respectively, the density and viscosity of the flowing fluid.

The other data are: $R = 225$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 120°$, $\rho_f/\rho_s = 0{\cdot}128$.

In Figure 8 it is shown that the non-linearity is of the softening type for the circumferential mode $n = 9$ and is of the hardening type for $n = 10$ for both flow and no-flow condition. We see also that the non-linear effect is more pronounced for the shell-fluid system when the fluid is moving. The difference between the cases is of the order of 10%.

7.4.3. *Effects of material properties*

With the same geometric data, in Figure 9 is shown the effect of non-linearities upon frequency for different material properties. When the open shell is simply supported at the four edges and is completely submerged in water, the data are as follows: $R = 450$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 180°$. The materials chosen are steel, acrylic, rubber and an orthotropic material, the physical properties of which are: $E_x = 1{\cdot}0 \times 10^{11}$ N/m^2, $E_\theta = 0{\cdot}05 \times E_x$, $v_x = 0{\cdot}2$, $v_\theta = 0{\cdot}05 \times v_x$, $G_{x\theta} = 0{\cdot}05 \times E_x$, $\rho_s = 7800$ N/m^3.

We observe for the mode ($m = 2, n = 3$) that the steel shell is the one on which the effect of non-linearity is more pronounced, the orthotropic shell is the one on which the effect of non-linearity is less pronounced.

7.4.4. *Effects of the circumferential mode n*

In Figure 10, we present the effect of large amplitude on the frequency ratio as a function of A/t for the axial mode $m = 1$ and the circumferential mode $n = 6, \ldots, 12$. The open shell is clamped along the straight edges and simply supported along its curved edges. The data for the steel shell are: $R = 225$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 120°$, $\rho_f/\rho_s = 0{\cdot}128$. In Figure 10 it is shown that the non-linearity is of the hardening type for circumferential modes $n = 10$, 11 and 12 and is of the softening type for n between 6 and 9. We see also that the non-linear effect is more pronounced for the mode $n = 6$ and the variation is small for the case of $n = 9$.

8. CONCLUSIONS

The method developed in this paper demonstrates the influence of the non-linearities

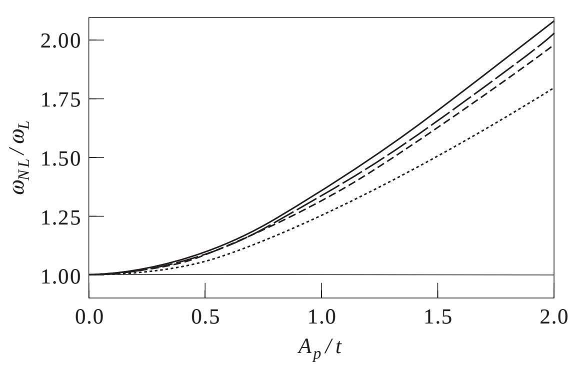

Figure 9. The influence of large amplitude on the natural frequency of a submerged simply supported open cylindrical shell with the fluid at rest for different material properties ($m = 2, n = 3$): $R = 450$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 100°$, $\rho_f = 1000$ kg/m^3. ——, steel; ----, rubber; — —, acrylic; ····, orthotropic material.

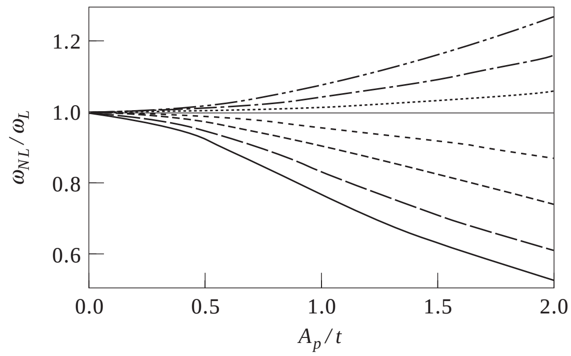

Figure 10. The influence of large amplitude on the natural frequency of a submerged clamped–clamped open cylindrical shell with the fluid at rest for various circumferential modes n and axial mode $m = 1$: $R = 225$ mm, $t = 1{\cdot}5$ mm, $L = 1350$ mm, $\phi_T = 120^\circ$, $\rho_f/\rho_s = 0{\cdot}128$. ——, $n=6$; — — $n = 7$; – – –, $n = 8$; - - - -, $n = 9$; ····, $n = 10$; - — -, $n = 11$; - - — - -, $n = 12$.

associated with the wall of the shell and with the fluid flow on the free vibrations of totally submerged open or closed cylindrical shells, subjected simultaneously to an internal and external flow. It is a hybrid method, based on a combination of non-linear thin shell theory, non-linear fluid theory and the finite element method.

An open cylindrical finite element is developed, in order that the displacement functions can be derived directly from classical thin shell theory. Mass and linear stiffness matrices are then obtained for the empty shell by the finite element method. With the modal coefficients derived from the Sanders–Koiter non-linear theory of thin shells and corresponding to non-linearities in strain–displacement relations, the second and third order non-linear stiffness matrices are then calculated using the finite element method.

The pressure exerted by the fluid is given using a non-linear development of Bernoulli's equation. From the solution of the potential equation we derive an expression of linear and non-linear pressure as a function of the nodal displacements of the fluid element, the inertial, centrifugal, Coriolis forces and a combination of non-linear effects. Through the finite element procedure, we obtain the linear mass, damping and stiffness matrices for the fluid as well as the non-linear matrices for damping and stiffness, and a combination of the two.

The non-linear equations of motion are solved by the fourth order Runge–Kutta numerical method. Variations in the free vibration frequencies are determined in conjunction with motion amplitude for a closed or open cylindrical shell, empty or submerged in flowing fluid. Deviations in terms of linear vibrations are observed.

This method combines the advantages of finite element analysis which deals with complex shells, and the precision of formulation which the use of displacement functions derived from shell and fluid theories contributes.

This area of investigation is still wide open and there is very little on the subject in the literature. We are unable, therefore, to confirm whether, in the context of a dynamic analysis, we are justified in completely neglecting the influence of non-linearities associated with fluid flow. On the other hand, the effect of geometric non-linearities associated with the walls is not negligible and should be taken into account in calculating the dynamic behaviour of shell–fluid interactions when the amplitude of vibration is greater than the thickness of the shell.

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APPENDIX A: EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin orthotropic cylindrical shell:

$$
\begin{aligned}
L_1(U, V, W, p_{ij}) = {} & p_{11}\frac{\partial^2 U}{\partial x^2} + \frac{p_{12}}{r}\left(\frac{\partial^2 V}{\partial x \partial\theta} + \frac{\partial W}{\partial x}\right) - p_{14}\frac{\partial^3 W}{\partial x^3} \\
& + \frac{p_{15}}{R^2}\left(\frac{\partial^3 W}{\partial x \partial\theta^2} + \frac{\partial^2 V}{\partial x \partial\theta}\right) + \left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\left(\frac{\partial^2 V}{\partial x \partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2}\right) \\
& + \left(\frac{p_{36}}{R^2} - \frac{p_{66}}{2R^3}\right)\left(-\frac{2\partial^3 W}{\partial x \partial\theta^2} + \frac{3}{2}\frac{\partial^2 V}{\partial x \partial\theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial\theta^2}\right),
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
L_2(U, V, W, p_{ij}) = {} & \left(\frac{p_{21}}{R} + \frac{p_{51}}{R^2}\right)\left(\frac{\partial^2 U}{\partial x \partial\theta}\right) + \frac{1}{R}\left(\frac{p_{22}}{R} + \frac{p_{52}}{R^2}\right)\left(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\right) \\
& - \left(\frac{p_{24}}{R} + \frac{p_{54}}{R^2}\right)\left(\frac{\partial^3 W}{\partial x^2 \partial\theta}\right) + \frac{1}{R^2}\left(\frac{p_{25}}{R} + \frac{p_{55}}{R^2}\right)\left(-\frac{\partial^3 W}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}\right) \\
& + \left(p_{33} + \frac{3p_{63}}{2R}\right)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 U}{R \partial x \partial\theta}\right) \\
& + \frac{1}{R}\left(p_{36} + \frac{3p_{66}}{2R}\right)\left(-2\frac{\partial^3 W}{\partial x^2 \partial\theta} + \frac{3}{2}\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 U}{2R \partial x \partial\theta}\right),
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
L_3(U, V, W, p_{ij}) = {} & p_{41}\frac{\partial^3 U}{\partial x^3} + \frac{p_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2 \partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - p_{44}\frac{\partial^4 W}{\partial x^4} \\
& + \frac{p_{45}}{R^2}\left(-\frac{\partial^4 W}{\partial x^2 \partial\theta^2} + \frac{\partial^3 V}{\partial x^2 \partial\theta}\right) + \frac{2p_{63}}{R}\left(\frac{\partial^3 U}{R \partial x \partial\theta^2} + \frac{\partial^3 V}{\partial x^2 \partial\theta}\right) \\
& + \left(\frac{2p_{66}}{R^2}\right)\left(-2\frac{\partial^4 W}{\partial x^2 \partial\theta^2} + \frac{3}{2}\frac{\partial^3 V}{\partial x^2 \partial\theta} - \frac{\partial^3 U}{2R \partial x \partial\theta^2}\right) + \frac{p_{51}}{R^2}\frac{\partial^3 U}{\partial x \partial\theta^2} \\
& + \frac{p_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3} + \frac{\partial^2 W}{\partial\theta^2}\right) + \frac{p_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right) - \frac{p_{21}}{R}\frac{\partial U}{\partial x} - \frac{p_{54}}{R^2}\frac{\partial^4 W}{\partial x^2 \partial\theta^2} \\
& - \frac{p_{22}}{R^2}\left(\frac{\partial V}{\partial\theta} + W\right) + \frac{p_{24}}{R}\frac{\partial^2 W}{\partial\theta^2} - \frac{p_{25}}{R^3}\left(-\frac{\partial^2 W}{\partial\theta^2} + \frac{\partial V}{\partial\theta}\right).
\end{aligned} \tag{A.3}
$$

APPENDIX B: CHARACTERISTIC EQUATION

Equation (3), the characteristic equation, is

$$h_8\eta^8 + h_6\eta^6 + h_4\eta^4 + h_2\eta^2 + h_0 = 0,$$

where

$$h_8 = f_1f_6f_{10} - f_1f_8^2,$$

$$h_6 = f_1f_6f_{11} + f_1f_7f_{10} - 2f_1f_8f_9 + f_2f_6f_{10} - f_2f_8^2 - f_3^2f_{10} + f_3f_8f_4 + f_4f_3f_8 - f_4^2f_6,$$

$$h_4 = f_1f_6f_{12} + f_1f_7f_{11} - f_1f_9^2 + f_2f_6f_{11} + f_2f_7f_{10} - 2f_2f_8f_9 - f_3^2f_{11} + f_3f_9f_4$$
$$+ f_3f_8f_5 + f_4f_3f_9 - f_4^2f_7 - f_4f_6f_5 + f_5f_3f_8 - f_5f_6f_4,$$

$$h_2 = f_1f_7f_{12} + f_2f_6f_{12} + f_2f_7f_{11} - f_2f_9^2 - f_3^2f_{12} + f_3f_9f_5 - f_4f_7f_5 + f_5f_3f_9 - f_5f_7f_4 - f_5^2f_6,$$

$$h_0 = f_2f_7f_{12} - f_7f_5^2.$$

The coefficients $f_i(i = 1, \ldots, 12)$ are given by

$$f_1 = \frac{1}{R}\left(P_{55} - \frac{1}{R}P_{36} + \frac{1}{4R^2}P_{66}\right), \qquad f_2 = -P_{11}\bar{m}^2,$$

$$f_3 = \bar{m}\left[\frac{1}{R}(P_{12} + P_{13}) + \frac{1}{R^2}(P_{15} + P_{36}) - \frac{3}{4R^3}P_{66}\right],$$

$$f_4 = -\frac{\bar{m}}{R^2}\left(P_{15} + 2P_{36} - \frac{1}{R}P_{66}\right), \qquad f_5 = \frac{P_{12}}{R}\bar{m} + P_{14}\bar{m}^3,$$

$$f_6 = -\frac{1}{R^2}\left(P_{22} + \frac{1}{R^2}P_{55} + \frac{2}{R}P_{25}\right), \qquad f_7 = \bar{m}\left(P_{33} + \frac{3}{R}P_{36} + \frac{9}{4R^2}P_{66}\right),$$

$$f_8 = \frac{1}{R^3}\left(P_{25} + \frac{1}{R}P_{55}\right),$$

$$f_9 = -\frac{1}{R^2}\left(P_{22} + \frac{1}{R}P_{52}\right) - \frac{\bar{m}^2}{R}\left(2P_{36} + P_{24} + \frac{3}{R}P_{66} + \frac{1}{R}P_{54}\right), \qquad f_{10} = -\frac{1}{R^4}P_{55},$$

$$f_{11} = \frac{2}{R^3}P_{25} + \frac{\bar{m}}{R^2}(2P_{45} + 4P_{66}), \qquad f_{12} = -\frac{1}{R}P_{22} - \frac{2}{R}P_{24}\bar{m}^2 - P_{44}\bar{m},$$

$$\bar{m} = m\pi/L.$$

APPENDIX C: MATRICES $[T_m]$, $[R]$, $[A]$, $[Q]$, $[M_s]$ AND $[k_s^{(L)}]$

Matrix $[T_m]_{(3\times 3)}$:

$$[T_m] = \text{Diag}\,[\cos\bar{m}x, \sin\bar{m}x, \sin\bar{m}x], \qquad \bar{m} = m\pi/L.$$

Matrix $[R]_{(3\times 8)}$:

$$R(1,j)=\alpha_j\,e^{\eta_j\theta},\quad R(2,j)=e^{\eta_j\theta},\quad R(3,j)=\beta_j\,e^{\eta_j\theta},\qquad j=1,\ldots,8.$$

Matrix $[A]_{(8\times 8)}$:

$$A(1,j)=\alpha_j,\qquad A(5,j)=\alpha_j\,e^{\eta_j\phi},\qquad A(2,j)=1,\qquad A(6,j)=e^{\eta_j\phi},\qquad A(3,j)=\eta_j,$$
$$A(7,j)=\eta_j\,e^{\eta_j\phi},\qquad A(4,j)=\beta_j,\qquad A(8,j)=\beta_j\,e^{\eta_j\phi}.$$

Matrix $[Q]_{(6\times 8)}$:

$$Q(1,j)=A_j\,e^{\eta_j\theta},\qquad Q(4,j)=D_j\,e^{\eta_j\phi},\qquad Q(2,j)=B_j\,e^{\eta_j\theta},$$
$$Q(5,j)=E_j\,e^{\eta_j\theta},\qquad Q(3,j)=C_j\,e^{\eta_j\theta},\qquad Q(6,j)=F_j\,e^{\eta_j\phi}.$$

The terms A_j, B_j, C_j, E_j and F_j ($j=1,\ldots,8$) may be expressed as follows:

$$A_j=-m\pi\alpha_j/L,\qquad B_j=-(\eta_j\beta_j+1)/R,\qquad C_j=-m\pi\beta_j/L+\eta_j\alpha_j/R,$$
$$D_j=-(m\pi)^2/L^2,\qquad E_j=-(\eta_j^2+\eta_j\beta_j)/R^2,$$
$$F_j=-2m\pi\eta_j/RL+3m\pi\beta_j/2RL-\eta_j\alpha_j/2R^2.$$

Matrices $[m]_{(8\times 8)}$ and $[k^L]_{(8\times 8)}$:

$$[m_S]=\rho_s t[A^{-1}]^{\mathrm{T}}[S][A^{-1}],\qquad [k_S^{(L)}]=[A^{-1}]^{\mathrm{T}}[G][A^{-1}],$$

where $[S]$ and $[G]$ are defined by

$$S(i,j)=\frac{RL}{2}\frac{(\alpha_i\alpha_j+\beta_i\beta_j+1)}{(\eta_i+\eta_j)}(e^{(\eta_i+\eta_j)\phi}-1),\qquad \text{if } \eta_i+\eta_j\neq 0;$$

$$S(i,j)=\frac{RL\phi}{2}(\alpha_i\alpha_j+\beta_i\beta_j+1),\qquad \text{if } \eta_i+\eta_j=0;$$

$$\begin{aligned}G(i,j)=\frac{RL}{2}(&p_{11}A_iA_j+p_{12}A_iB_j+p_{14}A_iD_j+p_{15}A_iE_j\\&+p_{21}B_iA_j+p_{22}B_iB_j+p_{24}B_iD_j+p_{25}B_iE_j\\&+p_{41}D_iA_j+p_{42}D_iB_j+p_{44}D_iD_j+p_{45}D_iE_j\\&+p_{51}E_iA_j+p_{52}E_iB_j+p_{54}E_iD_j+p_{55}E_iE_j\\&+p_{33}C_iC_j+p_{36}C_iF_j+p_{63}F_iC_j+p_{66}F_iF_j);\\&\frac{(e^{(\eta_i+\eta_j)\phi}-1)}{(\eta_i+\eta_j)},\qquad \text{if } \eta_i+\eta_j\neq 0);\end{aligned}$$

$$G(i,j)=\frac{RL\phi}{2}(p_{11}A_iA_j+\cdots+p_{66}F_iF_j),\qquad \text{if } \eta_i+\eta_j=0.$$

The terms A_i, B_i, C_i, D_i, E_i and F_i ($i=1,\ldots,8$) are listed with matrix $[Q]$.

APPENDIX D: DEFORMATION VECTORS

The deformation strain is defined by

$$\{\varepsilon\}=\{\varepsilon_L\}+\{\varepsilon_{NL}\},$$

where

$$
\{\varepsilon\} = \left\{ \begin{array}{c}
\dfrac{\partial U}{\partial x} \\[2ex]
\dfrac{1}{R}\left(\dfrac{\partial V}{\partial \theta} + W\right) \\[2ex]
\dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\[2ex]
-\dfrac{\partial^2 W}{\partial x^2} \\[2ex]
-\dfrac{1}{R^2}\left(\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial V}{\partial \theta}\right) \\[2ex]
-\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial x} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta}
\end{array} \right\}
$$

and

$$
\{\varepsilon_{NL}\} = \left\{ \begin{array}{c}
\frac{1}{2}\left(\dfrac{\partial W}{\partial x}\right)^2 + \frac{1}{8}\left(\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right)^2 \\[2ex]
\dfrac{1}{2R^2}\left(V - \dfrac{\partial W}{\partial \theta}\right)^2 + \frac{1}{8}\left(\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right)^2 \\[2ex]
\dfrac{1}{2R}\left(\dfrac{\partial W}{\partial x}\dfrac{\partial W}{\partial \theta} - V\dfrac{\partial W}{\partial x}\right) \\[2ex]
0 \\
0 \\
0
\end{array} \right\}.
$$

APPENDIX E: NOMENCLATURE

Symbols

A, B, C	constants in equation (2) defining U, V and W respectively
A_i	motion amplitude
a_p, b_p, c_p	modal coefficients determined by equation (19)
$a_p^{(1)}$, $b_p^{(1)}$, $c_p^{(1)}$	coefficient determined by equations (23)–(25)
$a_{rs}^{(1)}$, $b_{rs}^{(1)}$, $b_{rs}^{(2)}$, $c_{rs}^{(1)}$	coefficient determined by equations (26)–(28)
aA_{prs}, bB_{prs}, cC_{prs}, aB_{prs}, bA_{prs}	modal coefficients determined by equation (18)
A_{pq}, B_{pq}, C_{pq}	modal coefficients determined by equations (20)–(22)
A_{prsq}, B_{prsq}, C_{prsq}, AB_{prsq}, BA_{prsq}	modal coefficients determined by equation (18)
E	Young's modulus
e	exponential
f_i	function determined by equation (72)
$GG(p, q)$	coefficient determined by equation (32)
i	$\mathrm{i}^2 = -1$
$\mathrm{J}_{\mathrm{i}\eta_j}$	Bessel function of the first kind and of order i_{η_j}

L	length of the shell
m	axial mode number
N	number of finite elements
n	circumferential mode number
P_u	lateral pressure exerted on the shell; $u = i$ for internal pressure and $u = e$ for external pressure
P_{ij}	terms of elasticity matrix ($i = 1, \ldots, 6; j = 1, \ldots, 6$)
R	mean radius of the shell
R_j	solution of Bessel equation (47)
S_j	defined by equation (45)
$SS(p, q)$	coefficient determined by equation (36)
t	thickness of the shell
U, V, W	axial, tangential and radial displacements
U_{xu}	velocity of the liquid
V_x, V_θ, V_r	axial, tangential and radial fluid velocity (equation (38))
x	axial co-ordinate
$Y_{i\eta_j}$	Bessel function of the second kind and of order $i\eta_j$
Z_{u_j}	defined by equation (49) for $u = i$ and equation (50) for $u = o$
η_i	complex roots of the characteristic equation (3)
α_p, β_p	determined by equation (4)
θ	circumferential co-ordinate
v	Poisson ratio
ϕ	opening angle for one finite element
ϕ_T	opening angle for the whole open shell
Φ	velocity potential
ρ_s	density of the shell material
ρ_{fu}	density of fluid, $u = i$ for internal fluid and $\mu = e$ for external fluid
ω_L	linear frequency of free vibrations
ω_{NL}	non-linear frequency of free vibrations
τ	time related co-ordinates
ω_i, κ_i	coefficient determined by equation (74)
λ_i, σ_i	coefficient determined by equation (75)
ζ_i, ξ_i, γ_i	coefficient determined by equation (76)

Matrices

$[A]$	defined by equation (6)
$[B]$	defined by equation (8)
$[c_f^{(L)}], [c_f^{(NL)}]$	linear and non-linear damping matrices for a fluid finite element
$\{C\}$	vector of arbitrary constants
$[D_f^{(L)}]$	defined by equation (57)
$[D_f^{(NL)}]$	defined by equation (61)
$[G_f^{(L)}]$	defined by equation (58)
$[G_f^{(NL)}]$	defined by equation (62)
$[GD_f^{(NL)}]$	defined by equation (63)
$[k_f^{(L)}], [k_f^{(NL)}]$	linear and non-linear stiffness matrix for a fluid finite element
$[k_s^{(L)}], [k_s^{(NL2)}], [k_s^{(NL3)}]$	linear and non-linear stiffness matrix for a shell finite selement
$[kc_f^{(NL)}]$	defined by equation (60)

$[m_f^{(L)}]$	mass matrix for a fluid finite element
$[m_s]$	mass matrix for a shell finite element
$[N]$	displacement function defined by equation (7)
$[P]$	elasticity matrix
$[Q]$	defined by equation (8)
$\{q\}$	time-related vector co-ordinates
$[R]$	defined by equation (5)
$[S_f^{(L)}]$	defined by equation (56)
$[T_m]$	defined by equation (5)
$\{\delta_i\}$	vector of degrees of freedom at node i
$\{\delta\}$	vector of degrees of freedom for total shell
$\{\sigma\}$	stress vector
$\{\varepsilon_L\}, \{\varepsilon_{NL}\}$	linear and non-linear components of the deformation vector, respectively
$[\Phi]$	matrix of eigenvectors, equation (67)