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LETTERS TO THE EDITOR

TRANSVERSE VIBRATIONS OF SIMPLY SUPPORTED RECTANGULAR PLATES WITH RECTANGULAR CUTOUTS

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1. Introduction

Consider the structural system shown in Figure 1. The plate is characterized by the properties: ρ_1 (material density), h_1 (thickness), E_1 (Young's modulus) and μ_1 (Poisson ratio) while the corresponding properties of the insert are: ρ_2, h_2, E_2 and μ_2. The above characterization generates several types of situations, e.g., if $E_1 = E_2$, $\rho_1 = \rho_2$ and $\mu_1 = \mu_2$ but $h_2 > h_1$ one has the case of an overstepped plate [1–4]; if all the properties are different one has the case of a non-homogeneity which may be caused by a manufacturing process and if $\rho_2 = h_2 = E_2 = \mu_2 = 0$ a hole with free edges results [5–7]. The situations previously described are of interest in practically all fields of engineering: from naval and ocean engineering systems to structural elements used in aeronautical, civil and mechanical engineering.

Obtaining an exact solution is probably out of the question in view of the difficulty of satisfying the interface conditions in an exact fashion, as it has been pointed out by Warburton in an excellent discussion [2]. Hence, an approximate solution is proposed in the present paper whereby the displacement amplitude is expressed in terms of a double Fourier series which satisfies identically the boundary conditions at $x = 0$, a and $y = 0$, b and which constitutes the exact solution in the case of a solid, rectangular plate of uniform thickness. The Rayleigh-Ritz method is then applied to determine the fundamental frequency coefficient. This approach is, obviously, an extension of the methodology presented by the senior author and coworkers in reference [6]. Numerical results are presented only for the case of rectangular plates with holes but the methodology is proposed for the general structural case previously described and depicted in Figure 1. On the other hand, in several instances, frequency coefficients have been determined using a very accurate finite element code which utilizes the algorithm developed by Bogner *et al*. [8].

2. Approximate analytical solution

The Rayleigh-Ritz method requires minimization of the functional

$$J[W] = U_{max} - T_{max}, \tag{1}$$

where (see Figure 1)

$$U_{max} = \text{maximum strain energy}, \tag{2a}$$

$$T_{max} = \text{maximum kinetic energy}, \tag{2b}$$

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and

$$D_i = E_i h_i^3/12(1-\mu_i^2), \qquad \text{where } i = 1,\ 2.$$

Expressing the displacement amplitude $W(x, y)$ in terms of a double Fourier series

$$W \simeq W_a = \sum_{n=1}^{N} \sum_{m=1}^{M} b_{nm} \sin(n\pi x/a) \sin(m\pi y/b) \tag{3}$$

and substituting in equation (2a) one obtains

$$U_{max} = (D_1/2)[I_1 + I_2 + 2\mu_1 I_3 + 2(1-\mu_1)I_4] - [(D_1 - D_2)/2][I_1' + I_2'] \\ -[D_1\mu_1 - D_2\mu_2]I_3' - [D_1(1-\mu_1) - D_2(1-\mu_2)]I_4', \tag{4a}$$

where

$$I_1 = \int_0^b \int_0^a \left(\frac{\partial^2 W}{\partial x^2}\right)^2 \mathrm{d}x\,\mathrm{d}y, \qquad I_1' = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\frac{\partial^2 W}{\partial x^2}\right)^2 \mathrm{d}x\,\mathrm{d}y,$$

$$I_2 = \int_0^b \int_0^a \left(\frac{\partial^2 W}{\partial y^2}\right)^2 \mathrm{d}x\,\mathrm{d}y, \qquad I_2' = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\frac{\partial^2 W}{\partial y^2}\right)^2 \mathrm{d}x\,\mathrm{d}y,$$

$$I_3 = \int_0^b \int_0^a \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} \mathrm{d}x\,\mathrm{d}y, \qquad I_3' = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} \mathrm{d}x\,\mathrm{d}y,$$

$$I_4 = \int_0^b \int_0^a \left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y, \qquad I_4' = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

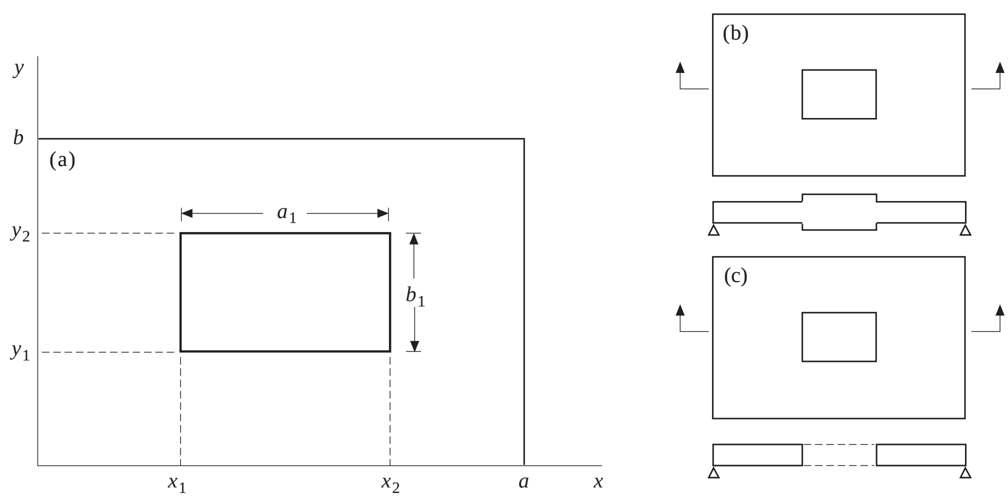

Figure 1. Vibrating system under consideration: (a) General case: plate characterized by (h_1, ρ_1, E_1, μ_1) with an insert (h_2, ρ_2, E_2, μ_2). Particular situations are (b) overstepped plate and (c) plate with a rectangular hole of free edges.

TABLE 1

Values of Ω_1 in the case of square plates with concentric square cutouts (Figure 2)

μ	a_1/a	Number of non-zero terms used: 1	4	9	16	Finite Element Solution	References: [5]	[7]
0·0	0	19·739	19·739	19·739	19·739	19·739	19·63	–
	0·1	19·940	19·935	19·922	19·902	–	–	–
	1/6	20·316	20·282	20·201	20·113	19·929	19·48	–
	0·2	20·589	20·525	20·390	20·271	–	–	–
	0·3	21·840	21·628	21·360	21·247	–	–	–
	1/3	22·438	22·170	21·890	21·790	21·657	21·45	–
	0·4	24·001	23·640	23·367	23·283	–	–	–
	0·5	27·660	27·197	26·950	26·869	–	26·05	–
0·30	0	19·739	19·739	19·739	19·739	19·739	19·63	–
	0·1	19·880	19·870	19·844	19·806	19·463	–	–
	1/6	20·145	20·070	19·905	19·712	19·205	–	–
	0·2	20·338	20·193	19·904	19·634	19·147	–	20·3
	0·3	21·232	20·700	20·096	19·853	19·722	–	20·8
	1/3	21·663	20·972	20·349	20·156	19·772	–	–
	0·4	22·806	21·824	21·279	21·152	20·773	–	22·1
	0·5	25·543	24·326	23·965	23·841	23·473	24·75	–

Similarly, substituting equation (3) in equation (2b) results in

$$T_{max} = (\omega^2/2)[\rho_1 h_1 Z - (\rho_1 h_1 - \rho_2 h_2) Z'], \tag{4b}$$

where

$$Z = \int_0^b \int_0^a W^2 \,\mathrm{d}x\,\mathrm{d}y, \qquad Z' = \int_{y_1}^{y_2} \int_{x_1}^{x_2} W^2 \,\mathrm{d}x\,\mathrm{d}y.$$

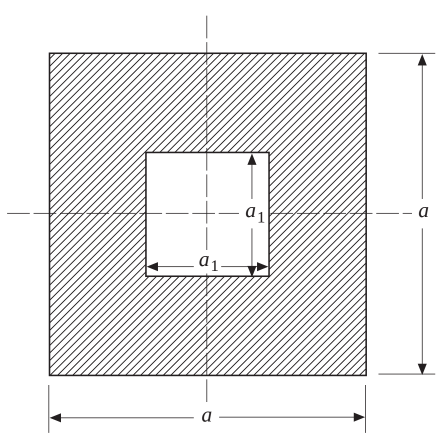

Figure 2. Vibrating simply supported plate of square shape with free, concentric; square cutout.

TABLE 2

Values of Ω_1 in the case of rectangular plates with concentric rectangular cutouts ($b/a = b_1/a_1$); see Figure 3

a_1/a	b/a 1	0·9	0·8	0·7	0·6	0·5
0·00	19·739	22·054	25·291	30·012	37·285	49·348
0·05	19·76	22·08	25·32	30·05	37·32	49·38
0·10	19·80	22·12	25·36	30·07	37·32	49·32
0·15	19·75	22·06	25·26	29·91	37·03	48·76
0·20	19·63	21·92	25·07	29·61	36·50	47·76
0·25	19·62	21·90	25·01	29·45	36·13	46·95
0·30	19·85	22·14	25·25	29·65	36·22	46·95
0·35	20·35	22·69	25·84	30·27	36·81	47·23
0·40	21·15	23·57	26·81	31·31	37·93	48·36
0·45	22·28	24·83	28·20	32·84	39·60	50·16
0·50	23·84	26·55	30·11	34·97	41·98	52·80

The Rayleigh-Ritz method requires minimization of the governing functional with respect to the b_{nm}'s:

$$\frac{\partial J[W]}{\partial b_{nm}} = \frac{\partial U_{max}}{\partial b_{nm}} - \frac{\partial T_{max}}{\partial b_{nm}} = 0, \tag{5}$$

which, taking into account equation (4a) and (4b) yields

$$\frac{D_1}{2}\left[\frac{\partial I_1}{\partial b_{nm}} + \frac{\partial I_2}{\partial b_{nm}} + 2\mu\frac{\partial I_3}{\partial b_{nm}} + 2(1-\mu_1)\frac{\partial I_4}{\partial b_{nm}}\right] - \frac{D_1 - D_2}{2}\left[\frac{\partial I_1'}{\partial b_{nm}} + \frac{\partial I_2'}{\partial b_{nm}}\right]$$

$$- [D_1\mu_1 - D_2\mu_2]\frac{\partial I_3'}{\partial b_{nm}} - [D_1(1-\mu_1) - D_2(1-\mu_2)]\frac{\partial I_4'}{\partial b_{nm}}$$

$$- \frac{\omega^2}{2}\left[\rho_1 h_1 \frac{\partial Z}{\partial b_{nm}} - (\rho_1 h_1 - \rho_2 h_2)\frac{\partial Z'}{\partial b_{nm}}\right] = 0 \qquad \begin{pmatrix} n = 1, 2, \ldots, N \\ m = 1, 2, \ldots, M \end{pmatrix}. \tag{6}$$

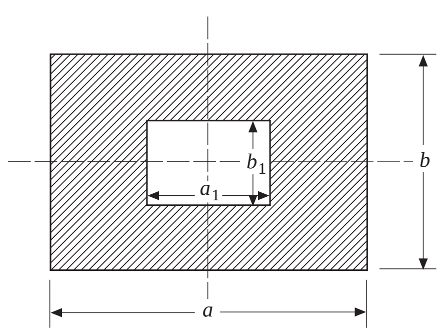

Figure 3. Vibrating rectangular plate with free, concentric rectangular cutout of the same aspect ratio ($b/a = b_1/a_1$).

TABLE 3

Values of Ω_1 in the case of rectangular plates with rectangular cutouts when the center of the hole displaces along the x-axis (Figure 4)

b/a	a_1/a	(a)	(b)	(c)	(d)
1	0·1	19·87	19·86	19·80	19·74 (19·72)
1	0·2	20·19	20·09	19·04	19·75 (19·52)
1	0·3	20·70	20·33	20·10	19·70 (19·13)
1	0·4	21·82	20·61	20·28	19·81
1	0·5	24·32	20·48	20·48	20·48
2/3	0·1	32·25	32·24	32·16	32·08 (32·05)
2/3	0·2	32·62	32·50	32·33	32·10 (31·80)
2/3	0·3	33·11	32·71	32·49	32·07 (31·40)
2/3	0·4	34·41	33·10	32·75	32·27
2/3	0·5	37·81	33·25	32·25	32·25
1/2	0·1	49·53	49·52	49·44	49·35
1/2	0·2	49·77	49·63	49·54	49·36
1/2	0·3	49·72	49·37	49·47	49·20
1/2	0·4	50·46	49·68	49·52	49·18
1/2	0·5	54·09	50·11	50·11	50·11

Note: values in parenthesis have been determined by means of the finite element method.

Expression (6) yields an $N \times M$ homogeneous, linear system of equations in the b_{nm}'s. A secular determinant in the natural frequency coefficients $\Omega_i = \sqrt{\rho h_1/D_1}\omega_i a^2$ of the system results from the non-triviality condition.

The present study is concerned with the determination of the fundamental frequency coefficient, Ω_1, in the case of plates with rectangular holes.

3. FINITE ELEMENT SOLUTION

The plate domain was subdivided into 144 rectangular elements, each element possessing 16 degrees of freedom [8]. It is important to point out that the element developed by Bogner *et al*. [8] has been proved to be extremely accurate when dealing with thin plates.

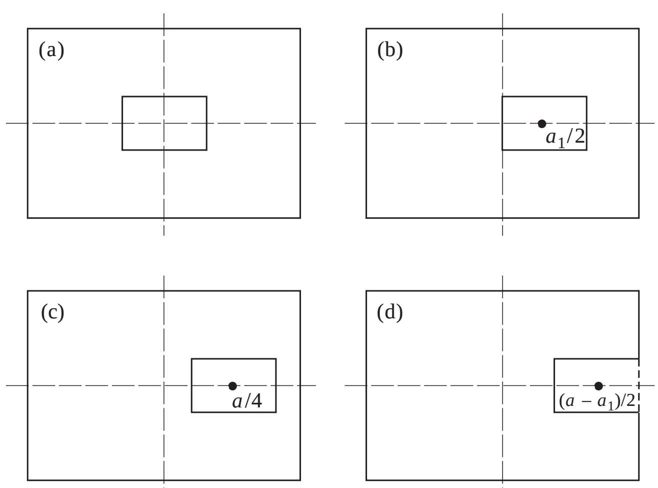

Figure 4. Mechanical system under analysis when the cutout is displaced along the x-axis. Positions of the cutout center: (a) $x_0 = 0$; (b) $x_0 = a_1/2$; (c) $x_0 = a/4$ and (d) $x_0 = (a - a_1)/2$.

TABLE 4

Values of Ω_1 in the case of rectangular plates with rectangular cutouts when the center of the hole is displaced along the y-axis (Figure 5)

b/a	a_1/a	(a)	(b)	(c)	(d)
1	0·1	19·87	19·86	19·80	19·74
1	0·2	20·19	20·09	19·04	19·75
1	0·3	20·70	20·33	20·10	19·70
1	0·4	21·82	20·61	20·28	19·81
1	0·5	24·32	20·48	20·48	20·48
2/3	0·1	32·25	32·24	32·16	32·08
2/3	0·2	32·62	32·42	32·31	32·06
2/3	0·3	33·11	32·47	32·30	31·64
2/3	0·4	34·41	32·54	32·05	31·13
2/3	0·5	37·81	31·48	31·48	31·48
1/2	0·1	49·53	49·52	49·43	49·35 (49·21)
1/2	0·2	49·77	49·43	49·47	49·27 (47·70)
1/2	0·3	49·72	48·78	48·91	48·04 (44·26)
1/2	0·4	50·46	48·06	47·43	45·88 (41·13)
1/2	0·5	54·09	44·94	44·94	44·94 (40·25)

Note: values in paranthesis have been determine by means of the finite element method.

4. NUMERICAL RESULTS

All calculations have been performed for a simply supported rectangular plate of uniform thickness taking $\mu_1 = \mu = 0{\cdot}30$; exception is made of a set of results presented in Table 1 where $\mu = 0$.

Using the Fourier series approach a 16×16 secular determinant was posed for all the situations. Obviously this means that when the mechanical configuration possesses 2 axes of symmetry, 49 terms of the series were employed but only those terms with odd subscripts contributed. Table 1 illustrates the convergence of the approach as the number of terms in the Fourier approximation is increased in the case of a square plate with a concentric

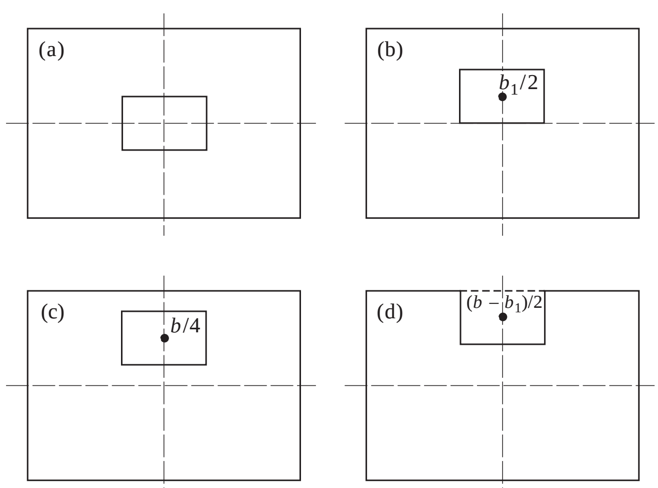

Figure 5. Mechanical system under analysis when the cutout is displaced along the y-axis. Positions of the cutout center: (a) $y_0 = 0$; (b) $y_0 = b_1/2$; (c) $y_0 = b/4$ and (d) $y_0 = (b - b_1)/2$.

TABLE 5

Values of Ω_1 in the case of rectangular plates with rectangular cutouts when the center of the hole is displaced along a diagonal of the rectangle (*Figure* 6)

b/a	a_1/a	(a)	(b)	(c)	(d)
1	0·1	19·87	19·85	19·73	19·59
1	0·2	20·19	19·99	19·69	19·22
1	0·3	20·70	19·99	19·57	18·71
1	0·4	21·82	19·79	19·28	18·21
1	0·5	24·32	18·25	18·25	18·25
2/3	0·1	32·25	32·23	32·07	31·87
2/3	0·2	32·62	32·29	31·99	31·35
2/3	0·3	33·11	32·13	31·73	30·53
2/3	0·4	34·41	31·78	31·14	29·49
2/3	0·5	37·81	29·20	29·20	29·20
1/2	0·1	49·53	49·51	49·33	49·12
1/2	0·2	49·77	49·27	49·18	48·50
1/2	0·3	49·72	48·56	48·63	47·22
1/2	0·4	50·46	49·90	47·37	44·80
1/2	0·5	54·09	43·49	43·41	43·41

square cutout. The results are compared with the finite element determinations performed in the present study and values available in the open literature for $\mu = 0$ and $\mu = 0{\cdot}30$.

The agreement between the analytical approach and the finite element results is excellent for all the situations considered (the maximum differences are of the order of 2%). The values obtained by Paramasivam [5] are considerably lower; exception is made for the case $a_1/a = 0{\cdot}5$ for $\mu = 0{\cdot}30$ where the eigenvalue determined in references [5] is considerably higher than the values determined in this study. The values determined in reference [7] are

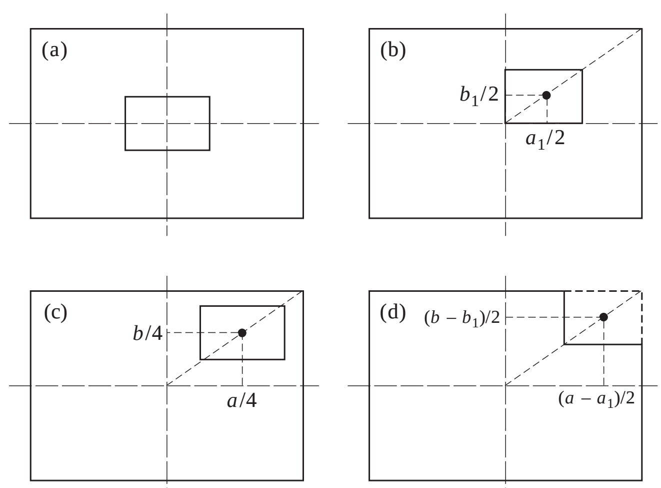

Figure 6. Mechanical system under analysis when the cutout is displaced along the diagonal. Positions of the cutout center: (a) $x_0 = y_0 = 0$; (b) $x_0 = a_1/2$, $y_0 = b_1/2$; (c) $x_0 = a/4$, $y_0 = b/4$ and (d) $x_0 = (a - a_1)/2$, $y_0 = (b - b_1)/2$.

always upper bounds (in general they are rather high upper bounds† but a single term polynomial approximation was used in that study).

Table 2 depicts fundamental frequency coefficients in the case of rectangular plates aspect ratio‡ as the center of the cutout displaces along the x-axis and for four locations: $x = 0$; $x = a_1/2$, $x = a/4$ and $x = (a - a_1)/2$. Some values have also been obtained by means of the finite element method (the maximum differences are of the order of 2%).

Table 4 depicts values of Ω_1 in the case of rectangular plates as the center of the cutout is displaced along the y-axis and again for four locations ($y = 0$, $y = b_1/2$, $y = b/4$ and $y = (b - b_1)/2$). As expected, the differences between the analytical results and the finite element determinations are now considerably higher: a maximum difference of the order of 10% is observed.

Table 5 presents results of Ω_1 when the hole center is displaced along a diagonal of the plate (see Figure 6) for the following positions: $x = y = 0$, $x = a_1/2$ and $y = b_1/2$, $x = a/4$ and $y = b/4$ and, finally, $x = (a - a_1)/2$ and $y = (b - b_1)/2$.

It is interesting to point out that in the case of a square plate with a square cutout (Table 1) the dynamic stiffening effect is quite apparent for $a_1/a > 0{\cdot}3$ for $\mu = 0{\cdot}30$. In other words, the fundamental frequency coefficient increases with respect to the value corresponding to a solid plate.

In the case of rectangular plates with concentric cutouts of the same aspect ratio (Table 2), the dynamic stiffening effect can also be observed. If one considers that they are upper bounds with respect to the exact eigenvalues, it seems reasonable to assume that the dynamic stiffening effect becomes noticeable for $a_1/a > 0{\cdot}45$ for the configurations under study.

Apparently no dynamic stiffening phenomenon takes place in the cases reported in Tables 3–5. On the other hand, for these configurations, the fundamental frequency coefficient attains a maximum value when the cutout is concentric with the plate outer boundary. It is interesting to notice the fact that for $a_1/a = 0{\cdot}5$ the parameter Ω_1 remains practically constant for each value of b/a, for the positions (b), (c) and (d), when using the double Fourier series approach. As expected, the analytical approximation yields extremely high values of frequency parameters when the cutout is very large and "acts" in a highly antisymmetric fashion, e.g., $a_1/a = 0{\cdot}50$ for position (d) in the case of Figure 5.

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† The maximum difference is of the order of 7% for $a_1/a = 0{\cdot}40$.

‡ This is also the case in Tables 4 and 5.

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