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A NONSTANDARD ANALYSIS OF A SIMPLE DISCONTINUOUS FORCE EQUATION MODELLING CONTINUOUS MOTION

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1. INTRODUCTION

Many problems in mechanical vibrations and control theory are modelled by idealized differential equations containing discontinuous terms. An example of this phenomena is found in dynamics problems involving friction damping (Wang [1]). Friction occurs in all real mechanical systems and is often described as a discontinuous force depending on the sign of velocity. Since the motions of all real mechanical systems are continuous in time, idealized differential equations modelling real systems must produce continuous solutions in time. This technical paper presents a mathematical analysis of a simple discontinuous force equation which may be used to describe continuous motion for a wide range of physical problems involving friction. The theoretical methods used in this study are applicable to general vibration and control problems containing discontinuities.

Consider the following ordinary differential equation

$$\mathrm{d}v/\mathrm{d}t + S(v) = 0 \tag{1}$$

describing the motion of a rigid-body of unit mass, where v and $S = -F$ are the velocity and the driving force of the body, respectively. Here S is assumed to be a generalized jump function defined by

$$S(v) = \begin{cases} \alpha & \text{if} \quad v > 0, \\ 0 & \text{if} \quad v = 0, \\ \beta & \text{if} \quad v < 0, \end{cases} \tag{2}$$

such that the real numbers α and β satisfy: $\alpha \neq \beta$. Moreover, $S(v) = 0$ for $v = 0$, since continuous motion is being studied. Equation (1) combined with an initial velocity,

$$v(0) = v_0, \tag{3}$$

yields an initial value problem governed by a simple discontinuous force equation which may be applied to model continuous motion.

The initial value problem defined by equations (1), (2) and (3) does not have a classical solution; that is, no mathematical description of the velocity exists which is differentiable on all motion time intervals and satisfies equation (1). This initial value problem does, however, have classical solutions on time intervals which do not contain the jump discontinuity of S. Time intervals without the jump discontinuity correspond to the times when the velocity is: $v > 0$, $v < 0$, or $v = 0$. If time intervals are considered which contain the discontinuity, both continuous and discontinuous generalized solutions of equation (1) may be found.

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In this study, modern mathematical techniques are used to approximate continuous solutions of the discontinuous model problem. Three mathematical theories are applied to obtain continuous solutions analytically and numerically: classical analysis, nonstandard analysis, and nonlinear generalized functions. The theories of classical and nonstandard analysis are used to approximate continuous solutions, while the theory of nonlinear generalized functions is used to determine how the microstructure of the generalized solutions must be specified to obtain continuous solutions.

2. GENERALIZED SOLUTIONS

The first step in the analysis of the model problem defined by equations (1), (2) and (3) is to apply heuristic arguments to obtain a generalized solution for all motion time intervals. The fundamental idea used here is to obtain classical solutions for each time interval and then patch these solutions together to obtain a continuous solution.

Combining equation (1) with the definition of the generalized jump discontinuity, equation (2), yields three ordinary differential equations:

$$\mathrm{d}v/\mathrm{d}t = -\alpha \text{ if } v > 0, \qquad \mathrm{d}v/\mathrm{d}t = 0 \text{ if } v = 0, \qquad \mathrm{d}v/\mathrm{d}t = -\beta \text{ if } v < 0. \tag{4}$$

The solutions to the first and third equations are given by:

$$v = -\alpha t + c_1 \quad \text{if} \quad v > 0, \qquad v = -\beta t + c_2 \quad \text{if} \quad v < 0, \tag{5}$$

where c_1 and c_2 are constants of integration, whilst, the solution to the second equation is $v = 0$. Now, specifying an initial value $v(0) = v_0$ and assuming that the velocity is zero only at a single instant of time, the constants c_1 and c_2 of equation (5) may be determined so that the velocity is a continuous function of time. Figure 1 shows an example of a continuous generalized solution of equation (1) formed by patching the two solutions of equation (5) together for $v(0) = 1$, $\alpha = 1$ and $\beta = 1/3$.

Figure 2 shows a second example of a continuous generalized solution of equation (1) formed by patching two solutions of equation (5) together for $v(0) = 1$, $\alpha = 1$ and $\beta = -1$.

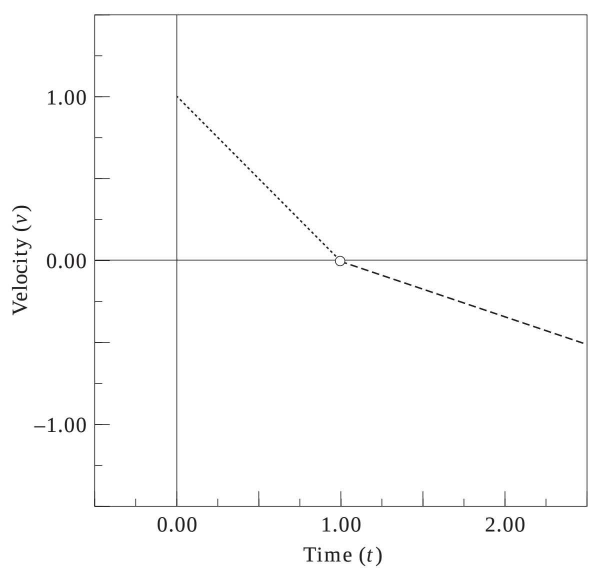

Figure 1. Continuous generalized solution of equation (1) for the values $\alpha = 1$, $\beta = 1/3$ and the initial condition $v(0) = 1$: · · · ·, solution for $v > 0$; ---, solution for $v < 0$.

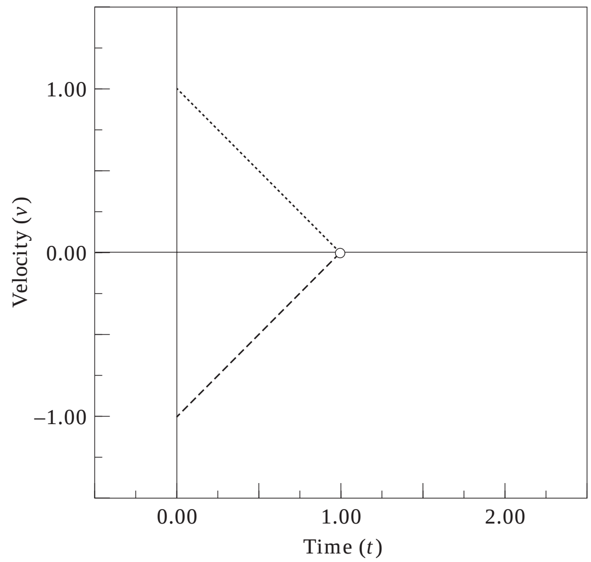

Figure 2. Continuous generalized solution of equation (1) for the signum function and the initial condition $v(0) = 1$. This generalized solution runs backwards in time and is physically unrealistic. Key as for Figure 1.

In this example, the jump function reduces to the signum function. Although this solution is continuous, it is physically unrealistic because it is multi-valued and, therefore, runs backwards in time. The backward time evolution of the solution is caused by the sign change of α and β in the jump discontinuity, here, $\alpha > 0 > \beta$. All jump discontinuities with a sign change yield a generalized solution with a branch that runs backwards in time. This suggests that either the positive branch, $v > 0$, or the negative branch, $v < 0$, may be combined with the zero solution, $v = 0$, to produce a single-valued continuous generalized solution for all time. Figure 3 shows the continuous generalized solution of equation (1)

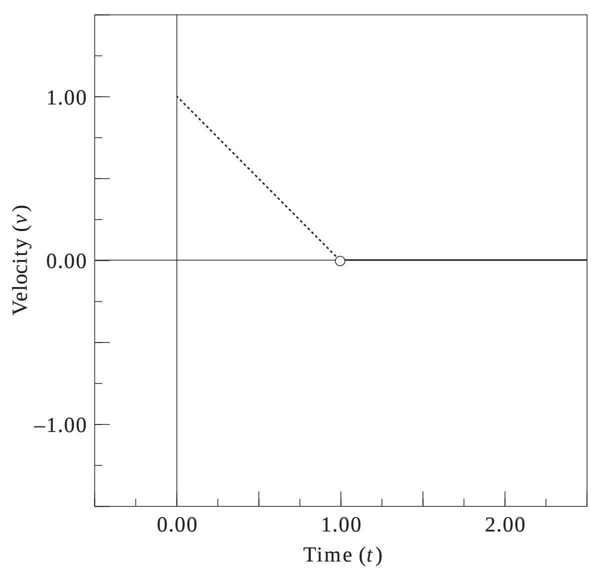

Figure 3. Continuous generalized solution of equation (1) for the signum function and the initial condition $v(0) = 1$. This generalized solution is valid for all time. · · · ·, solution for $v > 0$; ——, solution for $v = 0$.

formed by patching the solutions of $v > 0$ and $v = 0$ together for the signum function with the initial condition $v(0) = 1$. To contrast this example, if there is no sign change of α and β in the jump discontinuity, the two solutions of equation (5) may be patched together to form a continuous solution for all time as in the example of Figure 1.

The next step in the study of the model problem is to apply classical analysis to obtain smooth uniform approximations of continuous generalized solutions of equations (1), (2) and (3). To simplify the discussion, the case when S is the signum function with the initial value $v(0) = 1$ is considered. The goal is to approximate the continuous solution of Figure 3. Notice that the jump discontinuity S may be approximated as closely as desired by the smooth function

$$S(v) \cong (2/\pi) \arctan (mv) \tag{6}$$

for $v \neq 0$ and large values of m. Figure 4 shows the approximation of S for $m = 10$ and 100.

Substituting equation (6) into equation (1) and separating variables yields

$$-t = (\pi/2) \int_1^v \frac{1}{\arctan (m\zeta)} \, \mathrm{d}\zeta, \tag{7}$$

where the initial value $v(0) = 1$ has been applied. The integral on the right side of equation (7) may be written as

$$(\pi/2) \int_1^v \frac{1}{\arctan (m\zeta)} \, \mathrm{d}\zeta = (\pi/2m) \int_{\eta(1)}^{\eta(v)} \frac{1}{\eta \cos^2 \eta} \, \mathrm{d}\eta \tag{8}$$

by introducing the new variable

$$\eta(\zeta) = \arctan (m\zeta). \tag{9}$$

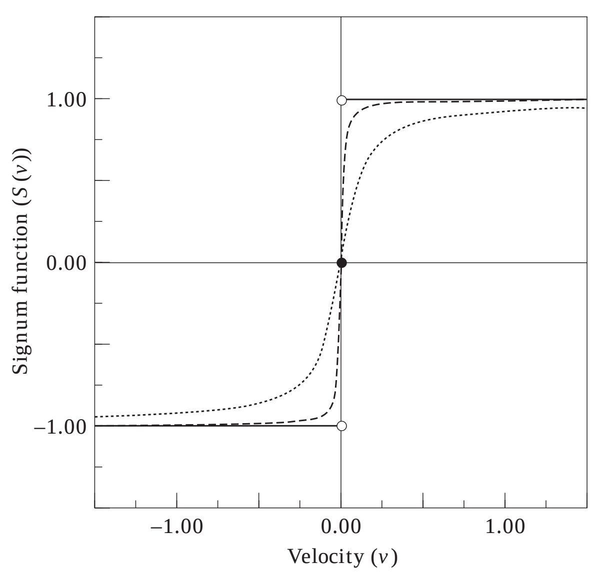

Figure 4. Smooth approximation of the signum function using the $(2/\pi) \arctan (mv)$ function: · · · ·, m = 10: ---, $m = 100$.

Then applying integral formula (10) from p. 190 of Gradshteyn and Ryzhik [2], the right side of equation (8) yields

$$(\pi/2)\int_1^v \frac{1}{\arctan(m\zeta)}\,d\zeta = (\pi/2m)\left\{\frac{\tan\eta}{\eta} + 6B_2\eta\ln\eta + \frac{1}{\eta^2}\sum_{k=2}^{\infty} f(k)(2\eta)^{2k}\right\}_{\eta(1)}^{\eta(v)}, \qquad (10)$$

where

$$f(k) = [(-1)^k(2^{2k}-1)/(2k-2)(2k)!]B_{2k} \qquad (11)$$

and where the B_{2k} for $k = 1, 2, 3, \ldots$ are the Bernoulli numbers. The integral formula defined in equations (10) and (11) converges for $|\eta| < \pi/2$.

Equation (10), when used in equation (7), supplies an analytic solution for equation (1) for $t \geqslant 0$ using the initial condition $v(0) = 1$. The infinite series in equation (10) is very complicated and appears to be unsuitable for numerical work. However, a useful asymptotic result may be obtained as follows. For $v > 0$, and large m, notice that

$$\arctan(mv) < \pi/2 \text{ for all } mv \quad \text{and} \quad \lim_{m\to\infty} \arctan(mv) = \pi/2. \qquad (12)$$

Then equations (10), (11) and (12) imply:

$$-t = (\pi/2)\int_1^v \frac{1}{\arctan(m\zeta)}\,d\zeta = v - 1 + O\left(\frac{1}{m}\right). \qquad (13)$$

Therefore, for large values of m, equation (7) approximates the continuous generalized solution

$$v = -t + 1 \quad \text{if} \quad t < 1, \qquad (14)$$

which gives $v(1) = 0$. To obtain a continuous generalized solution for $t > 1$, the solution $v = 0$ can be patched together with equation (14) to yield a solution for all $t \geqslant 0$. This produces the continuous generalized solution shown in Figure 3. A similar argument shows that equation (7) may be used to approximate a negative branch of a generalized solution.

Since the integral in equation (7) is very difficult to evaluate for the general jump function of equation (2), this problem has also been studied numerically. A fourth order, fixed-step size, Runge–Kutta scheme was used to integrate the differential form of equation (7) for $m = 5$, 10 and 100. Figure 5 shows that for values of m of the order of 100 or greater the numerical solution provides a very good approximation of the generalized solution. This figure illustrates the behavior of the numerical solution at the point where the positive branch of the generalized solution is mated with the zero solution.

In the above discussion, single-valued continuous generalized solutions of the model problem were defined and approximated. The smooth approximations of the generalized solutions developed here were mathematically difficult to construct, because they required both complicated integral formulae and limit arguments. The approximation of generalized solutions of the model problem may be simplified considerably by applying a linear description of the jump function together with nonstandard analysis.

3. NONSTANDARD SOLUTIONS

In this section, it will be shown that continuous generalized solutions of equations (1), (2) and (3) can be approximated to any order of accuracy by differentiable functions without using complicated integral formulae or numerical methods. Nonstandard analysis is a relatively new area of mathematics which emerged from basic research in mathematical

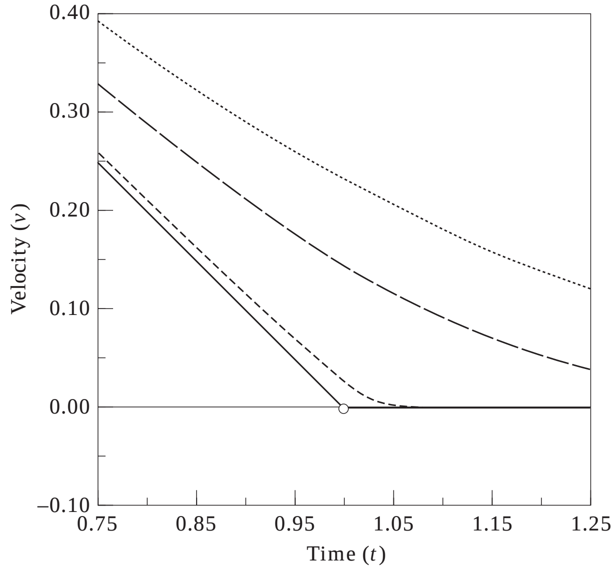

Figure 5. Numerical solution of equation (1) using equation (6) to describe the signum function: · · · · , $m = 5$; ——— , $m = 10$; – – –, $m = 100$.

logic. The subject was discovered in the early 1960s by Abraham Robinson [3]. The main contribution of nonstandard analysis to mathematics is the extension of the real numbers **R** to the hyperreal numbers ***R** which contain infinitely small (infinitesimals) and infinitely large numbers. The infinitesimals in the hyperreals have the properties that Newton and Leibniz discovered in their development of calculus, which justifies the algebraic manipulations of infinitesimals that engineers and physicists often use. Moreover, the existence of distinct infinitely large numbers that can be manipulated and used like finite numbers to solve problems provides a powerful new tool for applications. An excellent elementary introduction to nonstandard analysis is given by Henle and Kleinberg [4], while a brief overview of the properties of the hyperreal numbers needed for the present study may be found in Farassat and Myers [5].

The basic idea applied here to simplify the analysis of the model problem is to replace the smooth approximation of the jump function with a continuous linear approximation. The case when S is the signum function with the initial value $v(0) = 1$ is considered again. A continuous linear approximation of the signum function is obtained by joining the line passing through the origin of slope m to the constant functions $v = 1$ and $v = -1$:

$$S(v) \cong \begin{cases} 1 & \text{if} \quad v > 1/m, \\ mv & \text{if} \quad -1/m \leqslant v \leqslant 1/m, \\ -1 & \text{if} \quad v < -1/m. \end{cases} \tag{15}$$

Since equation (15) is valid for both the real and hyperreal numbers, m and v may assume any hyperreal value: infinitesimal, finite, or infinite. Now, if m is assumed to be an infinite hyperreal, say $^*m$, then $1/^*m$ and $-1/^*m$ are infinitesimal hyperreals such that $(^*m)(1/^*m) = 1$ and $(^*m)(-1/^*m) = -1$. This suggests that a line of infinite hyperreal slope $^*m$ may be used in equation (15) to approximate the signum function over an infinitesimal neighborhood containing zero: $[-1/^*m, 1/^*m]$. The main result of nonstandard analysis is used here: difficult limit processes are replaced by simple

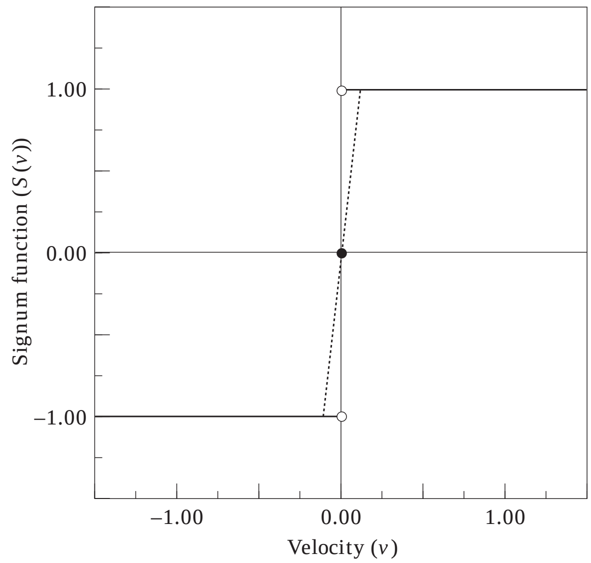

Figure 6. Continuous linear approximation of the signum function using the *mv* function on the neighborhood of zero: $[-1/m, 1/m]$; $\cdots\cdot$, $m = 10$.

arithmetic processes. Figure 6 shows the approximation of S defined by equation (15) for the real value of $m = 10$. A real number must be used to represent the plot, because infinitely small and infinitely large values cannot be plotted directly.

Then combining equation (1) with equation (15), for an infinite $^*m$, yields three ordinary differential equations:

$$\begin{aligned} \mathrm{d}v/\mathrm{d}t &= -1, &&\text{if} \quad v > 1/{}^*m; \\ \mathrm{d}v/\mathrm{d}t &= -{}^*mv, &&\text{if} \quad -1/{}^*m \leqslant v \leqslant 1/{}^*m; \\ \mathrm{d}v/\mathrm{d}t &= 1, &&\text{if} \quad v < -1/{}^*m. \end{aligned} \tag{16}$$

Integrating these equations produces

$$\begin{aligned} v &= -t + c_1, &&\text{if} \quad v > 1/{}^*m, \\ v &= c_2 \exp[-{}^*mt], &&\text{if} \quad -1/{}^*m \leqslant v \leqslant 1/{}^*m, \\ v &= t + c_3, &&\text{if} \quad v < -1/{}^*m, \end{aligned} \tag{17}$$

where c_1, c_2, and c_3 are constants of integration. Recall from section 2 that, since the signum function changes sign, one branch of the generalized solution evolves backwards in time, leading to a double-valued velocity. Hence, the approximation of the generalized solution will be obtained by patching two solution branches of equation (17) together which evolve forward in time and ignoring the branch which evolves backwards in time. Therefore, applying the initial condition $v(0) = 1$ and patching the first two solutions of equation (17) together yields

$$\begin{aligned} v &= -t + 1, &&\text{if} \quad t < 1 - 1/{}^*m, \\ v &= c_2 \exp[-{}^*mt], &&\text{if} \quad t \geqslant 1 - 1/{}^*m, \end{aligned} \tag{18}$$

where

$$c_2 = 1/{}^*m \exp[1 - {}^*m]. \tag{19}$$

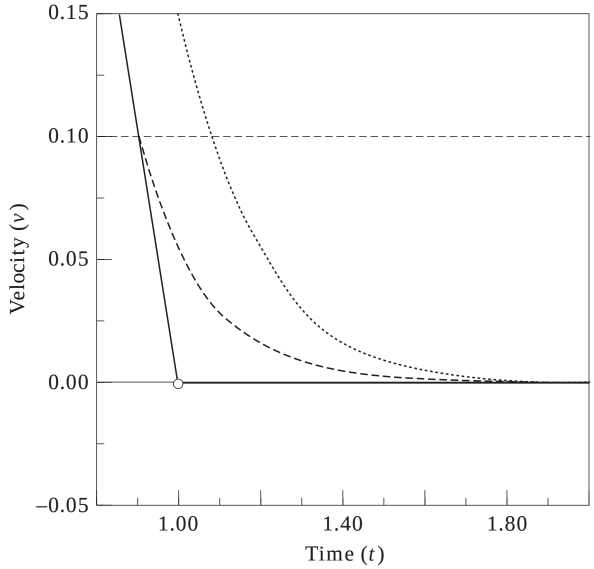

Figure 7. Comparison of the linear (———) and smooth (· · · ·) approximations of the continuous generalized solution of equation (1) for the signum function and the initial condition $v(0) = 1$. The nonstandard solution is the linear solution defined on an infinitesimally small neighborhood near zero. ---, 1/10 neighborhood boundary.

Equations (18) and (19) give a differentiable approximation of the continuous generalized solution of the model problem shown in Figure 3. This solution is of arbitrary accuracy, because the linear approximation of equation (15) assumes an infinitely large slope defined over an infinitesimally small neighborhood. Furthermore, equations (18) and (19) provide an approximation of equation (14) with a continuous first derivative, since the signum function was modeled using a continuous function.

Figure 7 compares the approximate solutions given by equations (18) and (19) and the numerical integration of equation (7), to the continuous generalized solution of the model problem given by equation (14) combined with the zero solution. The approximations of Figure 7 assume a real value of $m = 10$. A real number is used to represent the plot of the nonstandard solution, since the solution is valid for both real and hyperreal numbers. Recall that an infinitely small branch of the solution cannot be plotted directly. At this point the astute reader may ask: why use nonstandard analysis in this problem? Nonstandard analysis greatly simplifies the solution of the problem by encoding the limit processes into standard algebraic manipulations of hyperreal numbers. By assigning an infinite hyperreal value to m, $^*m$, the limit of the approximation is built into the solution. Therefore, the solution given by equations (18) and (19) is infinitely close to the generalized solution shown in Figure 3. On the other hand, if classical analysis is used, the resulting solution generated by a linear approximation of the jump function must be shown to approach the generalized solution of the model problem as m becomes large. Such limiting processes are often conceptually difficult and mechanically tedious to demonstrate.

4. JUMP FUNCTION MICROSTRUCTURE

Many linear differential equations with jump discontinuities may be analyzed using classical analysis. Classical analysis defines the discontinuous functions and their derivatives to be linear functionals acting on families of smooth test functions. Such linear functionals may be defined if the jump discontinuities depend on the independent time or space variables of the underlying differential equation.

In the present study, the model problem defined by equations (1), (2) and (3) is governed by a nonlinear ordinary differential equation. This equation is nonlinear, because the jump function is not linear in velocity:

$$S(v_1 + v_2) \neq S(v_1) + S(v_2) \tag{20}$$

for all $v_1, v_2 \neq 0$ satisfying $v_1 + v_2 \neq 0$. That is, the jump discontinuity is a function of the dependent variable, velocity, and not the independent variable, time. In general, nonlinearities involving jump discontinuities cannot be represented as simple linear functionals in terms of the unknown function and may lead to mathematically inconsistent results which depend on the specific method of approximation and not the defining differential equation.

Over the last two decades, Rosinger [6] and Colombeau [7], working independently, have developed mathematical theories of nonlinear generalized functions to analyze differential equations involving polynomial nonlinearities and jump discontinuities. The goal of their research has been to remove the mathematical inconsistencies that occur in problems that combine differentiation and multiplication with discontinuous functions. While the technical details are beyond the present discussion, the basic idea of nonlinear generalized functions is to define abstract solutions of differential equations as equivalence classes of reasonable approximations of a specific weak form of multiplication. By identifying all possible approximations for a given problem with an equivalence class or classes of abstract solutions, the mathematical inconsistencies are removed.

This approach yields a weak form of equality, $\sim$, called association, which allows a theory combining differentiation and multiplication with discontinuous functions. If equation (1) is cast in terms of the weak equality,

$$\mathrm{d}v/\mathrm{d}t + S(v) \sim 0, \tag{21}$$

nonlinear generalized solutions may be constructed using these ideas. These solutions will include both continuous and discontinuous motions. As shown by Colombeau *et al*. [8], the determination of a specific nonlinear generalized solution depends on the microstructure of the solution's jump function. Here, microstructure refers to how the nonlinear generalized solution is approximated in a neighborhood of a discontinuity. The upshot is that idealized problems like equation (1) require additional physical information to determine specific generalized solutions. By idealizing the jump as a discontinuity, equation (1) has discarded necessary physical information. For general nonlinear problems, such idealizations lead to mathematical inconsistencies which require the introduction of theories of nonlinear generalized functions in the analysis. While these theories admit mathematically consistent abstract solutions, they cannot specify individual generalized solutions without additional physical information.

For the model problem, individual generalized solutions are obtained by requiring that equations (1), (2) and (3) model single-valued continuous motion. Therefore, the generalized solutions of interest are continuous functions of time. This implies that the microstructure of the generalized solutions becomes trivial; no discontinuous solutions are allowed. Considering only continuous motion implies that continuous approximations of the jump discontinuity may be used to approximate generalized solutions that evolve forward in time. This property is the fundamental property used to construct the analytical and numerical approximations of the generalized solutions presented in this study.

5. CONCLUSIONS

Modern mathematical techniques have been applied to analyze a simple discontinuous force equation; the results of which are applicable to general friction problems containing discontinuities. Three mathematical theories were used to find and describe solutions of the model problem defined by equations (1), (2) and (3): classical analysis, nonstandard analysis, and nonlinear generalized functions.

Classical analysis was applied to approximate continuous generalized solutions of the model problem by using a smooth function to describe the jump discontinuity. Both analytical and numerical integration yielded very good approximations of single-valued continuous generalized solutions. Non-standard analysis was also applied to approximate generalized solutions of the model problem. Hyperreal numbers were combined with a linear approximation of the jump function to produce differentiable approximations of generalized solutions. Non-standard analysis greatly simplified the solution of the model problem by encoding the limit of the approximation into standard algebraic manipulations of hyperreal numbers. By using an infinite hyperreal slope in the description of the jump function, nonstandard solutions representing the limit were obtained that are infinitely close to continuous generalized solutions. The nonstandard solutions also provided finite approximations of continuous generalized solutions for real values. The nonstandard mathematical methods which were applied in this study yielded both insight and simplification in the analysis of the model problem and are expected to be of great utility for more complicated physical problems.

Basic ideas involving nonlinear generalized functions were then applied to argue that the model problem cannot produce *a priori* continuous solutions without additional physical information. Individual generalized solutions were obtained by requiring that the model problem describe continuous motion. This additional physical assumption forced the microstructure of the generalized solutions to be trivial and allowed the continuous approximations of the generalized solutions analyzed here.

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