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THE NON-LINEAR VIBRATION AND DYNAMIC INSTABILITY OF THIN SHALLOW SHELLS

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In this paper, the non-linear vibration and dynamic instability of thin shallow spherical and conical shells subjected to periodic transverse and in-plane loads are investigated. The Marguerre type dynamic equations used for the analysis of shallow shells, when treated by the Galerkin method, will result in a system of total differential equations in the time functions, known as Duffing and Mathieu equations, from which the various kinds of non-linear vibration and dynamic instability are determined by using numerical methods. Numerical results are presented for axisymmetric vibrations and dynamic instabilities of shallow spherical and conical shells with (a) clamped and (b) supported edge conditions. As numerical examples, non-linear vibration frequencies and instability regions for shells are determined. The effects of static load as well as static snap-through buckling on the instability are also investigated.

1. INTRODUCTION

To clarify the non-linear vibration and dynamic instability of shallow shells is of great importance for the design of aerospace structures, as shown in references [1–19]. However, only a few studies have been made in this subject, of pulsating external pressure, external excitation or parametric excitation, e.g., Chia and Chia [7], Tsai and Palazotto [5], Goncalves and Batista [7], Denisov and Zhinzher [14], Evansen [16], Leissa [17], Nath *et al*. [9], Jain and Nath [6], Mahrenholtz *et al*. [4], Yasuda and Kushida [10], Ye [19, 21]. They solved various kinds of non-linear problem, such as the problem of conical shells by applying Galerkin's method to the Donnell-type basic equations with one term approximate solutions for only the principal instability regions, the dynamic instability solutions of spherical shells under periodic load, and the solution of the non-linear dynamic response of doubly curved shallow shell on an elastic media. Some numerical results for non-linear vibrations of imperfect shells have been obtained by using finite element or collocation methods, etc. Recently, the author [19–21] analyzed the non-linear problems of circular plates and shallow shells with variable thickness by using a combined method involving an iterative method, Galerkin's method, and a numerical method. The results were satisfactory for some special cases.

In this paper, the non-linear vibration and dynamic instability of thin shallow spherical and conical shells subjected to periodic transverse and in-plane loads have been investigated. The Marguerre type dynamic equations used for the analysis of shallow shells when treated by Galerkin's method will result in a system of total differential equations in the time domain called Duffing and Mathieu equations, from which the various kinds of non-linear vibration and dynamic instability are determined by using numerical methods. Numerical results are presented for axisymmetric vibrations and dynamic

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instabilities of shallow spherical and conical shells with (a) clamped and (b) supported edge conditions. As numerical examples, non-linear vibration frequencies and instability regions for shells are determined. The effects of static load as well as static snap-through buckling on the instability are also investigated.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

A thin shallow shell with thickness h, base circle radius a and height of arch f, subjected to the uniformly distributed edge forces N and normal pressure q as shown in Fig. 1 will be considered. The Marguerre-type dynamic equations which can be found in references [4, 6, 8, 9], may be given by the following equation form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[D\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{v}{r}\frac{\partial w}{\partial r}\right)\right]\right\}+\frac{1}{r}\frac{\partial}{\partial r}\left[D(1-v)\left(\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w}{\partial r}\right)\right]+\rho h\frac{\partial^2 w}{\partial t^2}$$

$$=q(t)+\frac{1}{r}\frac{\partial}{\partial r}\left[rN_r\left\{\frac{\partial(w+z_0)}{\partial r}\right\}\right], \tag{1a}$$

$$\frac{\partial}{\partial r}\left\{r\left(\frac{\partial(rN_r)}{\partial r}-vN_r\right)\right\}-N_r+v\frac{\partial(rN_r)}{\partial r}=-\frac{Eh}{2}\left\{\left(\frac{\partial w}{\partial r}\right)^2+2\frac{\partial w}{\partial r}\frac{\partial z_0}{\partial r}\right\}. \tag{1b}$$

The relation between the incremental strain and deflection is given by

$$\partial(r\varepsilon_\theta)/\partial r-\varepsilon_r+\tfrac{1}{2}(\partial w/\partial r)^2=0 \tag{2}$$

while the relations between the incremental displacements and bending moments are given by

$$M_r=-D(r)(\partial^2 w/\partial r^2+(v/r)(\partial w/\partial r)),\qquad M_\theta=-D(r)((1/r)\partial w/\partial r+v\,\partial^2 w/\partial r^2). \tag{3a, b}$$

The non-linear strains in the radial and tangential directions are

$$\varepsilon_r=\partial u/\partial r+\tfrac{1}{2}(\partial w/\partial r)^2+(\partial w/\partial r)(\partial z_0/\partial r),\qquad \varepsilon_\theta=u/r \tag{4}$$

In the following equations, the non-dimensional notations are related to the physical variables as

$$x=r/a,\qquad W=\sqrt{12(1-v^2)}\,w/h,\qquad p=[12(1-v^2)]^{3/2}qa^4/Eh^4,$$

$$S=12(1-v^2)(N_r a^2/Eh^3)x,\qquad \tau=\frac{h}{a^2\sqrt{12(1-v^2)}}\sqrt{\frac{E}{\rho}}\,t,\qquad K=\sqrt{12(1-v^2)}\,\frac{f}{h}. \tag{5}$$

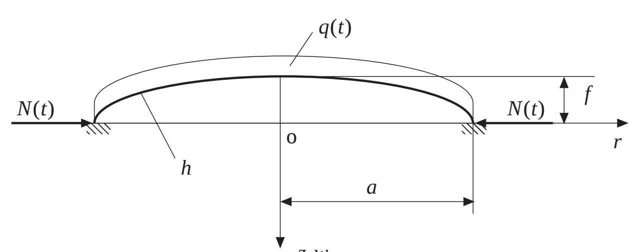

Figure 1. Geometric relations of thin shallow shell.

In these expressions, $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the shell, E, ν and ρ are Young's modulus, Poisson ratio and mass density, respectively.

Concerning the boundary conditions for the problems, the following two cases will be considered at $x = 0$ and 1,

$$\text{SS:}\quad W|_{x=1} = \left(\frac{\partial^2 W}{\partial x^2} + \frac{\nu}{x}\frac{\partial W}{\partial x}\right)\bigg|_{x=1} = 0, \qquad S|_{x=1} = S_s, \qquad S|_{x=0} - \textit{finite}, \tag{6}$$

$$\text{C:}\quad W|_{x=1} = \frac{\partial W}{\partial x}\bigg|_{x=1} = 0, \qquad S|_{x=1} = S_s, \qquad S|_{x=0} - \textit{finite}. \tag{7}$$

In these expressions, SS and C correspond to the simply supported and clamped cases, respectively. The governing equations in non-dimensional notations are

for the spherical case

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 W}{\partial x^2} + \frac{1}{x}\frac{\partial W}{\partial x}\right) + \frac{\partial^2 W}{\partial \tau^2} = p(\tau) + \frac{1}{x}\left\{S\left(\frac{\partial^2 W}{\partial x^2} + 2K\right) + \frac{\partial S}{\partial x}\left(\frac{\partial W}{\partial x} + 2Kx\right)\right\}, \tag{8a}$$

$$\frac{\partial^2 S}{\partial x^2} + \frac{1}{x}\frac{\partial S}{\partial x} - \frac{S}{x^2} = -\frac{1}{2x}\left(\frac{\partial^2 W}{\partial x^2} + 4\frac{\partial W}{\partial x}Kx\right), \tag{8b}$$

for the conical case

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 W}{\partial x^2} + \frac{1}{x}\frac{\partial W}{\partial x}\right) + \frac{\partial^2 W}{\partial \tau^2} = p(\tau) + \frac{1}{x}\left\{S\frac{\partial^2 W}{\partial x^2} + \frac{\partial S}{\partial x}\left(\frac{\partial W}{\partial x} + K\right)\right\}, \tag{8c}$$

$$\frac{\partial^2 S}{\partial x^2} + \frac{1}{x}\frac{\partial S}{\partial x} - \frac{S}{x^2} = -\frac{1}{2x}\left(\frac{\partial^2 W}{\partial x^2} + 2\frac{\partial W}{\partial x}K\right). \tag{8d}$$

Here, $z_0 = -f(1 - x^n)$, in which $n = 1$, represents a conical case and $n = 2$ a spherical case.

The problem consists of finding the limiting values of non-linear natural frequencies, forced vibration responses and dynamic instability behavior for which the basic equations have some solutions under the given loading and boundary conditions.

3. SOLUTIONS

An approximate solution is obtained by assuming the non-linear vibrations to have the same spatial shape, i.e.,

$$W(x, \tau) = T(\tau)Y(x), \tag{9}$$

or

$$W(x, \tau) = T(\tau)(1 - a_1 x^2 + a_2 x^4). \tag{10}$$

equations (6, 7) then yield

$$\text{C:}\quad a_1 = 2, \qquad a_2 = 1; \tag{11}$$

$$\text{SS:}\quad a_1 = (6 + 2\nu)/(5 + \nu), \qquad a_2 = (1 + \nu)/(5 + \nu). \tag{12}$$

TABLE 1
Value of functions A_1–A_8 ($\nu = 0{\cdot}3$)

		A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
Spherical	C	0·1000	10·667	−3·333	0·1778	−0·3333	0·1429	0·1667	0·6667
	SS	0·1472	3·6037	−3·144	0·1673	−0·2550	0·0867	0·2296	0·9182
Conical	C	0·1000	10·667	−3·333	0·1689	−0·3278	0·1429	0·1667	0·5333
	SS	0·1472	3·6037	−3·144	0·1228	−0·2162	0·0867	0·2296	0·6340

Substituting equation (9) or (10) into equations (8b), (8d), integrating the resulting equations and making use of boundary conditions (6) or (7) yields $S(x)$ for the spherical or conical case, respectively. Substituting $W(x, \tau)$ and $S(x)$ from equations (10) and (13) into the equations of motion (8a), (8c) and applying the Galerkin's procedure (multiplying both sides by $xW(x, \tau)$ and integrating from $x = 0$ to $x = 1$), one obtains the following equation

$$A_1\,\mathrm{d}^2T/\mathrm{d}\tau^2 + \{A_2 + A_3S_s(\tau) + A_4K^2\}T + A_5KT^2 + A_6T^3 = A_7p(\tau) + A_8S_s(\tau)K, \quad (13)$$

in which, A_1, A_2, A_3, A_4, A_5, A_6, A_7 and A_8 are the functions of a_1 and a_2. Some numerical results of these are listed in Table 1.

4. NUMERICAL RESULTS

The responses for different static and dynamic cases are obtained as follows.

4.1. STATIC SNAP-THROUGH BUCKLING

When $S_s = 0$, the snap-through buckling response is obtained from equation (13) by setting $\ddot{T} = S_s = 0$, to yield

$$p = (1/A_7)\{(A_2 + A_4K^2)T + A_5KT^2 + A_6T^3\}. \quad (16)$$

The snap-through buckling load p_{cr} is given by

$$K_{cr} = \sqrt{3A_2A_6/(A_5^2 - 3A_4A_6)}, \qquad T_{p_{cr}} = -A_5K_{cr}/3A_6,$$
$$p_{cr} = (1/A_7)\{(A_2 + A_4K_{cr}^2)T_{p_{cr}} + A_5K_{cr}T_{p_{cr}}^2 + A_6T_{p_{cr}}^3\}, \quad (17)$$

with the numerical results listed in Table 2. These results are the same as some of author's work in reference [19–21]. When $S_s \neq 0$, or $S_s = S_0$, which is a constant, we have the snap-buckling loads Table 3.

TABLE 2
The snap-through buckling loads

		K_{cr}	$T_{p_{cr}}$	p_{cr}
Spherical	C	11·45240346	8·903863009	605·2157232
	SS	8·956470345	6·109380293	403·7400021
Conical	C	11·41238498	8·728997049	570·2533343
	SS	9·230984567	6·485637082	427·6700267

TABLE 3

The snap-through buckling loads ($S_s = S_0$)

		S_0	K_{cr}	$T_{p_{cr}}$	p_{cr}
Spherical	C:	−1	9·496114551	7·382913413	307·0454035
	C:	1	13·12017144	10·20049718	962·4768566
Conical	C:	−1	9·462931990	7·237917889	294·8202023
	C:	1	13·07432523	10·00016618	899·2599879

TABLE 4

ω_N/ω_0 ($v = 0{\cdot}3,\ S_s = 0$) *for spherical shells*

		T_0			
	K	0·5	1·0	2·0	3·0
C	0	1·00253	1·00788	1·02190	1·04577
	5	1·16783	1·14793	1·09849	1·05843
	10	1·59909	1·56701	1·48441	1·40696
SS	0	1·00630	1·01811	1·04192	1·08320
	5	1·42531	1·38303	1·28528	1·19747
	10	2·31285	2·27008	2·13775	2·00977

4.2. FREE NON-LINEAR VIBRATION

The free non-linear vibrations are governed by

$$\ddot{T} + \omega_0^2\{(1 + (A_4/A_2)K^2)T + (A_5/A_2)KT^2 + (A_6/A_2)T^3\} = 0, \qquad (18)$$

where ω_0 is the non-dimensional linear frequency, which is $\omega_0 = \sqrt{A_2/A_1}$. The ratio of non-linear frequency ω_N to linear ω_0 is given by

$$\frac{\omega_N}{\omega_0} = \lim_{T_1 \to T_0} \frac{2\pi}{4\displaystyle\int_0^{T_1} \frac{\mathrm{d}T}{\sqrt{\left(1 + \dfrac{A_4}{A_2}K^2\right)(T_0^2 - T^2) + \dfrac{2A_5}{3A_2}\,K(T_0^3 - T^3) + \dfrac{A_6}{4A_2}(T_0^4 - T^4)}}}. \qquad (19)$$

The numerical results of non-linear frequencies and periods are listed in Tables 4 and 5.

TABLE 5

ω_N/ω_0 ($v = 0{\cdot}3,\ S_S = 0$) *for conical shells*

		T_0			
	K	0·5	1·0	2·0	3·0
C	0	1·00253	1·00788	1·02190	1·04577
	5	1·16954	1·16265	1·10642	1·06384
	10	1·58724	1·54114	1·45873	1·38150
SS	0	1·00630	1·01811	1·04192	1·08320
	5	1·32030	1·28252	1·19672	1·12318
	10	2·04734	1·99708	1·87193	1·75179

Table 6

Bifurcation points for spherical case

	$S^* = 1$		$S^* = 2$	
	Stable	Unstable	Stable	Unstable
(a) $K = 0$				
C	0·8438	1·1562	0·6876	1·3124
S	0·5638	1·4362	0·1276	1·8724
(b) $K = 5$				
C	1·4167	1·7291	1·2605	1·8853
S	0·7243	1·5967	0·2881	2·3210

From (18), one has the following statement of conservation of energy:

$$H = \tfrac{1}{2}\dot{T}^2 + \omega_0^2\left\{\left(1 + \frac{A_4}{A_2}K^2\right)\frac{T^2}{2} + \frac{A_5}{3A_2}KT^3 + \frac{A_6}{4A_2}T^4\right\}, \quad (20)$$

where H is the constant determined from the initial conditions and is the energy level. When $K \leqslant K_0$ (C: $K_0 = 25{\cdot}24$, SS: $K_0 = 13{\cdot}42$ for the spherical case and C: $K_0 = 22{\cdot}52$, SS: $K_0 = 18{\cdot}58$ for the conical case), the cases of H have the stable equilibrium state as indicated in Figs. 2 and 3.

4.3. Forced non-linear vibration

The case in which the shallow shells are continuously excited will now be considered. In this paper, two types of excitations are investigated: (a) $p(\tau) \neq 0$, $S_s = 0$ and (b) $p(\tau) \neq 0$, $S_s = S_s(\tau)$.

(a) The excitation appears as an inhomogeneous term in the equation governing the motion of the shell. The basic equation reduces to a kind of Duffing equation:

$$\ddot{T} + \omega_0^2\{1 + (A_4/A_2)K^2)T + (A_5/A_2)KT^2 + (A_6/A_2)T^3\} = p_0 \cos \Omega\tau \quad (21)$$

Equation (21) could be rewritten as

$$\dot{T}_1 = \omega_0 T_2, \qquad \dot{T}_2 = -\omega_0\left\{\left(1 + \frac{A_4}{A_2}K^2\right)T_1 + \frac{A_5}{A_2}KT_1^2 + \frac{A_6}{A_2}T^3\right\} + \frac{p_0}{\omega_0}\cos \Omega\tau, \quad (22)$$

Table 7

Bifurcation points for conical shape

	$S^* = 1$		$S^* = 2$	
	Stable	Unstable	Stable	Unstable
(a) $K = 0$				
C	0·8438	1·1562	0·6876	1·3124
S	0·5638	1·4362	0·1276	1·8724
(b) $K = 5$				
C	1·2396	1·5520	1·0834	1·7082
S	1·4157	2·2881	0·9795	2·7243

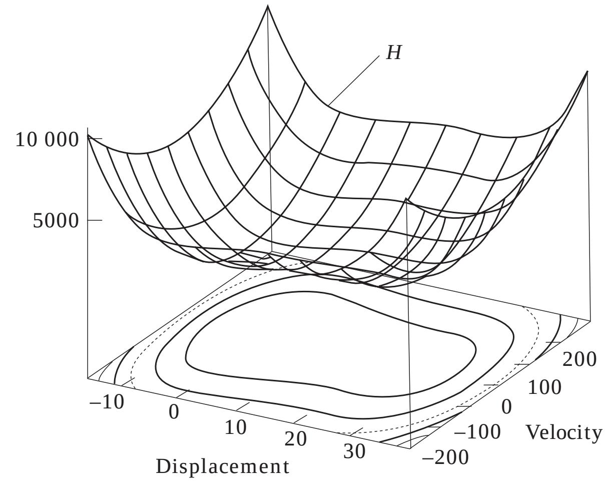

Figure 2. The energy level H of stable equilibrium state ($K_0 = 25{\cdot}24$ or $22{\cdot}52$).

where the initial conditions are

$$T|_{\tau=0} = T_0, \qquad \dot{T}|_{\tau=0} = 0, \qquad \text{or} \qquad T_1|_{\tau=0} = T_0, \qquad T_2|_{\tau=0} = 0. \qquad (23, 24)$$

A variation of the Runge-Kutta method is used to solve the above Duffing equation. Taking $K = 5$ the forced vibration response is shown in Fig. 4, for superharmonic resonances, i.e., $\Omega = \omega_0/3$. In studying the forced responses, large values were also found when $\Omega/\omega_0 = 1, 1/2, 1/3, 1/4, 2, 3, 2/3, 3/2, \ldots$. In this paper, only those results for which $\Omega = \omega_0/3$ are presented.

(b) The excitations appear as coefficients in the governing differential equation. Mettlet's result is used to analyze the problem. When $\omega_0 = \bar{\omega}$ (main resonance), there is a kinetic snap-through. Otherwise, $\omega_0 = m\bar{\omega}$ (m is integer), are parametrically excited vibrations and there are some instability regions.

$$\ddot{T} + \omega_0^2\left\{\left(1 + \frac{A_4}{A_2}K^2 + \frac{A_3}{A_2}S^* \cos\bar{\omega}\tau\right)T + \frac{A_5}{A_2}KT^2 + \frac{A_6}{A_2}T^3\right\} = 0 \qquad (25)$$

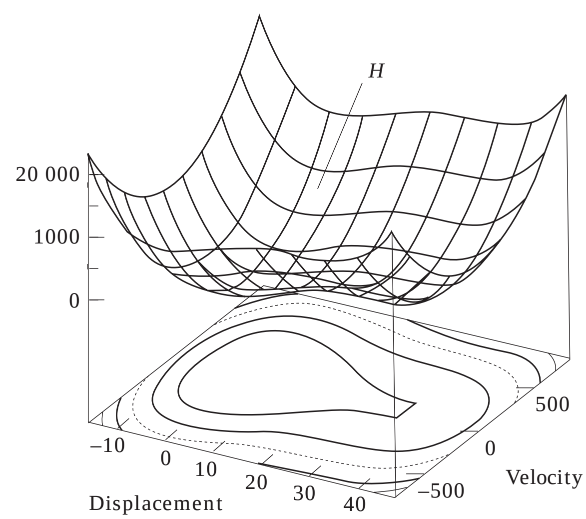

Figure 3. The energy level H of stable equilibrium state ($K_0 = 25{\cdot}24$ or $22{\cdot}52$).

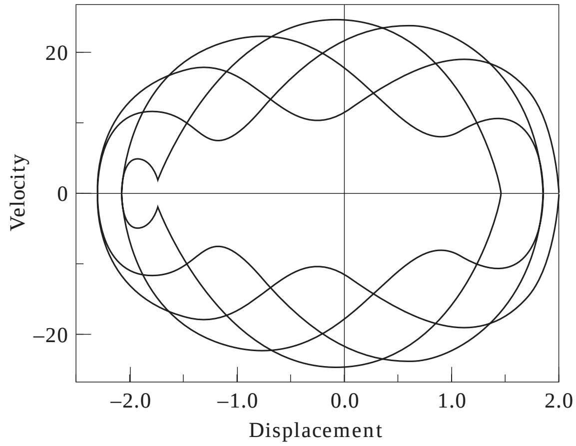

Figure 4. Forced vibration of shallow spherical shell ($K = 5$, $\Omega = \bar{\omega}/3$).

Equation (25) can be rewritten as

$$\ddot{T} + \omega_0^2\left[\left(1 + \frac{A_4}{A_2}K^2\right)T + \frac{A_5}{A_2}T^2\right] + \frac{A_6}{A_1}T^3 + \frac{A_3}{A_1}TS^* \cos\bar{\omega}\tau = 0. \tag{26}$$

Since the resonance coefficient A_3/A_1 is non-zero, the general theory predicts that a main resonance must occur if $\bar{\omega} = \omega_0$ and a subharmonic resonance of order 2 with $\bar{\omega} = 2\omega_0$, and these conditions yield the following relation

$$\bar{\omega}/\omega_0 = 1 + (A_4/A_2)K^2 + (3A_6/8A_1)(1/\omega_0^2)Q^2 \mp A_3S^*/2A_1(1/\omega_0^2) \tag{27}$$

Eq. (27) indicates that an increasing load S^* causes an increment of the amplitude Q, until a point with a horizontal tangent line is reached. Then with further increment of S^* the Q jumps to a higher value. This is an analogy to the static snap-through and therefore one calls it kinetic snap-through. By analogy with the theory of static stability, the two points where the solution $Q \neq 0$ branches off from the solution $Q = 0$ are called bifurcation points and kinetic bifurcations. The bifurcation points are listed in Tables 6a, b and 7a, b. But as has been shown, the forced vibrations and parametrically excited vibrations show no clear distinction if one excludes main resonances.

CONCLUSION

This paper presents a brief look at the non-linear vibration and dynamic stability problems of shallow spherical shells. The results are also given for the static snap-through buckling, free non-linear vibration and forced non-linear vibration which includes two types of excitations.

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APPENDIX: NOMENCLATURE

h	shell thickness
a	base circle radius
f	height of arch
N_r	radial membrance force
w	radial deflection of shell
z_0	initial deflected shape of spherical and conical shells
ρ	mass density
$q(t)$	transverse distributed loading
D	$= Eh^3/12(1-v^2)$, flexural rigidity of the shell
E	Young's modulus
v	Poisson's ratio
M_r, M_θ	radial and circumferential moments