Journal of Sound and Vibration (1997) **202**(3), 417–426

THE INFLUENCE OF ELASTIC BOTTOM PLATE MOTION ON THE RESONANT RESPONSE OF A LIQUID FREE SURFACE IN A CYLINDRICAL CONTAINER: A LINEAR ANALYSIS

M. Chiba

Department of Mechanical Engineering, Iwate University, Morioka 020, *Japan*

(*Received* 18 *September* 1995, *and in final form* 6 *November* 1996)

Linear response analyses were carried out to investigate the influence of an elastic bottom plate motion on a liquid free surface resonant response amplitude in a cylindrical container with a rigid wall. The bottom plate is harmonically excited with a constant amplitude in the frequency range of the lower sloshing mode, assuming that liquid responses are axisymmetric. Resonant response amplitudes of the free surface were found to vary with liquid heights; i.e., with the increase of the liquid height from zero, response amplitudes increase, taking a maximum at some height, and then decrease with the liquid height, which indicates that there is a most effective liquid height on the response amplitude, and the influence of the bottom motion decreases with the liquid height. The calculated results obtained here showed qualitative agreement with experimental results.

1. INTRODUCTION

A liquid storage tank, e.g., a large capacity oil tank, is one of the most popular fluid–structure coupled dynamical systems, with which engineering problems are often encountered that cannot be solved within a linear analysis. In such cases, a non-linear dynamic analysis for both the liquid and an elastic container has to be employed.

To investigate the non-linear fluid–structure coupled problem both theoretically and experimentally, the author has been conducting systematic studies on a cylindrical container with a rigid wall and an elastic bottom (base), which is one of the fluid–elastic container coupled systems (see Chiba [1, 2]). There have been many studies on vibration analysis within a linear range for such a coupled system; i.e., an elastic container with an elastic bottom. Studies on a rigid wall cylindrical tank with a flexible bottom have been conducted by Bhuta and Koval [3, 4], Tong [5], Siekmann and Chang [6], Nagaya and Takeuchi [7], Takada and Ono [8] and Chiba [2], and on a rectangular tank by Bauer [9]. For an elastic container with a flexible wall and a flexible bottom, studies have been carried out by Bauer *et al*. [10], Bauer and Siekmann [11] and Bauer *et al*. [12]. Coal and Nagano [13] have analyzed an elastic cylindrical tank with an elastic semi-spherical bottom. Fujita [14] has carried out an aseismatic response analysis for an elastic cylinder with a flexible bottom.

Interesting results have been obtained from the experimental study, carried out by Chiba [1], on the non-linear vibration of a liquid-containing cylindrical tank with an elastic bottom. They are presented in Figure 1, in which is shown the variation of the non-dimensional liquid free surface amplitude $(\delta'/h)_{rms}$ with liquid height parameter l_i, when the center of the elastic bottom plate is harmonically excited with constant amplitude

0022–460X/97/180417 + 10 $25.00/0/sv960855

$\Delta\varepsilon = 0{\cdot}3$ mm, $0{\cdot}6$ mm at the sloshing resonant frequency ranges of the first and the second axisymmetric modes. From the results, one can see the following: the response amplitude has a peak at some liquid height l_{ic}, and the amplitude decreases with the increase of l_i. Furthermore, the amplitude of the second mode is much less than that of the first mode, under the same excitation amplitude.

In this paper is presented a linear analysis of the influence of an elastic bottom motion on the response of liquid free surface, as the first step in the non-linear analysis, clarifying the above experimental results. The aim of this study is also to provide an answer to the following question: How much does an elastic bottom motion influence the liquid free surface response with different liquid height? In other words, when we excite the liquid in a container by tapping on an elastic bottom, how much does the free surface response amplitude depend on the liquid height?

In the present analysis, the effects of a static deflection of the bottom plate are considered. From the results, the influence of the motion of the elastic bottom plate was found to have a peak, and decreases with liquid height, and the present results showed substantial agreement with the previous experimental results [1], even if this is a linear analysis. It should be noted here that, concerning the study of the response of a liquid free surface, Bhuta and Yeh [15] have analyzed the axisymmetric motion of a liquid free surface due to the outlet velocity fluctuations at the bottom, which relates to the problem of a fuel tank connected to a propulsion system.

It should be emphasized that, in the present analysis, axisymmetric excitation was considered, because the center of the bottom plate was excited in the experiment. In actual engineering problems, i.e., a liquid storage tank under vertical earthquake excitation, a liquid fuel tank in a rocket under axial acceleration during launching, and so on, axisymmetric excitation is often encountered.

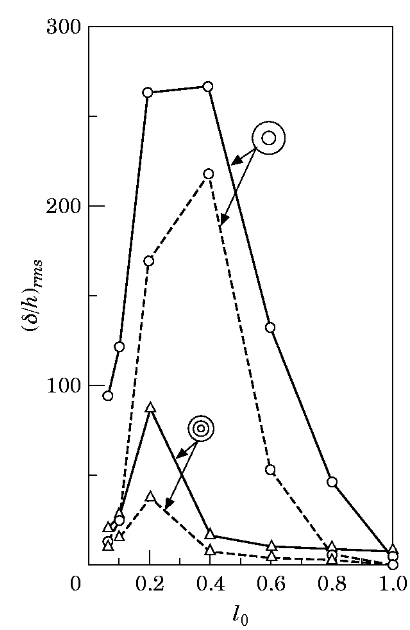

Figure 1. The amplitude of a liquid free surface for the first and second axisymmetric sloshing mode. ——, $\Delta\varepsilon = 0{\cdot}6$ mm; – – –, $\Delta\varepsilon = 0{\cdot}3$ mm, experiment.

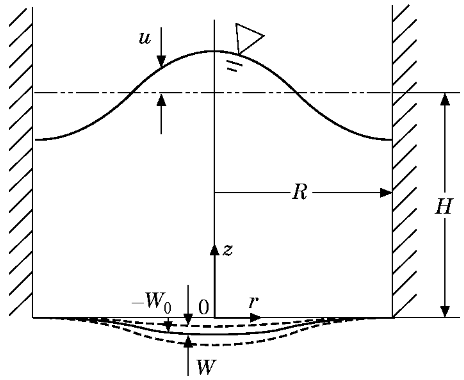

Figure 2. A cylindrical container with an elastic bottom.

2. BASIC EQUATION AND BOUNDARY CONDITION

Incompressible and inviscid liquid is contained, to a height H, in a circular cylindrical container of radius R with a rigid wall and an elastic bottom (base). We shall consider the effect of bottom plate motion on the liquid free surface response, when the bottom plate is harmonically excited at small amplitudes. The co-ordinate system is defined as shown in Figure 2. At the beginning, the bottom plate is deflected by $-W_0$ due to static liquid pressure, and then small amplitude forced motion with amplitude W is enforced around it.

The liquid is assumed to make an irrotational axisymmetric motion. Therefore the basic equation and the boundary conditions for the liquid velocity potential ϕ will be as follows (see reference [2]):

$$\nabla^2\phi = 0, \tag{1}$$

$$\phi_{,\tau\tau} + \bar{g}\phi_{,\xi} = 0 \qquad \text{at } \xi = l_i, \tag{2}$$

$$\phi_{,\rho} = 0 \qquad \text{at } \rho = 1, \tag{3}$$

$$w_{,\tau} - \phi_{,\xi} - w_{0,\rho}\phi_{,\rho} + w_0\phi_{,\xi\xi} = 0 \qquad \text{at } \xi = 0. \tag{4}$$

Equation (2) is a free surface condition, equation (3) is the rigid wall condition and equation (4) is a velocity matching condition at the bottom. In these equations, the following non-dimensional parameters are used:

$$\xi = z/R, \qquad \rho = r/R, \qquad l_i = H/R, \qquad \tau = \Omega_0 t, \qquad \bar{g} = g/(R\Omega_0^2), \qquad \omega = \Omega/\Omega_0,$$

$$\Omega_0^2 = \lim_{l_i\to\infty} (g\varepsilon_1/R)\tanh(\varepsilon_1 l_i), \qquad (w, w_0) = (W, W_0)/R, \qquad \phi = \Phi/(\Omega_0 R^2). \tag{5}$$

In these equations, t is time, l_i is the non-dimensional parameter concerned with liquid height H, and ε_1 is a parameter that satisfies equation (9). A list of notation is given in Appendix II.

3. METHOD OF SOLUTION

The static deflection $w_0(\rho)$ and forced vibration $w(\rho, \tau)$ of the bottom plate, and the velocity potential $\phi(\rho, \xi, \tau)$ are assumed in the following form:

$$w_0(\rho) = aw(\rho) = a(1-\rho^2)^2, \qquad w(\rho, \tau) = bw(\rho)\cos\omega\tau, \tag{6, 7}$$

$$\phi(\rho, \xi, \tau) = -\omega\left[A_0(\tau)\xi + B_0(\tau) + \sum_k J_0(\varepsilon_k\rho)\left(\frac{\sinh(\varepsilon_k\xi)}{\cosh(\varepsilon_k l_i)}A_k(\tau) + \frac{\cosh(\varepsilon_k\xi)}{\sinh(\varepsilon_k l_i)}B_k(\tau)\right)\right], \tag{8}$$

where a and b are amplitude parameters, and $A_0(\tau)$, $B_0(\tau)$, $A_k(\tau)$ and $B_k(\tau)$ are unknown time functions. Equations (6) and (7) satisfy the clamped boundary condition along the edge (see reference [2]). The ε_k are the values that satisfy

$$J_1(\varepsilon_k) = 0. \tag{9}$$

First, substituting equations (6), (7) and (8) into the velocity matching condition (4), we obtain $A_0(\tau)$ and $A_k(\tau)$ in the following forms:

$$A_0(\tau) = \left(\alpha_1 b + a\sum_k \beta_k B_k^s(\tau)\right)\sin\omega\tau, \tag{10}$$

$$A_k(\tau) = \left(\alpha_2 b + a\sum_p \gamma_{kp} B_p^s(\tau)\right)\sin\omega\tau, \tag{11}$$

where α_1, α_2, β_k and γ_{kp} are coefficients.

Then, putting the damping term, $\delta\phi_{,\tau}$, into the free surface condition equation (2) for convenience, and assuming that

$$B_0(\tau) = B_0^c\cos\omega\tau + B_0^s\sin\omega\tau, \qquad B_k(\tau) = B_k^c\cos\omega\tau + B_k^s\sin\omega\tau, \tag{12, 13}$$

yields the free surface condition in the form

$$\begin{bmatrix} \omega^2 & -\omega\delta & 2aX_k(\omega^2 l_i - \bar{g}) & -2aX_k\omega\delta l_i \cdots \\ \omega\delta & \omega^2 & 2aX_k\omega\delta l_i & 2aX_k(\omega^2 l_i - \bar{g}) \cdots \\ 0 & 0 & \begin{gathered}(\omega^2 CTH_p - \bar{g}\varepsilon_p)\delta_{pq} \\ + 2aY_pZ_{kp}(\omega^2 - \bar{g}\varepsilon_p CTH_p)\end{gathered} & -\omega\delta(CTH_p\,\delta_p + 2aY_pZ_{kp}) \ldots \\ 0 & 0 & \omega\delta(CTH_p\,\delta_{pk} + 2aY_pZ_{kp}) & \begin{gathered}(\omega^2 CTH_p - \bar{g}\varepsilon_p)\,\delta_{pk} \ldots \\ + 2aY_pZ_{kp}(\omega^2 - \bar{g}\varepsilon_p CTH_p)\end{gathered} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\times \begin{Bmatrix} B_0^c \\ B_0^s \\ B_k^c \\ B_k^s \\ \vdots \end{Bmatrix} = 2b \begin{Bmatrix} \omega\delta l_i Q \\ (\bar{g} - \omega^2 l_i)Q \\ Y_pP_p\omega\delta \\ Y_pP_p(\bar{g}\varepsilon_p CTH_p - \omega^2) \\ \vdots \end{Bmatrix} \tag{14}$$

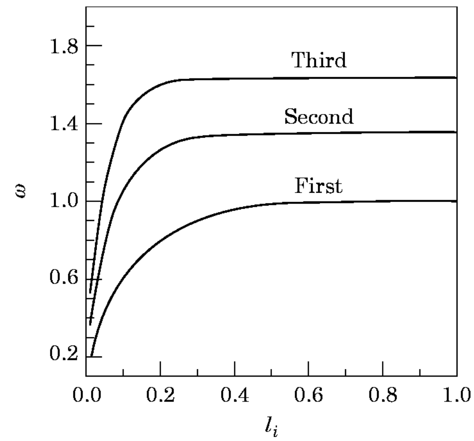

Figure 3. The natural frequency of an axisymmetric sloshing mode.

where δ_{pk} is the Kronecker delta, and CTH_p, Y_p, P_p, etc. are presented in Appendix I.

This is a $(2k + 2)$ coupled linear equation with respect to B_0^c, B_0^s, B_k^c and B_k^s, and this equation can be solved when ω, l_i, a and b are given. From the above solutions, the non-dimensional free surface response, $d = u/R$, can be obtained using the following equation:

$$d_{,\tau}(\rho, \tau) = \phi_{,\xi}(\rho, \tau) \qquad \text{at } \xi = l_i. \tag{15}$$

4. NUMERICAL RESULTS

First, the variations of the sloshing natural frequencies of the axisymmetric mode with the liquid height, l_i, are shown in Figure 3, for the lower three modes. As shown in the figure, the natural frequency increases from zero and tends to a saturated value, with an increase in the liquid height l_i.

Then, we shall examine the response amplitude variation of each mode with the liquid height l_i, when the bottom plate is harmonically excited with a constant amplitude. To this end, the frequency response curve of the liquid free surface at the middle of the tank was obtained in the proximity of the natural frequency. As an example, the result when $l_i = 0{\cdot}2$

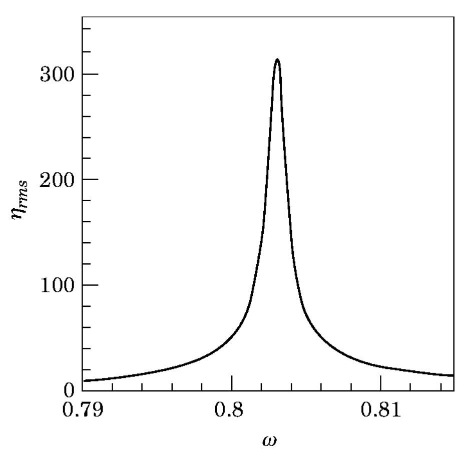

Figure 4. The frequency response curve at the center of the free surface: $l_i = 0{\cdot}2$, $\delta = 0{\cdot}001$.

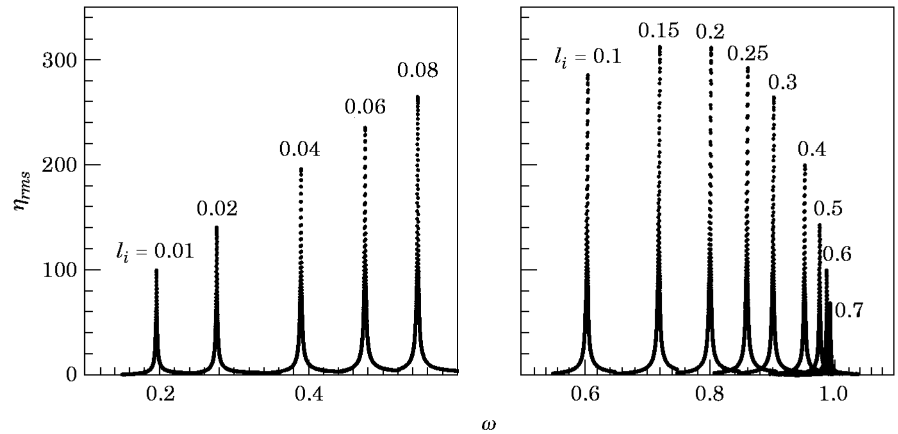

Figure 5. Frequency response curves of the first mode with l_i: $\delta = 0{\cdot}001$.

and $\delta = 0{\cdot}001$ is shown in Figure 4, for the first mode. In the calculations, the unknown parameter k in equation (14) was taken as $k = 1$, 2, 3. In the ordinate, the r.m.s. value of the non-dimensional free surface amplitude normalized by the excitation amplitude b, $\eta = u/b = Rd/b$, is taken. Since the analysis is linear, i.e., the response is linearly proportional to the excitation amplitude and is independent of b (see equation (14)), it is convenient to use this non-dimensional amplitude, η. At $\omega = 0{\cdot}803$ one can see a sharp resonance peak. Similar response curves can be calculated by changing the l_i value. These are shown in Figure 5 for the first mode, when $\delta = 0{\cdot}001$. With the increase in l_i, the resonance frequency increases, as is shown in Figure 3. From these results, taking the maximum points for each l_i result for the modes from the first to the third, we obtain Figure 6. From the figure one can see the following.

(1) The response amplitude increases with l_i and has its maximum at $l_{ic} = 0{\cdot}17$ for the first mode, and at $l_{ic} = 0{\cdot}09$ for the second mode; it then decreases. This means that there is a most effective liquid height that is influenced by the elastic bottom plate motion, and that the influence of the bottom plate motion becomes insignificant for higher liquid heights.

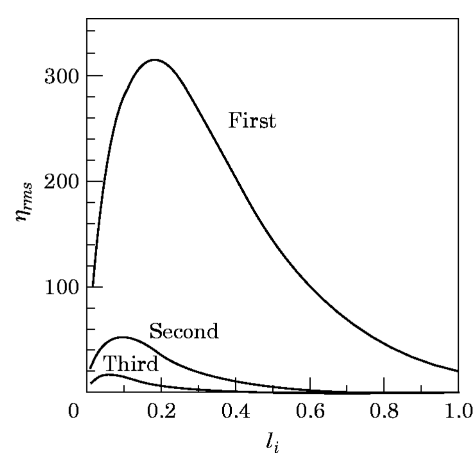

Figure 6. The variation of the response amplitude with l_i:$\delta = 0{\cdot}001$.

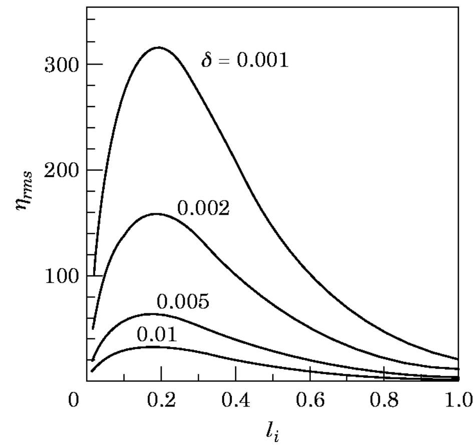

Figure 7. The influence of damping δ on the response amplitude.

(2) We can conclude that the higher the vibration mode, the smaller is the maximum amplitude. It can also be said that the l_{ic} that corresponds to the maximum amplitude decreases slightly for higher modes.

Next, in order to see the effect of the damping parameter δ, response curves were obtained for varying values of $\delta = 0{\cdot}001$, $0{\cdot}002$, $0{\cdot}005$ and $0{\cdot}01$. The results are shown in Figure 7 for the first mode. Then, the effect of the initial static deflection of the bottom plate, a, is shown in Figure 8 for the first mode, from which one can see that there is little influence of the static deflection on the amplitude of the response.

Comparison with experimental results in reference [1] are presented in Figure 9, re-normalizing the experimental data by the excitation amplitude. In the figure, the solid lines are the present results when $\delta = 0{\cdot}0016$ for the first mode and the second mode, while ○ and ⊙ represent the experimental data when $\Delta\varepsilon = 0{\cdot}3$ mm, and □ and △ are the data when $\Delta\varepsilon = 0{\cdot}6$ mm, respectively. The value $\delta = 0{\cdot}0016$ was chosen so as to match the response amplitude in the experiments with that of the numerical results.

The variations of the response amplitude with l_i are in good agreement, except that the l_{ic} of the experimental results are a little higher than the calculated ones. This may be due

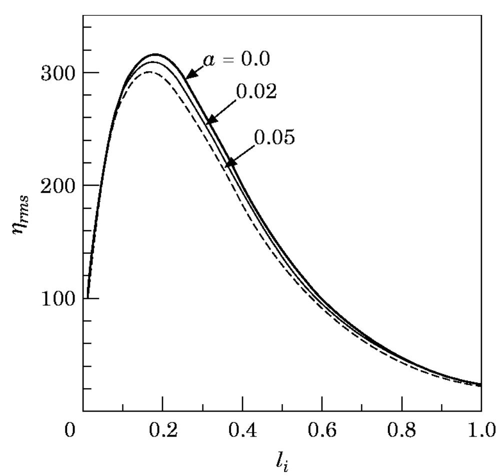

Figure 8. The effect of the initial deflection of the bottom plate on the response amplitude: first mode.

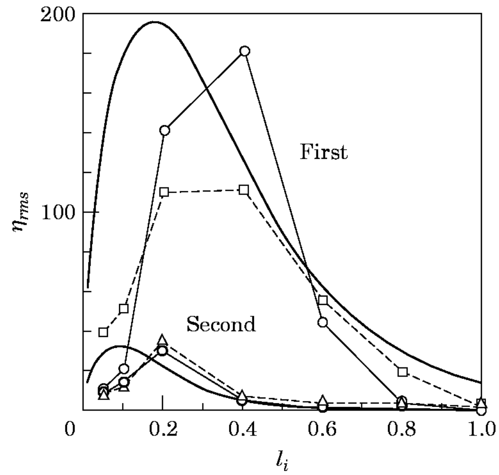

Figure 9. A comparison with experimental results: $\delta = 0{\cdot}0016$.

to two factors: the first is the non-linear effect in the experimental data, and the second is the difference of the bottom plate deflection; i.e., in the experiment the bottom plate is excited at its center, and the spatial distribution of the deflection along the radial direction is a little different from that used in the analysis.

Finally, in Figures 10(a) and 10(b), free surface vibration modes with l_i are presented for the first and second modes, respectively.

5. CONCLUSIONS

A linear response analysis was carried out to study the influence of the elastic bottom plate motion on the free liquid surface axisymmetric resonant response amplitude in a cylindrical container with a rigid wall, when the bottom plate is harmonically excited with a constant amplitude. The results obtained in the present analysis are as follows.

(1) The response amplitude of the free surface depends on the liquid height; it reaches its maximum at the approximate non-dimensional liquid height of $l_{ic} = 0{\cdot}17$ and then

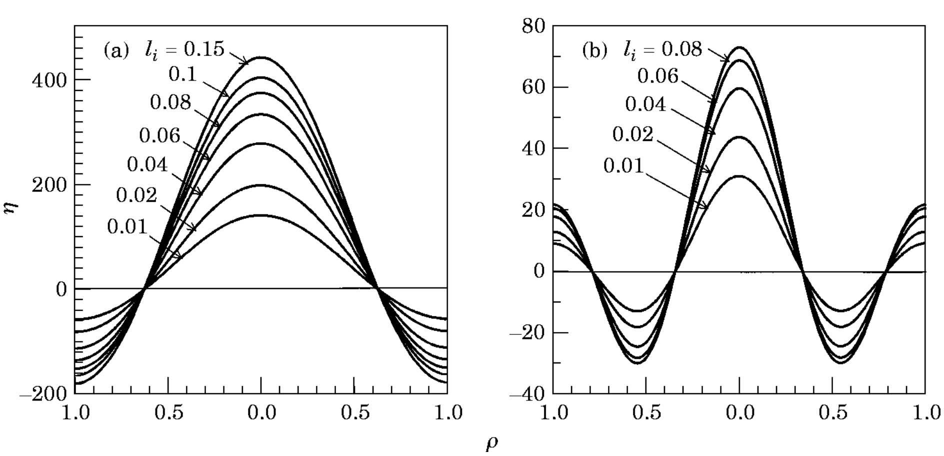

Figure 10. Mode variation with l_i: (a) first mode; (b) second mode.

decreases with liquid height, in the first axisymmetric mode. This means that there is a most effective liquid height that is influenced by the elastic bottom plate motion, and that the influence of the bottom plate motion becomes insignificant for higher liquid heights.

(2) In the second and third modes, the maximum amplitude decreases with an increase in the order of vibration mode, shifting the corresponding liquid height l_{ic} to a slightly lower one.

(3) The influence of the static deflection of the bottom plate on the response amplitude is small.

(4) Results (1) and (2) agree well with previous experimental results.

The present linear analysis will provide fundamental information for non-linear analysis in the future.

REFERENCES

1. M. Chiba 1992 *Journal of Fluids and Structures* **6**(2), 181–206. Non-linear hydroelastic vibration of a cylindrical tank with an elastic bottom, containing a liquid, part I: experiment.
2. M. Chiba 1993 *Journal of Fluids and Structure* **7**(1), 57–73. Non-linear hydroelastic vibration of a cylindrical tank with an elastic bottom, containing a liquid, part II: linear axisymmetric analysis.
3. P. G. Bhuta and L. R. Koval 1964 *Zeitschrift für Angewante Mathematik und Physik* **15**, 466–480. Coupled oscillations of a liquid with a free surface in a tank having a flexible bottom.
4. P. G. Bhuta and L. R. Koval 1964 *Journal of the Acoustical Society of America* **36**, 2071–2079. Hydroelastic solution of the sloshing of a liquid in a cylindrical tank.
5. P. Tong 1967 *American Institute of Aeronautics and Astronautics* **5**(10), 1842–1848. Liquid motion in a circular cylindrical container with a flexible bottom.
6. J. Siekmann and S.-C. Chang 1968 *Ingenieur-Archiv* **37**, 99–109. On the dynamics of liquids in a cylindrical tank with a flexible bottom.
7. K. Nagaya and J. Takeuchi 1984 *Journal of the Acoustical Society of America* **74**(5), 1511–1518. Vibration of a plate with arbitrary shape in contact with a fluid.
8. H. Takada and K. Ono 1985 *Transactions of the Japan Society of Mechanical Engineers* **51**, No. 469, 2393–2396. Natural frequency of an edge-fixed disc in contact with a liquid (in Japanese).
9. H. F. Bauer 1981 *International Journal of Solids and Structures* **17**, 639–652. Hydroelastic vibrations in a rectangular container.
10. H. F. Bauer, T.-H. Hsu and J. T.-S. Wang 1968 *Journal of Basic Engineering*, *Transactions of the American Society of Mechanical Engineers*, Paper No. 68, FE-20, 1–5. Interaction of a sloshing liquid with elastic containers.
11. H. F. Bauer and J. Siekmann 1969 *Zeitschrift für Angemante Mathematik und Mechanik* **49**, 577–589. Note on linear hydroelastic sloshing.
12. H. F. Bauer, J. T.-S. Wang and P. Y. Chen 1972 *Aeronautical Journal* 704–712. Axisymmetric hydroelastic sloshing in a circular cylindrical container.
13. C. W. Coal and M. Nagano 1965 *AIAA Symposium on Structural Dynamics and Aeroelasticity*, *Boston*, *MA*, *U.S.A.*, 169–176. Axisymmetric modes of an elastic cylindrical–hemispherical tank partially filled with a liquid.
14. K. Fujita 1982 *Transactions of the Japan Society of Mechanical Engineers* **48**, No. 428, 516–525. A seismic response analysis of a cylindrical liquid storage tank on an elastic foundation (in Japanese).
15. P. G. Bhuta and G. C. K. Yeh 1965 *International Journal of Mechanical Science* **7**, 475–488. Liquid sloshing due to a time-dependent discontinuous boundary.

APPENDIX I: PARAMETERS IN EQUATION (14)

$$CTH_p = \coth(\varepsilon_p l_i), \qquad X_k = \varepsilon_k (R_k + \varepsilon_k P_k)/\sinh(\varepsilon_k l_i), \qquad Y_p = \frac{2\sinh(\varepsilon_p l_i)}{\varepsilon_p J_0^2(\varepsilon_p)},$$

$$Z_{kp} = \frac{\varepsilon_p}{\sinh(\varepsilon_p l_i)} \left(\int_0^1 w(\rho)_{,\rho} J_1(\varepsilon_k \rho) J_0(\varepsilon_p \rho) \rho \, \mathrm{d}\rho + \varepsilon_k \int_0^1 w(\rho) J_0(\varepsilon_p \rho) J_0(\varepsilon_k \rho) \rho \, \mathrm{d}\rho \right),$$

$$Q=\int_0^1 w(\rho)\rho\ \mathrm{d}\rho=\frac{1}{6},\qquad P_p=\int_0^1 w(\rho)J_0(\varepsilon_p\rho)\rho\ \mathrm{d}\rho,\qquad R_k=\int_0^1 w(\rho)_{,\rho}J_1(\varepsilon_k\rho)\rho\ \mathrm{d}\rho.$$

Z_{kp}, P_p and R_k are numerically integrated.

APPENDIX II: NOTATION

A_0, A_k, B_0, B_k	unknown time functions in equation (8)
a	initial deflection amplitude parameter
b	excitation amplitude parameter
H (l_i)	liquid height (non-dimensional form)
h	thickness of bottom plate (experiment)
g ($\bar{g}$)	gravitational acceleration
R	radius of container
t (τ)	time
$u(d, \eta)$	free surface amplitude
W (w)	excitation amplitude
W_0 (w_0)	static deflection of bottom plate
δ	damping parameter
δ'	free surface amplitude (experiment)
δ_{pk}	Kronecker delta
$\Delta\varepsilon$	excitation amplitude (experiment)
ϕ	velocity potential
Ω (ω)	frequency