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EXPERIMENTAL INVESTIGATIONS AND SHAPE FUNCTIONS FOR LATERAL VIBRATION OF AXIALLY CONSTRAINED BEAMS WITH A CONCENTRATED MASS AT THE CENTRE

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1. INTRODUCTION

The classical theory of vibrations of beams is based on a number of assumptions, such as small deflections, supports free to move in the direction of the longitudinal axis, inextensional deflections, etc. If the ends of the beam are prevented from moving axially, lateral deflections of the beam are accompanied by length changes in the beam fibers that are in addition to the changes caused by bending. Due to the lateral deflection, the length dx of the beam is increased by an amount $[\sqrt{1+(\mathrm{d}y/\mathrm{d}x)^2}-1]\,\mathrm{d}x \cong \frac{1}{2}(\mathrm{d}y/\mathrm{d}x)^2\,\mathrm{d}x$ [1–3]. The tensile strains resulting from these axial extensions add to the strains caused by bending. This implies that more strain energy is stored in an axially constrained beam than in a beam with ends free to move axially.

Large amplitude free flexural vibrations of slender beams have been investigated by several researchers using continuum [4, 5] and finite element [6] methods. The effect of axial displacement has been considered by Raju *et al*. [4], in which the strain energy U and the kinetic energy T of the unloaded beam are given by

$$U=\int_0^l\left[EA\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left\{\frac{\partial w}{\partial x}\right\}^2\right)^2+EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right]\mathrm{d}x,\qquad T=\tfrac{1}{2}m\omega^2\int_0^l(u^2+w^2)\,\mathrm{d}x,$$

where u is the axial displacement due to a tensile force. Raju *et al*. concluded that the effect of the longitudinal deformation and inertia is to reduce the non-linearity, and the longitudinal inertia is negligible for slender beams [4].

Singh *et al*. [5] studied and compared various analytical formulations for the problem of large amplitude free vibrations of simply supported beams with immovable ends based on the Rayleigh–Ritz method with one-term approximations for axial and transverse displacements. It was found that the formulation wherein the quadratic term in the strain displacement relation is linearized leads to an equation of motion, rather than energy balance equation [5].

In a recent work [7], both experimental and theoretical results were presented for beams carrying a concentrated mass at mid-span. It was found that experimental data for the thicker beams correlated well with theory of linear deformation, but no so well for the thinner beams. Chai *et al*. [8] later provided an improved model by adding a tensile force.

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By virtue of the Rayleigh–Ritz procedure, a multi-term series with sine function was used for deflection curves for modes higher than the first [8].

In what follows here, different assumed shape functions are used to obtain the natural frequency of a centrally loaded, fixed–fixed beam. By comparing each corresponding result with that obtained experimentally, the effect of shape functions on the frequency of vibration of axially constrained fixed–fixed beams under transverse load can then be examined.

2. ASSUMED SHAPE FUNCTIONS

Rayleigh's method states that a reasonable mode shape satisfying at least the geometric support conditions leads to a good approximation for the natural frequency [1–3]. By equating the maximum kinetic and potential energies of a loaded beam, we obtain Rayleigh's quotient for the natural circular frequency of the beams carrying a concentrated mass at $x = a$:

$$\omega^2 = \frac{\int EI(\mathrm{d}^2y/\mathrm{d}x^2)^2 \ \mathrm{d}x + \int (EA/4)(\mathrm{d}y/\mathrm{d}x)^4 \, \mathrm{d}x}{\int \rho A y^2 \, \mathrm{d}x + My^2|_{x=a}} = \frac{K_{ei} + K_{ea}}{M^*}, \tag{1}$$

where EI is the flexural rigidity of the beam, A is the cross-sectional area, ρ is the mass density of the beam, and M is the mass of the load alone. It is shown in equation (1) that axial constraints, as a function of bending slope ($\mathrm{d}y/\mathrm{d}x$), act to increase the natural frequency of beams, whereas a concentrated mass on the beam decreases the system frequency.

In this work, each of the following commonly used shape functions [3] is incorporated into equation (1) to obtain the fundamental frequency:

$$y_W = \frac{Wx^2}{48EI}(3l - 4x), \qquad \text{for } 0 \leqslant x \leqslant l/2; \tag{2}$$

$$y_m = \frac{wx^2}{24EI}(x^2 - 2lx + l^2), \qquad \text{for } 0 \leqslant x \leqslant l; \tag{3}$$

$$y_c = y_m + y_W, \qquad \text{for } 0 \leqslant x \leqslant l; \tag{4}$$

$$y_t = C[1 - \cos(2\pi x/l)], \qquad \text{for } 0 \leqslant x \leqslant l. \tag{5}$$

Note that y_W is a static-deflection curve by considering only the load, W, while the deflection curve y_m is defined in terms of the distributed beam mass, m, only. The function y_c involves the combined contribution from both the distributed mass and the load. The fourth function, y_t, composed of one-minus-cosine, is the most commonly used and most useful for such fixed-end cases [1, 3]. The parameter C in equation (5) refers to both the displacement of the beam at $x = l/2$ and the displacement owing to the load. The effect of this parameter on the frequency will be discussed in section 3.

3. ANALYTICAL AND EXPERIMENTAL FREQUENCIES

A fixed–fixed loaded beam was tested by using a shaker system with an accelerator, as shown in Figure 1. A sharp pivot hanger was used to carry the concentrated mass. The geometries and properties of the four tested beams are listed in Table 1, in which is also

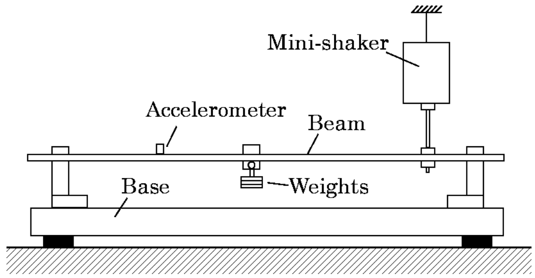

Figure 1. The loaded beam system considered.

indicated the corresponding slenderness ratio μ. Note that $\mu = l/k$, where k is the radius of gyration of the beam's cross-section. All the beams are of the same length and width, but with different thicknesses, and are made of different materials (aluminum or steel).

Let us now define ω_W, ω_m, ω_c, and ω_t as the associated frequencies obtained from equation (1) by using y_W, y_m, y_c, and y_t, respectively. By representing as ω_0 the fundamental frequency for an unloaded beam and as ω the frequency of the beam carrying a central mass, the relationship between the frequency ratio β $(=\omega/\omega_0)$ and the mass ratio α $(=M/m)$ is studied by both analytical and experimental methods. Comparative results with those obtained experimentally for beams 1 and 3 are shown in Figures 2 and 3, respectively. The results for beams 2 and 4 are omitted here, as their comparison trends are similar to those between beams 1 and 3. The ratio of K_{ea}/K_{ei} for beams 1 and 3 is depicted in Figure 4 to illustrate the significance of axial stiffness as functions of the mass ratio. Results obtained by using equation (1) without the term in EA are compared with the same experimental results, as shown in Figure 5.

Several points are worth noting from Figures 2 to 5.

1. For thicker beams with a lower slenderness ratio (such as beam 3 in Figure 3) the analytical frequencies obtained by using all the proposed assumed shape functions are almost identical, and can predict the experimental results well. However, this is not the case for beam 1, with a higher slenderness ratio. As shown in Figure 2, the models obtained by using y_c, y_W, and y_t, predict the experimental frequency well. Nevertheless, the analytical frequency obtained from y_t in equation (5) matches the experimental result very well.

2. Let us now investigate the effect of the parameter C in equation (5) on the frequency result. It is defined as the amplitude of vibration of the beam at $x = l/2$ [3]. The result in Figure 2 for β_t was generated by assuming the parameter C as $y_m|_{x=l/2} + (13/35)\, y_W|_{x=l/2}$, by virtue of the concept of equivalent mass, m_{eq} [9]. Note that the factor of 13/35 is associated with the shape function, y_W [10]. It is interesting to see how other assumptions would affect the results. It was found that the frequency ratio curve with y_t would be much

Table 1

Geometric properties and dimensions of the tested beams

	Beam 1	Beam 2	Beam 3	Beam 4
Length (m)	1·0	1·0	1·0	1·0
Width (mm)	12·7	12·7	12·7	12·7
Thickness (mm)	3·175	3·175	4·7625	4·7625
E (GPa)	71	207	71	207
ρ (Mg/m^3)	2·71	7·81	2·71	7·81
μ $(=\sqrt{Al^2/I})$	1091	1091	727	727

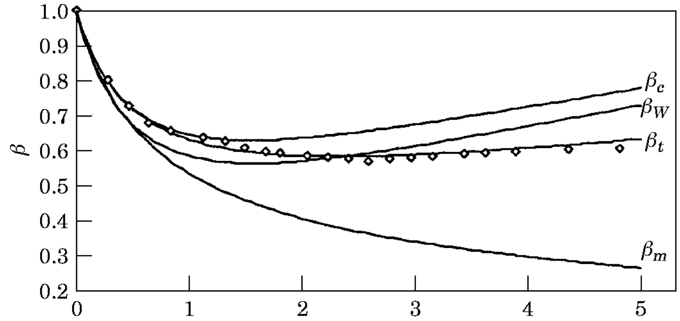

Figure 2. The frequency ratio versus the mass ratio for beam 1. —, Analytical; ◇, experimental.

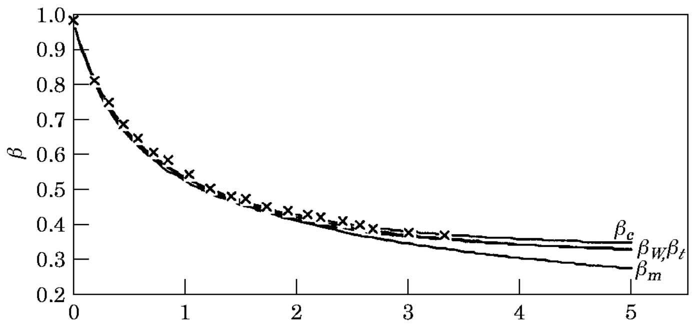

Figure 3. The frequency ratio versus the mass ratio for beam 3. —, Analytical; ×, experimental.

higher than y_c if $C = y_W|_{x=l/2}$, while the curve would be similar to that obtained with y_m if $C = y_m|_{x=l/2}$. In commenting on the work by Bert [11], Maurizi *et al.* [12] pointed out that the trigonometric function (with $C = y_m$) is a good enough approximation for beams with a uniformly distributed load, but not for those beams carrying one or two concentrated loads. This is especially true for cases of thinner beams, where a very poor prediction is given by y_m as shown in Figure 2.

3. As shown in Figure 4, the ratio of K_{ea}/K_{ei} for beam 3 is less than one, whereas the ratio for beam 1 becomes very large in comparison with unity as the concentrated mass

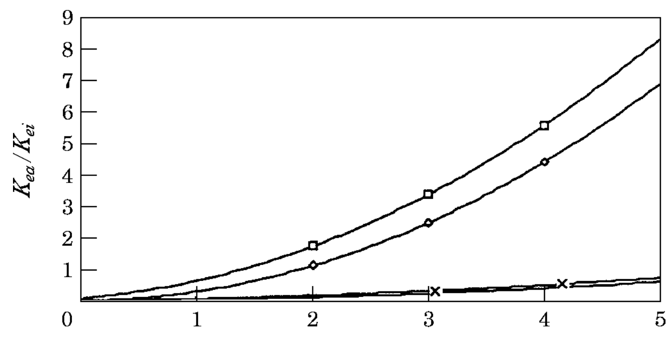

Figure 4. The stiffness ratio (K_{ea}/K_{ei}) versus the mass ratio. ⊟, y_c (beam 1); ◇, y_W (beam 1); ×, y_c (beam 3); —, y_W (beam 3).

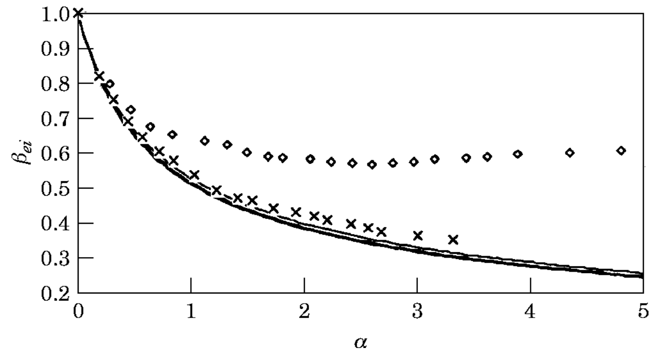

Figure 5. The frequency ratio versus the mass ratio for beams 1 and 3 (without the *EA* term). —, Analytical; ◇, experimental (beam 1); ×, experimental (beam 3).

increases. This explains why the contribution of the *EA* term should be considered in predicting the experimental frequency for beam 1, as shown in Figure 2.

4. The frequencies without considering the effect of axial force (only the term in *EI*) are also compared with the experimental results (see Figure 5). If the mass ratio is less than one, all of the analytical models give the same result. It was also found that the analytical curves of β_{ei} for beams 1 and 3 are the same, owing to their non-dimensional characteristics and the absence of K_{ea}.

5. It is shown in Figure 5 that the discrepancy between the analytical and experimental frequency ratios (β_{ei}) is wider for thinner beams such as beam 1. As can be seen in equation (1), these frequencies are smaller than those obtained before, when the axial effect was included. The observation made in reference [13] for beam models without a tension effect is no longer valid for thinner beams.

6. In general, the frequency obtained by y_c is higher than that of y_W, while the result given by y_m is always the lowest.

4. CONCLUDING REMARKS

Four assumed shape functions, given in equations (2)–(5), have been used to predict the frequency of four fixed–fixed beams with different slenderness ratios. The results with and without an axial effect are both compared with experimental results. It is found that the selection of the four considered functions has very little effect on the fundamental frequencies of thicker beams (for example, for a slenderness ratio of 727), whereas the effect is quite pronounced for thinner beams with a high slenderness ratio. The frequencies obtained by using y_t, y_W and y_c are closer to the experimental results. The choice of the one-minus-cosine trigonometric function (y_t) appears to yield values that are in remarkable agreement with those obtained experimentally. It is recommended to apply y_t with an appropriate amplitude parameter, in association with the concept of equivalent mass, to such non-linear beams with large deflection. In general, the model when using y_m gives poor estimates, except for beams with a low slenderness ratio.

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