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ON THE FREE VIBRATION ANALYSIS OF CIRCULAR PLATES WITH STEPPED THICKNESS OVER A CONCENTRIC REGION BY THE DIFFERENTIAL QUADRATURE ELEMENT METHOD

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1. INTRODUCTION

Recently, the differential quadrature (DQ) has been used to obtain the fundamental frequency of annular plates with uniform and non-uniform thickness and shown to be computationally efficient [1, 2]. However, there is a certain lack of flexibility when applying this method to real life structural analysis. Additionally, difficulties arise when the method is used to analyze structures or structural elements with discontinuous geometries or discontinuous distributed loads. To overcome the difficulties and to extend the applicability range of the DQ method to these problems, Striz *et al*. proposed a method called the quadrature element method (QEM) [3]. The QEM was successfully used to perform static analyses of truss and beam structures under various loads. However, the QEM is not convenient and accurate enough in the analysis due to the introduction of two points representing one end point.

The present authors recently proposed a new method called the differential quadrature element method (DQEM) and performed static, buckling and free vibrational analyses of frame structures by using DQEM [4]. Numerical examples indicate that the DQEM is more convenient and accurate than the QEM. The essential difference between the DQEM and the QEM is that two degrees of freedom are used at each end point in the DQEM instead of two points used in the QEM for a fourth order differential equation. In view of the fact that the previous papers [1, 2] have not analyzed the fundamental frequency of circular plates with stepped thickness, the writers have computed frequencies of such problems using DQEM. The problem considered in this paper involves plates of isotropic material only. For completeness, the DQEM is described and the weighting coefficient matrix of the circular plate and annular plate elements are derived. Then the title problem is analyzed and the DQEM results are compared with the results in the open literatures [5, 6] to show the efficiency of the DQEM.

2. ANALYSIS OF DIFFERENTIAL QUADRATURE ELEMENT METHOD

In the case of free vibration analysis of circular plates, one takes

$$w(r, \theta, t) = w(r, \theta)\, \mathrm{e}^{\mathrm{i}\omega t} \tag{1}$$

and then introduces the following convenient approximation, namely,

$$w(r, \theta) = \cos(k\theta) W(r), \tag{2}$$

where k is the number of nodal diameters.

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The governing differential equation for free vibration analysis of a circular plate with stepped thickness over a concentric circular region shown in Figure 1 becomes

$$\frac{d^4W}{dr^4} + \frac{2}{r}\frac{d^3W}{dr^3} - \frac{1+2k^2}{r^2}\frac{d^2W}{dr^2} + \frac{1+2k^2}{r^3}\frac{dW}{dr} + \frac{(k^4-4k^2)W}{r^4} = \frac{\rho_i h_i \omega^2 W}{D_i}, \tag{3}$$

where $i=2$ when $0 \leqslant r \leqslant r_0$ and $i=1$ when $r_0 \leqslant r \leqslant a$. At $r=a$, the boundary conditions are:

$$D_1\left(\frac{d^2W}{dr^2} + \frac{v}{a}\frac{dW}{dr} - \frac{vk^2W}{a^2}\right) + \Phi\frac{dW}{dr} = 0,$$

$$D_1\left(\frac{d^3W}{dr^3} + \frac{1}{a}\frac{d^2W}{dr^2} + \frac{vk^2-1-2k^2}{a^2}\frac{dW}{dr} + \frac{3k^2-vk^2W}{a^3}\right) + KW = 0, \tag{4}$$

where Φ and K are rotational and translational spring constants, respectively, and $D_1 = E_1 h_1^3/12(1-v_1^2)$ is the flexural rigidity. At $r=0$, the regularity condition can be written as

$$dW/dr + K_0 W = 0, \tag{5}$$

where K_0 is set to zero when $k=0$ and ∞ when $k>0$.

The problem can be analyzed by the DQEM. The DQE method described in detail in [4] is a modification of the differential quadrature method and is similar to the QEM [3]. As was pointed out previously, the essential difference between DQEM and QEM is that

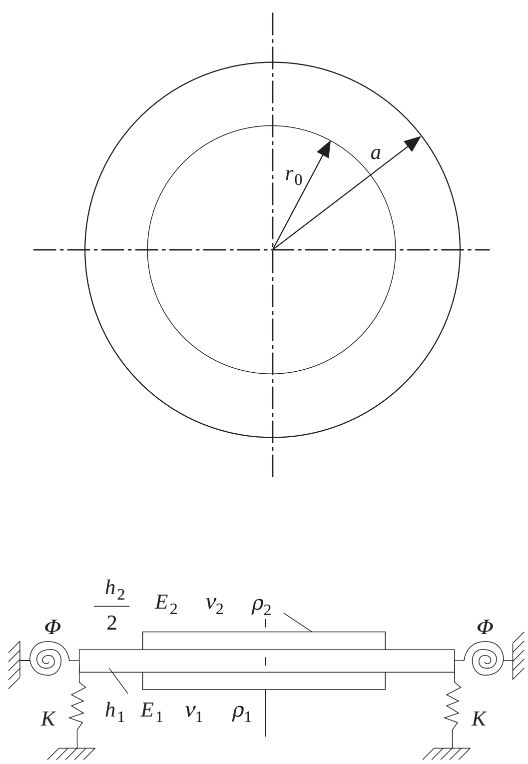

Figure 1. Vibrating circular plate studied.

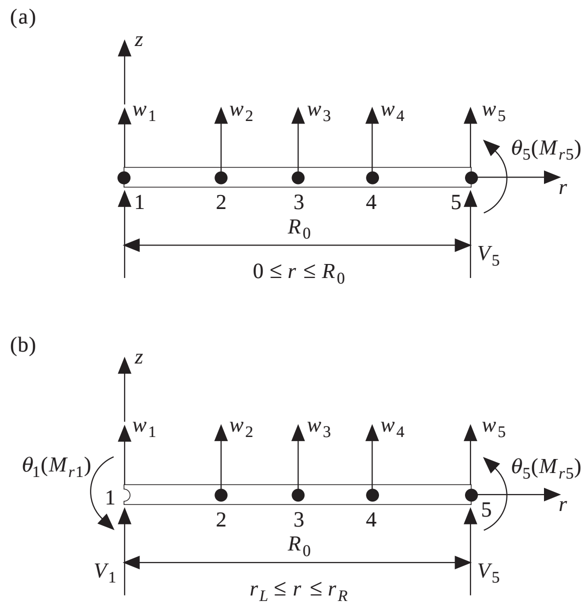

Figure 2. (a) Differential quadrature circular plate element; (b) differential quadrature annular plate element.

two degrees of freedom are introduced at each end grid for a fourth order differential equation. In this paper, the DQ circular element and the DQ annular element, shown in Figure (2a) and Figure (2b), are established for the first time.

Assume that

$$W(r) = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \cdots + \alpha_M r^M = [1, r, r^2, \ldots, r^M]\{\alpha\} \tag{6}$$

where M is equal to N for the DQ circular element and to $N+1$ for the DQM annular element, and N is the number of the grid point for the element. Let

$$\{\boldsymbol{\delta}_c\} = \{W_1, W_2, W_3, \ldots, W_N, \theta_N\}^{\mathrm{T}}, \qquad \{\boldsymbol{\delta}_a\} = \{W_1, \theta_1, W_2, W_3, \ldots, W_N, \theta_N\}^{\mathrm{T}}, \tag{7}$$

where subscripts c and a represent the circular plate with its center at $r = 0$ and annular plate, and W_i and θ_i are the deflection and rotation at grid point **i** shown in Figure 2.

After some manipulations, equation (6) can then be rewritten as

$$W(r) = [f_1(r), f_2(r), \ldots, f_N(r), f_{\theta N}(r)]\{\boldsymbol{\delta}_c\} = \sum_{j=1}^{M+1} f_j^*(r)\boldsymbol{\delta}_j \tag{8}$$

for the circular plate element and

$$W(r) = [f_1(r), f_{\theta 1}(r), f_2(r), \ldots, f_N(r), f_{\theta N}(r)]\{\boldsymbol{\delta}_a\} = \sum_{j=1}^{M+1} f_j^*(r)\boldsymbol{\delta}_j \tag{9}$$

for the annular plate element. The weighting coefficients for the first, second, third and fourth order derivatives, A_{ij}, B_{ij}, C_{ij}, and D_{ij} can be readily determined by using equations (8) and (9). For example, the weighting coefficients for the first order derivatives can be computed by

$$A_{ij} = \mathrm{d}f_j^*(r_i)/\mathrm{d}r \tag{10}$$

To expedite the convergence rate, r_i are selected as [2]

$$r_i = -\frac{r_R + r_L}{r_R - r_L} + \frac{2y_i}{r_R - r_L}, \qquad y_i = [-1, -\cos(2i-3)(\pi/(2N-4)), 1], \tag{11}$$

where r_L and r_R are the co-ordinates of the left and right ends of the DQ element in the radial direction, respectively.

Similarly to the finite element method, the equation for a DQ element can be symbolically written as

$$\{\mathbf{f}\} = [\mathbf{e}]\{\boldsymbol{\delta}\} \tag{12}$$

where $\{\mathbf{f}\}$, $[\mathbf{e}]$ and $\{\boldsymbol{\delta}\}$ are the generalized force vector, weighting coefficient matrix and generalized displacement vector of the DQ element, respectively.

For the DQ circular plate element shown in Figure 2(a), $[\mathbf{e}]$ is derived by using equations (3), (4) and (5), namely,

$$\begin{aligned} e_{1j} &= A_{1j} + \{K_0 \quad \text{when} \quad j = 1\}, \\ e_{2j} &= D[D_{2j} + 2C_{2J}/r_2 - (1+2k^2)B_{2j}/r_2^2 + (1+2k^2)A_{2j}/r_2^3] \\ &\quad + \{D(k^4 - 4k^2)/r_2^4 \quad \text{when} \quad j = 2\}, \\ &\vdots \\ e_{Mj} &= -D[C_{Nj} + B_{Nj}/r_N - (1+2k^2 - \nu k^2)A_{Nj}/r_N^2] - \{D(3k^2 - \nu k^2)/r_N^3 \quad \text{when} \quad j = M\}, \\ e_{M+1j} &= D(B_{Nj} + B_{Nj}\nu/r_N) - \{D\nu k^2/r_N^2 \quad \text{when} \quad j = M\}, \end{aligned} \tag{13}$$

and

$$\{\mathbf{f}\} = [0, \omega^2\rho h, \ldots, V_M, M_{rM}]^{\mathrm{T}}. \tag{14}$$

For the DQ annular plate element shown in Figure (2b), [e] is derived by using equations (3) and (4), namely,

$$\begin{aligned} e_{1j} &= D[C_{1j} + B_{1j}/r_1 - (1+2k^2 - \nu k^2)A_{1j}/r_1^2] + \{D(3k^2 - \nu k^2)/r_1^3 \quad \text{when} \quad j = 1\}, \\ e_{2j} &= -D(B_{1j} + B_{1j}\nu/r_1) + \{D\nu k^2/r_1^2 \quad \text{when} \quad j = 1\} \\ e_{3j} &= D[D_{2j} + 2C_{2j}/r_2 - (1+2k^2)B_{2j}/r_2^2 + (1+2k^2)A_{2j}/r_2^3] \\ &\quad + \{D(k^4 - 4k^2)/r_2^4 \quad \text{when} \quad j = 3\} \\ &\vdots \\ e_{Mj} &= -D[C_{Nj} + B_{Nj}/r_N - (1+2k^2 - \nu k^2)A_{Nj}/r_N^2] \\ &\quad - \{D(3k^2 - \nu k^2)/r_N^3 \quad \text{when} \quad j = M+1\} \\ e_{M+1j} &= D(B_{Nj} + B_{Nj}\nu/r_N) - \{D\nu k^2/r_N^2 \quad \text{when} \quad j = M+1\} \end{aligned}$$

and

$$\{\mathbf{f}\} = [V_1, M_{r1}, \omega^2\rho h, \ldots, V_M, M_{rM}]^{\mathrm{T}} \tag{16}$$

In equations (13) and (15), the term in the parentheses is included only for the indicated jth column. In equations (14) and (16), V and M_r are the effective shear force and bending moment, D, h and ρ are the flexural rigidity, plate thickness and density respectively.

The procedures to formulate the system equations for the DQEM are similar to the finite element method, i.e., combine all generalized forces together and then set the forces equal to the applied loads. Details are omitted here for simplicity. Consider the free vibration analysis of a circular plate with stepped thickness over a concentric circular region, as shown in Figure 1. The thickness, mass density, Poisson ratio and Young's modulus are denoted by h_1, ρ_1, v_1 and E_1, for the outer region ($r_0 \leqslant r \leqslant a$) and by $h_1 + h_2$, ρ_2, v_2 and E_2, for the inner regoin ($0 \leqslant r \leqslant r_0$). To simplify the representation, let $E_1 = E_2 = E$, $\rho_1 = \rho_2 = \rho$ and $v_1 = v_2 = v$. The flexural rigidity for the outer and inner portions are, respectively, $D_1 = Eh_1^3/12(1 - v^2)$ and $D_2 = E(h_1 + h_2)^3/12(1 - v^2)$. One DQ circular plate element ($0 \leqslant r \leqslant r_0$) and one DQ circular annular plate element ($r_0 \leqslant r \leqslant a$) are used in the present analysis and the common boundary is at $r = r_0$. It should be pointed out that if the thickness of the circular plate changes smoothly, as in the cases in references [1] and [2], only one DQ circular or annular plate element is necessary.

The system equation is given by

$$\{\mathbf{F}\} = \{\mathbf{E}\}\{\boldsymbol{\Delta}\} \tag{17}$$

where $\{\mathbf{F}\}$ and $\{\boldsymbol{\Delta}\}$ are

$$\{\mathbf{F}\} = [0, \hat{\omega}^2, \ldots, \hat{\omega}^2, 0, 0, \beta\hat{\omega}^2, \beta\hat{\omega}^2, \ldots, \beta\hat{\omega}^2, 0, 0],$$

$$\hat{\omega}^2 = \rho h_1 \omega^2, \qquad \beta = (1 + h_2/h_1),$$

$$\{\boldsymbol{\Delta}\} = [W_1, W_2, \ldots, W_{N-1}, W_N, \theta_N, W_{N+2}, \ldots, W_{2N-2}, W_{2N-1}, \theta_{2N-1}] \tag{18}$$

and $[\mathbf{E}]$ are obtained by using equations (13) and (15). To account for the elastic constraints at the boundary, the last two rows in matrix $[\mathbf{E}]$ are modified as

$$E_{2N,2N} = E_{2N,2N} + K \qquad E_{2N+1,2N+1} = E_{2N+1,2N+1} + \Phi \tag{19}$$

where $K = 0$ corresponds to $V = 0$, $K = \infty$ corresponds to $W = 0$, $\Phi = 0$ corresponds to $M_r = 0$ and $\Phi = \infty$ corresponds to $\theta = 0$. In this way, various boundary conditions, such as simply supported, clamped, freely and elastically constrained boundary conditions, can be easily applied.

Before solving the eigenvalues, equation (17) is modified, namely, $W_1, W_N, \theta_N, W_{2N-1}, \theta_{2N-1}$ are expressed in terms of the other generalized displacements by using the first, Nth, $(N + 1)$th, $(2N)$th and $(2N + 1)$th equations of equation (17). After some further manipulations, equation (17) becomes

$$[\bar{\mathbf{E}}]\{\bar{\boldsymbol{\Delta}}\} = \bar{\omega}^2\{\bar{\boldsymbol{\Delta}}\} \tag{20}$$

where $\bar{\omega}^2 = \omega^2 a^4 \rho h_1/D_1$ (a is the radius of the plate) is the non-dimensionalized frequency. Equation (20) can be solved for each k (the number of the nodal diameter) and a particular boundary condition to obtain the frequencies.

3. RESULTS AND DISCUSSION

A computer program was written to solve the same problem ($r_0 = 0{\cdot}5a$) as the one in reference [5] by using the DQEM developed in this paper. Various boundary conditions can be applied by simply inputting different values of K and Φ. After a convergence study, the number of the grid point N is taken as 9 for both DQ circular and annular elements. Thus, the dimension of the matrix in equation (20) is 15×15. Table 1 lists results for uniform circular plates with clamped and simply supported boundary conditions. As can be seen from Table 1, the present results agree very well with the Ritz data [5] for the plate with uniform thickness. Table 2 shows three frequencies for plates with uniform or stepped thickness, when the number of the nodal diameter is 1. The DQEM results are slightly lower than the Ritz data in reference [5], which are the upper bounds. A larger discrepancy exists when compared with the Ritz results for the higher order frequencies. Increasing the

TABLE 1

Comparison of values of $\bar{\omega}$ in the case of a uniform plate

No. of nodal diameters	Clamped			Simply supported		
	Present	Ref. [5]	Ref. [6]	Present	Ref. [5]	Ref. [6]
0	10·2158	10·215	10·2158	4·935	4·935	4·977
1	21·260	21·260	21·26	13·898	13·898	13·94
2	34·877	34·877	34·88	25·613	25·613	25·65
3	51·030	51·030	51·04	39·957	39·957	NA

number of grid points for the DQEM improves only the higher order frequencies other than those presented in this paper. Thus, one may conclude that the DQEM results are more accurate than the Ritz data [5]. To confirm this, the problem was reanalyzed by the ordinary Ritz method with eight terms, the results of which are also included in Table 2. It can be seen that the new Ritz data are closer to the present results than the Avalos data [5] for certain cases. The lowest frequency for each fixed number of nodal diameters is shown in Table 3 for both uniform and stepped thicknesses and various boundary conditions. It can be seen that the present results compare well with the Avalos data [5]. The small difference for the uniform case is due to round-off errors in numerical calculations. It is found that some higher order frequencies in reference [5] may not be accurate due to the small number of terms used in the analysis.

4. CONCLUSION

The differential quadrature element method has been developed and then used to analyze the free vibration of circular plate with stepped thickness for the first time. It can be seen that DQEM retains all advantages of the differential quadrature method, extends the applicability range of the DQ method and is more convenient for treating various boundary conditions.

TABLE 2

Comparison of values of $\bar{\omega}$ in the case of a stepped plate

h_1/h_2	No. of nodal circles	Clamped			Simply supported		
		Present	Ref. [5]	Ritz†	Present	Ref. [5]	Ritz†
0·0	0	21·260	21·26	21·260	13·898	13·898	13·90
	1	60·829	60·85	60·86	48·479	48·48	48·48
	2	120·80	126·2	121·76	102·77	105·3	102·81
0·5	0	23·276	24·02	23·93	14·950	15·40	15·34
	1	74·665	74·71	74·89	58·827	60·01	60·04
	2	141·67	152·3	152·74	121·39	127·7	127·83
1·0	0	23·45	25·88	25·520	14·930	16·32	16·07
	1	85·31	90·30	90·356	65·823	71·73	71·63
	2	158·94	177·8	176·66	139·54	149·8	145·94

† Results are obtained by Ritz method with eight-term polynomial.

TABLE 3

Comparison of values of $\bar{\omega}$ in the case of a stepped plate

			Φ_a/D_1							
			∞		8		1		0	
h_2/h_1	Ka^3/D_1	k	DQEM	Ref. [5]	DQEM	Ref. [5]	DQEM	Ref. [5]	DQEM	Ref. [5]
0	∞	1	21·26	21·26	18·15	18·14	14·97	14·96	13·893	13·898
		2	34·88	34·87	30·33	30·32	26·67	26·66	25·61	25·61
		3	51·03	51·03	45·02	45·02	41·00	40·99	39·96	39·95
	32	1	9·569	9·569	9·508	9·507	9·398	9·398	9·339	9·339
		2	13·60	13·59	13·50	13·49	13·34	13·34	13·27	13·27
		3	20·35	20·35	19·72	19·72	18·92	18·91	18·60	18·59
	1	1	3·513	3·513	3·229	3·229	2·537	2·537	1·988	1·988
		2	8·983	8·983	8·166	8·165	6·670	6·670	5·835	5·835
		3	17·02	17·02	15·60	15·60	13·58	13·57	12·69	12·69
	0	1	3·082	3·082	2·710	2·709	1·680	1·679	0·0	0·0
		2	8·785	8·784	7·913	7·913	6·290	6·290	5·358	5·358
		3	16·90	16·90	15·44	15·44	13·35	13·35	12·44	12·43
0·5	∞	1	23·28	24·02	19·57	20·15	16·07	16·55	14·95	15·40
		2	38·53	39·20	33·42	33·85	29·53	19·82	28·45	28·71
		3	57·33	57·71	50·38	50·57	45·93	46·05	44·80	44·91
	32	1	9·556	9·616	9·515	9·582	9·443	9·522	9·406	9·492
		2	15·00	15·00	14·83	14·83	14·57	14·57	14·46	14·46
		3	22·32	22·36	21·49	21·53	20·44	20·49	20·03	20·08
	1	1	3·549	3·580	3·240	3·261	2·513	2·518	1·959	1·959
		2	10·76	10·76	9·851	9·867	8·276	8·308	7·447	7·492
		3	19·10	19·15	17·44	17·50	15·16	15·22	14·19	14·26
	0	1	3·131	3·164	2·732	2·754	1·669	1·673	0·0	0·0
		2	10·58	10·59	9·632	9·648	7·955	7·990	7·057	7·105
		3	18·99	19·04	17·29	17·35	14·94	15·01	13·95	14·02
1·0	∞	1	23·45	25·88	19·58	21·42	16·65	17·52	14·93	16·32
		2	41·98	43·42	36·77	37·57	32·89	33·39	31·83	32·26
		3	64·81	65·34	56·81	57·14	51·85	52·16	50·62	50·93
	32	1	9·415	9·586	9·377	9·565	9·311	9·528	9·278	9·510
		2	16·93	17·02	16·64	16·73	16·21	16·32	16·02	16·34
		3	24·26	24·92	23·36	23·80	22·02	22·44	21·51	21·92
	1	1	3·515	3·603	3·203	3·262	2·478	2·493	1·930	1·931
		2	13·02	13·19	11·96	12·16	10·18	10·45	9·293	9·610
		3	21·32	21·85	19·36	19·88	16·75	17·28	15·68	16·21
	0	1	3·106	3·199	2·706	2·766	1·647	1·660	0·0	0·0
		2	12·87	13·04	11·76	11·97	9·900	10·18	8·956	9·290
		3	21·21	21·74	19·21	19·73	16·54	17·07	15·44	15·98

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REFERENCES

1. X. WANG, A. G. STRIZ and C. W. BERT 1993 *Journal of Sound and Vibration* **164**, 173–175. Free vibration analysis of annular plates by the DQ method.
2. X. WANG, J. YANG and J. XIAO 1995 *Journal of Sound and Vibration* **184**, 547–551. On free

vibration analysis of circular annular plates with non-uniform thickness by the differential quadrature method.
3. A. G. Striz, W. Chen and C. W. Bert 1994 *International Journal of Solids and Structures* **31**, 2807–2818. Static analysis of structures by the quadrature element method (QEM).
4. X. Wang and H. Gu 1997 *International Journal for Numerical Methods in Engineering* (in press). Static analysis of frame structures by the differential quadrature element method.
5. D. R. Avalos, P. A. A. Laura and H. A. Larrondo 1988 *Journal of Acoustical Society of America* **84**, 1181–1185. Vibrating circular plates with stepped thickness over a concentric circular region: a general, approximation solution.
6. A. W. Leissa 1969 *NASA SP*-160. Vibration of plates.