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A METHOD OF ANALYZING FINITE PERIODIC STRUCTURES, PART 1: THEORY AND EXAMPLES

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The application of the Rayleigh–Ritz and extended Rayleigh–Ritz energy methods to finite periodic structures with sinusoidal displacement functions is discussed. The coupling relationships among the assumed functions for a stiffened beam are derived. It is shown that, by neglecting a secondary coupling, the coupling relationships become relatively simple. For a periodically constrained beam the geometric constraints can be replaced by a set of equivalent constraints, each of which only involves the functions in one coupling group dictated by the coupling relationships when the stiffeners are also added at the constrained points. A method of analyzing finite periodic structures is therefore proposed by making use of the coupling relationships. The benefits of using the proposed method are demonstrated with numerical examples.

1. INTRODUCTION

Many engineering structures are found to be periodical or pseudo-periodic. The fuselages of civil transport airplanes, railway bridges and railway carriages are examples. In practice, these structures may often be modelled as periodically stiffened and/or supported beams, plates or shells. Research into the vibration of these structures have been carried out by many investigators. The mathematical principle of wave propagation in periodic structures was discussed extensively by Brillouin [1]. The early applications to structural engineering include the works of Heckl [2], Ungar [3], Bobrovnitskii and Maslov [4] and Mead and Wilby [5]. They studied the flexural wave motion in periodically supported beam structures. Mead and Sen Gupta [6–11] studied the free and forced vibrations of periodic beams and rib–skin structures. The behaviour of propagation and attenuation of waves in mono-coupled and multi-coupled periodic structures were discussed by Mead [12, 13], using characteristic receptance functions. Orris and Petyt [14, 15] used finite element techniques to obtain the equations of motion of a periodic element and to study the free wave propagation and response due to a convected random pressure field in one-dimensional periodic systems. Their work was further extended by Rahman [16, 17] to two-dimensional periodic systems. More recently, Mead and Bardell [18–21] extended the analysis to stringer or ring stiffened shells, and combined the hierarchical finite element method with wave propagation theory to calculate the phase constant surfaces of orthogonally stiffened shell and plates.

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The feature of the analysis mentioned above is that the periodicity of a structure is assumed to be extended to infinity. The characteristics of wave propagation along such an infinite periodic structure can then be extracted by studying only a single element of the system in terms of propagation constants. The natural frequencies and normal modes of a finite periodic structure can be calculated from the propagation constants [7]. This is advantageous when the number of periodic elements is not small and when each element is subject to the same type of loading. However, when the number of elements is small and/or the excitation has not the same periodicity as the structure, the analysis procedure becomes less efficient. This problem can be avoided if the analysis is based directly on a finitely long periodic structure model. However, very few studies have been found in the literature in this regard. Miles [22] proposed a set of assumed displacement functions and used them to formulate the mass and stiffness matrices of periodic beams and plates of finite length, but no numerical calculations were presented.

It has long been noticed by many investigators [23–26] that when sinusoidal displacement functions are used in the Rayleigh–Ritz method to analyze shells with equally spaced stiffeners, not all of the functions are coupled together. Thus, including an uncoupled function in the eigenvalue equations makes no improvement to the convergence of the modal frequency concerned. Miller [25] theoretically derived the coupling relationships found for an orthogonally stiffened shell, although an error exists in his formulae. Wei and Petyt [26] have re-examined the coupling relations found for a ring-stiffened shell and have used the coupling relationships to reduce the size of eigenvalue equation. They have shown that the technique is an efficient method for periodically ring-stiffened shells of finite length. The technique has been further developed, and it has been established that the proposed analysis is a very efficient approximate method to analyze periodic structures of finite length. The study reported in this paper presents the first part of the work and describes the methodology of the proposed method.

In order to demonstrate the methodology of the analysis clearly, the presentation in this paper is based on periodically stiffened or supported beams. After a brief discussion of the procedure of applying the Rayleight–Ritz and extended Rayleigh–Ritz methods, the coupling relationships among the assumed sinusoidal displacement functions for a periodically stiffened beam, simply supported at the two extreme ends, are first examined. A periodically simply supported beam is then studied in the next section. In the accompanying paper [27], the relation between the present analysis and the analyses based on an infinite periodic structural model is discussed. It will be demonstrated that the method can also be used to calculate the propagation constants in the frequency pass-bands of an infinite periodic structure. The applications of the method to point supported plates, and to orthogonally stiffened shells with or without an interior plate, can be found in reference [28].

2. THE RAYLEIGH–RITZ AND EXTENDED RAYLEIGH–RITZ METHODS

The Rayleigh–Ritz method is a procedure that produces an approximate solution for a vibration problem. The main feature of this method is to expand each displacement component of the structure into a linear combination of a series of prescribed functions. Each of the prescribed functions should satisfy the geometric boundary conditions and internal kinematic compatibility conditions. In the extended Rayleigh–Ritz method, the latter requirement is dropped, but constraint equations must be applied to enforce these conditions.

The general procedure of applying the extended Rayleigh–Ritz method to a vibration problem is outlined in the following steps. (1) Assume a series expansion of prescribed functions for each displacement component. (2) Substitute the expansions into the kinetic and strain or potential energy expressions of the system. For a forced response problem, the work done by applied load is also calculated. (3) Apply constraints to enforce those geometric boundary conditions and/or kinematic compatibility conditions which are not satisfied by the assumed displacement functions. (4) Derive the equations of motion using Hamilton's principle. (5) Solve the equations of motion. For the Rayleigh–Ritz method, the third step is dropped, since the assumed functions already satisfy these conditions. In later discussions, not all of these steps are presented and only those necessary are retained to save space. The equations of motion are often expressed in matrix form in terms of the generalized co-ordinates, which are the unknown coefficients associated with the prescribed functions in the assumed series. For a free vibration problem, these are given by

$$[\mathbf{K} - \omega^2\mathbf{M}]\{\mathbf{u}\} = \{\mathbf{0}\}, \tag{1}$$

where [**K**] and [**M**] are the system stiffness and mass matrices and ω is the circular frequency. The vector $\{\mathbf{u}\}$ contains the generalized co-ordinates.

3. THE COUPLING RELATIONSHIP FOR A BEAM WITH EQUALLY SPACED STIFFENERS

The layout of a simply supported beam with equally spaced stiffeners is shown in Figure 1. All of the stiffeners are identical and each stiffener consists of a mass M_t, rotational inertia I_r, and vertical and rotational stiffnesses K_t and K_r (not shown). The assumed displacement expression in this case is

$$w(x) = \sum_{m=1}^{\infty} w_m \sin(m\pi x/L), \tag{2}$$

where $w(x)$ is the beam flexural displacement and L is the beam length. m indicates the number of half sine waves in the shape of the corresponding prescribed function, and w_m is an unknown coefficient associated with this function. The kinetic and strain energy of the beam itself are

$$T_b = \tfrac{1}{2}\int_0^L \rho[A\dot{w}^2 + I\dot{w}_{,x}^2]\,\mathrm{d}x, \qquad U_b = \tfrac{1}{2}\int_0^L EIw_{,xx}^2\,\mathrm{d}x, \tag{3, 4}$$

where ρ is the beam mass density. A and I are the cross-sectional area and the second moment of area of the beam, respectively. E is Young's modulus. The second term in equation (3) is the contribution of beam rotational inertia. $w_{,x}$ and $w_{,xx}$ are the derivatives of w with respect to x, once and twice respectively. A dot over w indicates the derivative with respect to time.

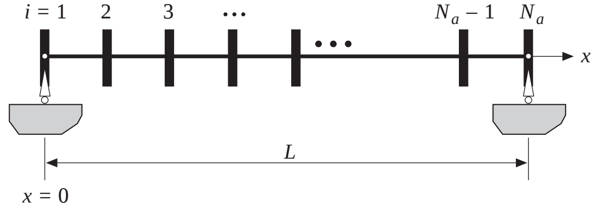

Figure 1. A periodically stiffened beam simply supported at the two ends.

The total kinetic and potential energy of the stiffeners are

$$T_a = \tfrac{1}{2}\sum_{k=1}^{N_a}[M_t\dot{w}_k^2 + I_r(\dot{w}_{k,x})^2], \qquad U_a = \tfrac{1}{2}\sum_{k=1}^{N_a}[K_t w_k^2 + K_r(w_{k,x})^2], \tag{5a, b}$$

where N_a is the number of stiffeners and subscript k denotes the kth stiffener. Substituting equation (2) into equtions (3), (4) and (5), the system mass and stiffeness matrices are obtained as

$$[\mathbf{M}] = [\mathbf{M}]_b + [\mathbf{M}]_a, \qquad [\mathbf{K}] = [\mathbf{K}]_b + [\mathbf{K}]_a, \tag{6a, b}$$

where $[\mathbf{M}]_b$ and $[\mathbf{K}]_b$ are diagonal, since the prescribed functions are the natural modes of the unstiffened beam. The diagonal elements are

$$M_{ii} = 0{\cdot}5\rho LA[1 + (I/A)(m_i\pi/L)^2], \qquad K_{ii} = 0{\cdot}5\rho LEI(m_i\pi/L)^4. \tag{7a, b}$$

The expressions for $[\mathbf{M}]_a$ and $[\mathbf{K}]_a$ are as follows:

$$[\mathbf{M}]_a = [M_{ij}], \qquad M_{ij} = M_t S_{ij} + I_r C_{ij}(m_i\pi/L)(m_j\pi/L), \tag{8a}$$

$$[\mathbf{K}]_a = [K_{ij}], \qquad K_{ij} = K_t S_{ij} + K_r C_{ij}(m_i\pi/L)(m_j\pi/L), \tag{8b}$$

with

$$C_{ij} = \sum_{k=1}^{N_a}\cos(m_i\pi x_k/L)\cos(m_j\pi x_k/L), \tag{9a}$$

$$S_{ij} = \sum_{k=1}^{N_a}\sin(m_i\pi x_k/L)\sin(m_j\pi x_k/L), \tag{9b}$$

where in equations (7) and (8), $i, j = 1, 2, 3, \ldots$, etc., and m_i and m_j are the corresponding values of m at the ith row and jth column respectively. x_k is the location of kth stiffener. Equations (7) and (8) demonstrate clearly that stiffeners cause the coupling between the functions having different values of m and the coupling is represented by two summations, C_{ij} and S_{ij}, in equation (8). It appears that in order to account for the effect of this coupling all the functions of $m < M^*$, where M^* is the largest value of m which is included in the solution, have to be included in the final equation simultaneously, or at least half of them have to be included if the symmetric property of the structure is used. However, it can be shown that, for a periodically stiffened beam, explicit expressions for these two summations may be derived. The details of evaluating these summations are given in Appendix A. Only the final results are listed here, as follows:

$$C_{ij} = \begin{cases} N_b/2 \pm 1 & \text{if } |m_i - m_j| = 2pN_b \text{ or } |m_i + m_j| = 2qN_b, \quad (10a)\\ N_b \pm 1 & \text{if } |m_i - m_j| = 2pN_b \text{ and } |m_i + m_j| = 2qN_b, \quad (10b)\\ 0 & \text{if } |m_i - m_j| \text{ is odd;} \quad (10c)\\ \pm 1 & \text{otherwise;} \quad (10d)\end{cases}$$

$$S_{ij} = \begin{cases} N_b/2 & \text{if } |m_i - m_j| = 2pN_b \text{ and } |m_i + m_j| \neq 2qN_b, \quad (11a)\\ -N_b/2 & \text{if } |m_i - m_j| \neq 2pN_b \text{ and } |m_i + m_j| = 2qN_b, \quad (11b)\\ 0 & \text{otherwise;} \quad (11c)\end{cases}$$

where N_b is the number of beam segments. p and q are any integers or zero. The value $+1$ in equation (10) is for the cases in which there is a stiffener on either end of the beam ($N_a = N_b + 1$), while -1 is for the cases in which there is nothing on both ends ($N_a = N_b - 1$). If m_i is odd and m_j is even (or vice versa), both $|m_i - m_j|$ and $|m_i + m_j|$ are odd. Therefore equations (10c) and (11c) indicate that there is no coupling between odd and even functions, where odd or even functions are the prescribed functions having odd or even values of m, and they represent symmetric or anti-symmetric beam motions, respectively. Equations (10a, b) and (11a, b) subdivide the even and odd terms into groups. The functions within each group are coupled together. The coupling relations governed by these equations can be stated as follows:

For a given m, if $m < N_b$, the coupling exists between the functions which have

$$m_1 = m, \qquad m_2 = 2N_b - m, \qquad m_2 = 2N_b + m, \qquad m_3 = 4N_b - m, \qquad \ldots, \qquad \text{etc.} \tag{12a}$$

with

$$C_{ij} = N_b/2 \pm 1, \qquad S_{ij} = \begin{cases} N_b/2 & \text{if } |m_i - m_j| = 2pN_b, \\ -N_b/2 & \text{if } |m_i + m_j| = 2qN_b. \end{cases}$$

If $m = N_b$, the coupling exists between

$$m_1 = N_b, \qquad m_2 = 3N_b, \qquad m_3 = 5N_b, \qquad m_4 = 7N_b, \qquad \ldots, \qquad \text{etc.} \tag{12b}$$

If $m = 2N_b$, then

$$m_1 = 2N_b, \qquad m_2 = 4N_b, \qquad m_3 = 6N_b, \qquad m_4 = 8N_b, \qquad \ldots, \qquad \text{etc.} \tag{12c}$$

For the latter two cases, $C_{ij} = N_b \pm 1$ and $S_{ij} = 0$.

Equations (12) indicate that, for a beam of N_b segments, there is a total of $N_b + 1$ coupling groups. The lowest value of m in each group is $m_1 = 1, 2, 3, \ldots, N_b$ and $2N_b$, respectively. As a convention, these values will be used to identify each group. For example, in the case of $N_b = 5$, there are six coupling groups; namely, $m_1 = 1$, 2, 3, 4, 5 and 10. The first seven values of m in each group are listed in Table 1.

In the cases in which $N_a = N_b + 1$ or $N_a = N_b - 1$, equation (10d) exists, which indicates that there are couplings between the even or odd groups (e.g., between even groups of $m_1 = 2$, 4 and 10, or between odd groups of $m_1 = 1$, 3 and 5 in Table 1). This coupling is dependent on the end situations of the beam. The value of C_{ij} due to this coupling is always one, no matter what N_b is. Naturally, it becomes less important when N_b increases

TABLE 1

Example of coupling groups for $N_b = 5$

Group	The first seven values of m_i						
m_1	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
1	1	9	11	19	21	29	31
2	2	8	12	18	22	28	32
3	3	7	13	17	23	27	33
4	4	6	14	16	24	26	34
5	5	15	25	35	45	55	65
10	10	20	30	40	50	60	70

and therefore may be neglected. This kind of coupling may be called secondary coupling. In the cases in which the structure is perfectly periodic, i.e., there is a half-stiffener, which has half the mass and stiffness of those in between, at each end of the beam ($N_a = N_b$), this coupling does not exist and all ± 1 in equations (10) and (12) will be dropped (see Appendix A). In general, if the secondary coupling due to C_{ij} is neglected, equation (12) becomes the only coupling relationship between the prescribed functions. Therefore it is only necessary to solve a problem which is formed by retaining only the functions coupled to each other. This means that the original problem can be resolved into $N_b + 1$ sub-problems, but the order of each sub-problems is only $1/N_b$ of the original one, if the same accuracy is maintained in both cases. This will obviously save computer time, since the solution time for an eigenvalue problem is approximately proportional to the cube of the order of the problem. Moreover, when the number of stiffeners is not small, it becomes necessary to adopt this technique and to include the coupled functions only in the solution, the order of which will otherwise be unmanageable.

The general effect of omitting the secondary coupling due to C_{ij} is very difficult to estimate, since it depends not only on the number of stiffeners but also on the relative mass and stiffness ratios between beam and stiffener. However, it becomes decreasingly important as the number of beam segments N_b increases. This is demonstrated in Table 2,

Table 2

The first two natural frequencies of periodically stiffened beams with three different end situations

		$N_a = N_b - 1$			$N_a = N_b + 1$			$N_a = N_b$,
N_b	M^*	B	A	δ (%)	B	A	δ (%)	A and B
1	30	9·869604	9·869604	0·0	4·740209	4·740209	0·0	5·301018
	30	39·47842	39·47842	0·0	6·331425	6·331425	0·0	7·980884
2	30	2·839505	2·839505	0·0	3·416108	3·416108	0·0	3·241466
	15	5·853870	6·118407	4·52	4·919312	4·947100	0·57	5·306099
3	20	2·166674	2·218630	2·40	2·586745	2·600920	0·55	2·451258
	20	4·241264	4·242725	0·03	4·126148	4·126360	$<0·01$	4·166614
4	15	1·994388	2·040499	2·31	2·234144	2·249184	0·67	2·159696
	15	3·071456	3·093687	0·72	3·337003	3·343643	0·20	3·242572
5	12	1·922615	1·957677	1·82	2·067161	2·078972	0·57	2·024566
	12	2·565528	2·605482	1·56	2·830464	2·844412	0·49	2·741451
10	6	1·826781	1·836376	0·53	1·852246	1·855097	0·15	1·846103
	6	1·970905	1·995941	1·27	2·049867	2·058326	0·41	2·028655
15	4	1·806627	1·810426	0·21	1·815470	1·816500	0·06	1·813530
	4	1·872772	1·884636	0·63	1·903209	1·906776	0·19	1·895992
20	3	1·798379	1·800186	0·10	1·802363	1·802832	0·03	1·801528
	3	1·837098	1·843286	0·34	1·851632	1·853366	0·09	1·848409
30	2	1·795109	1·795877	0·04	1·796866	1·797032	$<0·01$	1·796458
	2	1·815285	1·818029	0·15	1·821237	1·821891	0·04	1·819976
40	1	1·792714	1·793046	0·02	1·793403	1·793485	$<0·01$	1·793266
	1	1·804347	1·805606	0·07	1·807014	1·807337	$<0·01$	1·806472
30	3		1·792454			1·793245		1·792853
	3		1·812114			1·815210		1·813678
40	3		1·789649			1·789984		1·789817
	3		1·800862			1·802185		1·801528

A, The results excluded the effect of secondary coupling due to C_{ij}.
B, The results included the effect of secondary coupling due to C_{ij}.
M^* = the number of the coupled terms used in the calculation of A.

in which the first two lowest natural frequencies of a stiffened beam are calculated for three end situations, $N_a = N_b - 1$, $N_a = N_b + 1$ and $N_a = N_b$. A non-dimensional frequency $\Omega = \omega l^2(\rho A/EI)^{0\cdot 5}$ is used, where l is the length of a beam segment. The properties of the stiffeners are $M_t = 0{\cdot}2\rho Al$, $I_r = 0{\cdot}25\rho Al^2$, $K_t = 4EI/l^3$ and $K_r = 4EI/l$. The secondary coupling is included in the results shown in column B, but it is excluded in the results in column A. These two frequencies are the lowest natural frequencies of the coupling groups for $m_1 = 1$ and 2, respectively. For the results which include the secondary coupling, 30 odd ($m = 1, 3, 5, \ldots, 59$) or even ($m = 2, 4, 6, \ldots, 60$) functions were used in the calculation. Therefore, when N_b increases, the number of the coupled functions as defined by equation (12) decreases within these 30 functions. The actual number of the coupled functions included in the 30 odd or even functions is indicated in the second column of the table. The results excluding the secondary coupling were calculated by using the coupling functions for $m_i < 61$ in each coupling group.

For $N_b = 1$, there are only two coupling groups containing either even or odd functions, and hence the results which include and exclude the secondary coupling are the same. For $N_b = 2$, the coupling group of $m_1 = 1$ contains all odd functions and thus gives the same frequency as those of B. The effect of the secondary coupling does exist in the rest of the results. The frequencies that include the secondary coupling are smaller than those where it is excluded. The relative differences are gradually reduced as N_b increases. When N_b is large, say, larger than ten, the differences are less than 1%. However, the number of coupled functions used in A is much less than those for B. When the number of coupled functions used for A is allowed to increase, the frequencies for A become smaller than those in B, as shown at the bottom of the table for $N_b = 30$ and 40. This shows the necessity of using the proposed technique in these cases if a very large order of eigenvalue problem is to be avoided.

Table 2 can also be used to examine the effect of three end situations by comparing the results in columns A or B horizontally. It can be seen that the different end situations do result in different frequencies and that the differences decrease as N_b increases. The proposed method can effectively take this into account by including end situations in the calculation of C_{ij} in equation (12), even if the secondary coupling due to C_{ij} is neglected.

4. CONSTRAINT EQUATIONS FOR PERIODICALLY SIMPLY SUPPORTED BEAMS

Consider a periodically simply supported beam of finite length (see Figure 2). The prescribed displacement functions in equation (2) may still be used in this case, since they satisfy the boundary conditions at two ends. However, the geometric conditions imposed by the supports between the ends are not satisfied. These conditions are

$$w(x_k) = 0 \qquad \text{at } x_k = kL/N_b, \qquad \text{for } k = 1, 2, \ldots, N_b - 1, \tag{13}$$

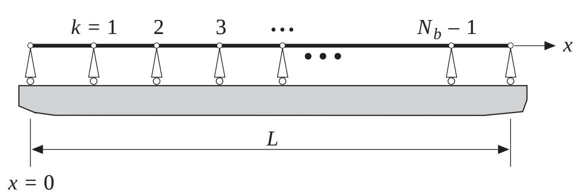

Figure 2. A periodically simply supported beam.

where N_b is the number of beam segments and L is the beam length. Substituting the displacement expression into equation (13) gives

$$\sum_{m=1}^{\infty} w_m \sin(m\pi k/N_b) = 0, \qquad k = 1, 2, \ldots, N_b - 1. \tag{14}$$

There are $N_b - 1$ constraints that need to be imposed. However it is very difficult, if not impossible, to apply equation (14) directly to impose the constraints. Nor do these constraints in the present form provide any insight into the coupling relations between the assumed functions. In order to apply the extended Rayleigh–Ritz method to this problem efficiently and with good effect, an alternative approach must be found to impose the constraints. The idea is to try to find a new set of constraints which is equivalent to equation (14) but is easy to implement. The least squares technique may be used for this purpose. The constraints in equation (14) are first relaxed and the total squared error due to the relaxation is then calculated as follows:

$$\varepsilon = \sum_{k=1}^{N_b-1} w_k^2 = \sum_{k=1}^{N_b-1} \left(\sum_{m=1}^{\infty} w_m \sin(m\pi k/N_b) \right)^2. \tag{15}$$

To minimize the error, we have

$$\frac{\partial \varepsilon}{\partial w_{m_j}} = 2 \sum_{m_i=1}^{\infty} w_{m_i} \sum_{k=1}^{N_b-1} \sin(m_i \pi k/N_b) \sin(m_j \pi k/N_b)$$

$$= 2 \sum_{m_i=1}^{\infty} w_{m_i} S_{ij} = 0, \qquad \text{with } j = 1, 2, 3, \ldots, \text{etc.}, \tag{16}$$

where the summation S_{ij} is given in equation (9b), with N_a being replaced by $N_b - 1$. According to the discussion in the above section, S_{ij} divides the displacement functions into $N_b + 1$ coupling groups, as stated in equation (12). In the coupling group for $m_1 = N_b$ or $m_1 = 2N_b$, the functions already satisfy all the constraints. The remaining functions fall into $N_b - 1$ coupling groups, as indicated by equation (12a). For any given m_i, S_{ij} is zero if m_j is not in the same coupling group as m_i. This gives rise to $N_b - 1$ independent constraint equations from equation (16). Each equation only involves the functions in only one coupling group. Equation (16) becomes

$$\sum_{i=1}^{\infty} w_{m_i} (-1)^{i-1} = 0, \qquad \text{with } m_1 = 1, 2, 3, \ldots, N_b - 1, \tag{17}$$

and m_i is defined by equation (12a).

The significance of the above conclusion is that the coupling relationships discussed in the last section for a periodically stiffened structure are still applicable when discrete constraints are imposed at the same locations of the stiffeners. Thus, the problem can still be divided into $N_b + 1$ sub-problems. For each sub-problem, only the prescribed functions in one coupling group are involved, and the geometric conditions are imposed by using equation (17) to eliminate any one of the unknown coefficients associated with these functions. It should be pointed out that equation (17) is exactly equivalent to the original set of geometric constraints of equation (14), that is, equation (14) is satisfied if and only if equation (17) is met, although from the derivation of equation (17) here, it seems that

TABLE 3

The natural frequencies of a five-bay, periodically simply supported beam: comparison with Sen Gupta's results

m_1	A	B
1	4·5519	4·55
2	4·1537	4·15
3	3·7007	3·70
4	3·3092	3·30
5	3·1416	3·14

A, The present results with nine coupled terms; B, Sen Gupta's results.

the geometric conditions might be only approximately satisfied. The proof of this is in Appendix B.

To demonstrate the above discussion, the first five natural frequencies of a five-bay periodically simply supported beam are calculated using the proposed method. The frequencies are listed in Table 3. The example is taken from the work of Sen Gupta [7]. He used the periodical structure theory to calculate the natural frequencies of the beam. His results are also shown in the table. It can be seen that the two sets of results agree with each other very well. Nine coupled functions are used in the calculation of the present results. The convergence of the calculation is also examined by increasing M^*, the number of the coupled functions in the calculation. The results are shown in Table 4. Since the functions in the coupling group $m_1 = 5$ already satisfy all of the constraints, there is no coupling among them and therefore the frequency in the last row is independent of M^*. The last column gives the percentage differences between the results of $M^* = 9$ and $M^* = 21$. The differences are very much negligible and therefore nine coupled functions provide adequate results in this case.

When external loading is added, the response of a finite periodic structure can be readily solved once the system mass, stiffness and damping matrices and the generalized force vector are calculated. The procedure for calculating the generalized force vector is again standard, regardless of the geometric distribution of the load.

5. CONCLUSION

The application of the Rayleigh–Ritz and extended Rayleigh–Ritz energy methods to finite periodic structures with sinusoidal displacement functions is discussed. The coupling relationships among the assumed functions for a stiffened beam are derived. It has been

TABLE 4

The convergence of natural frequencies of a five-bay, periodically simply supported beam

	M^*							
m_1	3	5	7	9	11	17	21	%
1	4·5843	4·5583	4·5534	4·5519	4·5512	4·5506	4·5505	0·030
2	4·1725	4·1575	4·1546	4·1537	4·1534	4·1531	4·1530	0·017
3	3·7093	3·7026	3·7012	3·7007	3·7006	3·7004	3·7004	0·008
4	3·3096	3·3097	3·3093	3·3092	3·3091	3·3091	3·3091	0·003
5	3·1416	3·1416	3·1416	3·1461	3·1416	3·1416	3·1416	—

shown that by neglecting a secondary coupling the coupling relationships become relatively simple. For a periodically constrained beam the geometric constraints can be replaced by a set of equivalent constraints, each of which only involves the functions in one coupling group dictated by the coupling relationships when the stiffeners are also added at the constrained points. A method of analyzing finite periodic structures is therefore proposed by making use of the coupling relationships. The benefits of using the proposed method are demonstrated with the numerical examples. The proposed method inherits the main advantages of the Rayleigh–Ritz and extended Rayleigh–Ritz methods. The concept and procedure of the analysis is simple. It also enables these methods to be applied effectively to structures with many stiffeners and/or supports. It is shown, and will be further demonstrated in the accompanying paper [27], that the method is a very useful approximate approach for the analysis of periodically stiffened or constrained structures of both finite and infinite length.

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APPENDIX A: EVALUATION OF C_{ij} AND S_{ij}

Two summations given in equation (9) are

$$C_{ij} = \sum_{k=1}^{N_a} \cos(m_i \pi x_k/L) \cos(m_j \pi x_k/L), \tag{A1a}$$

$$S_{ij} = \sum_{k=1}^{N_a} \sin(m_i \pi x_k/L) \sin(m_j \pi x_k/L), \tag{A1b}$$

where L is the length of the structure over which N_a stiffeners are equally spaced. x_k is the location of the kth stiffener. First examine the case in which there is no stiffener at either end of the structure. In this case, $N_a = N_b - 1$ and

$$x_k = lk, \qquad k = 1, 2, \ldots, N_b - 1, \tag{A2}$$

with $L = N_b l$ and l is the pitch of stiffeners or the length of a segment. Substituting equation (A2) into equation (A1) gives

$$C_{ij} = \sum_{k=1}^{N_b-1} \cos(m_i \pi k/N_b) \cos(m_j \pi k/N_b),$$

$$= \tfrac{1}{2}\left[\sum_{k=1}^{N_b-1} \cos((m_i - m_j)\pi k/N_b) + \sum_{k=1}^{N_b-1} \cos((m_i + m_j)\pi k/N_b)\right], \tag{A3a}$$

$$S_{ij} = \sum_{k=1}^{N_b-1} \sin(m_i \pi k/N_b) \sin(m_j \pi k/N_b),$$

$$= \tfrac{1}{2}\left[\sum_{k=1}^{N_b-1} \cos((m_i - m_j)\pi k/N_b) - \sum_{k=1}^{N_b-1} \cos((m_i + m_j)\pi k/N_b)\right], \tag{A3b}$$

It is clear that if we can evaluate a summation which is defined as

$$\text{SUMC} = \sum_{k=1}^{N_b-1} \cos(M\pi k/N_b), \tag{A4}$$

where M is an integer between $-\infty$ and $+\infty$, C_{ij} and S_{ij} can then be calculated readily. From equation (A.4), if $M = 0$, $2N_b$, $4N_b$, . . . , etc., then $\cos(M\pi k/N_b) = 1$; thus

$$\text{SUMC} = N_b - 1. \tag{A5}$$

If $M \neq 0$, $2N_b$, $4N_b$, . . . , etc. then

$$\text{SUMC} = \text{Re}\left\{\sum_{k=1}^{N_b-1} \exp(jM\pi k/N_b)\right\}, \tag{A6}$$

where $j = \sqrt{-1}$ and Re{} indicates the real part of a complex number. Since

$$\sum_{k=1}^{N_b-1} x^k = (x - x^{N_b})/(1-x), \tag{A7}$$

the complex expression in equation (A6) becomes

$$\begin{aligned}\sum_{k=1}^{N_b-1} \exp(\mathrm{j}M\pi k/N_b) &= \frac{\exp(\mathrm{j}M\pi/N_b) - \exp(\mathrm{j}M\pi)}{1 - \exp(\mathrm{j}M\pi/N_b)} \\ &= \frac{\exp(\mathrm{j}M\pi/2N_b) - \exp(\mathrm{j}M\pi)\exp(-\mathrm{j}M\pi/2N_b)}{\exp(-\mathrm{j}M\pi/2N_b) - \exp(\mathrm{j}M\pi/2N_b)} \\ &= \frac{\exp(\mathrm{j}M\pi/2N_b) - (-1)^M \exp(-\mathrm{j}M\pi/2N_b)}{-2\mathrm{j}\sin(M\pi/2N_b)} \\ &= \begin{cases} -1 & \text{if } M \text{ is even;} \\ \mathrm{j}\tan^{-1}(M\pi/2N_b) & \text{if } M \text{ is odd.} \end{cases}\end{aligned} \tag{A8}$$

Combining equations (A5), (A6) and (A8) gives

$$\text{SUMC} = \begin{cases} N_b - 1 & \text{if } |M| = 2pN_b, \\ 0 & \text{if } |M| \text{ is odd,} \\ -1 & \text{otherwise,} \end{cases} \tag{A9}$$

where p is any integer or zero. Likewise,

$$\sum_{k=1}^{N_b-1} \cos((m_i - m_j)\pi k/N_b) = \begin{cases} N_b - 1 & \text{if } |m_i - m_j| = 2pN_b, \\ 0 & \text{if } |m_i - m_j| \text{ is odd,} \\ -1 & \text{otherwise,} \end{cases} \tag{A10}$$

and

$$\sum_{k=1}^{N_b-1} \cos((m_i + m_j)\pi k/N_b) = \begin{cases} N_b - 1 & \text{if } |m_i + m_j| = 2qN_b, \\ 0 & \text{if } |m_i + m_j| \text{ is odd,} \\ -1 & \text{otherwise,} \end{cases} \tag{A11}$$

where q is any integer. Noticing that if $|m_i - m_j|$ is odd, $|m_i + m_j|$ is odd as well, and if $|m_i - m_j|$ is even $|m_i + m_j|$ is also even, substituting equations (A10) and (A11) into equation (A3) yields

$$C_{ij} = \begin{cases} N_b - 1 & \text{if } |m_i - m_j| = 2pN_b \text{ and } |m_i + m_j| = 2qN_b, \quad \text{(A12a)} \\ N_b/2 - 1 & \text{if } |m_i - m_j| = 2pN_b \text{ or } |m_i + m_j| = 2qN_b \text{ only}, \quad \text{(A12b)} \\ 0 & \text{if } |m_i - m_j| \text{ is odd}; \quad \text{(A12c)} \\ -1 & \text{otherwise}; \quad \text{(A12d)} \end{cases}$$

$$S_{ij} = \begin{cases} N_b/2 & \text{if } |m_i - m_j| = 2pN_b; \quad \text{(A13a)} \\ -N_b/2 & \text{if } |m_i + m_j| = 2qN_b; \quad \text{(A13b)} \\ 0 & \text{if } |m_i - m_j| = 2pN_b \text{ and } |m_i + m_j| = 2qN_b \text{ or otherwise} \quad \text{(A13c)} \end{cases}$$

For the case in which there are stiffeners at both ends of the structure, S_{ij} is unchanged while C_{ij} becomes

$$C_{ij} = \sum_{k=1}^{N_b-1} \cos(m_i \pi k/N_b)\cos(m_j \pi k/N_b) + 1 + \cos(m_i\pi)\cos(m_j\pi)$$

$$= \begin{cases} N_b + 1 & \text{if } |m_i - m_j| = 2pN_b \text{ and } |m_i + m_j| = 2qN_b, \quad \text{(A14a)} \\ N_b/2 + 1 & \text{if } |m_i - m_j| = 2pN_b \text{ or } |m_i + m_j| = 2qN_b \text{ only}, \quad \text{(A14b)} \\ 0 & \text{if } |\mathrm{m_i} - \mathrm{m_j}| \text{ is odd}; \quad \text{(A14c)} \\ +1 & \text{otherwise}; \quad \text{(A14d)} \end{cases}$$

If the stiffeners at the ends have half of the mass and stiffness properties of those stiffeners in between, C_{ij} becomes

$$C_{ij} = \begin{cases} N_b & \text{if } |m_i - m_j| = 2pN_b \text{ and } |m_i + m_j| = 2qN_b, \quad \text{(A15a)} \\ N_b/2 & \text{if } |m_i - m_j| = 2pN_b \text{ or } |m_i + m_j| = 2qN_b \text{ only}, \quad \text{(A15b)} \\ 0 & \text{otherwise}, \quad \text{(A15c)} \end{cases}$$

Equation (A13) is equation (11), while equations (A12), (A14) and (A15) give equation (10).

APPENDIX B: THE EQUIVALENT CONSTRAINTS OF PERIODICAL SIMPLY SUPPORTS

The constraints in equation (15) for a periodically simply supported beam are

$$\sum_{m=1}^{\infty} w_m \sin(m\pi k/N_b) = 0, \qquad k = 1, 2, \ldots, N_b - 1. \tag{B1}$$

Divide $\sin(m\pi k/N_b)$ into $N_b + 1$ groups according to the coupling relations given by equation (12). The lowest values of m in each group are $m_1 = 1, 2, \ldots, N_b - 1, N_b$ and $2N_b$, respectively. For the groups with $m_1 = N_b$ and $2N_b$, $\sin(m\pi k N_b) = 0$; therefore the functions in this group do not appear in the constraints. In other words, the motions represented by them pre-satisfy the constraints. For each of the remaining $N_b - 1$ groups, if $0 < m < N_b$,

$$m_1 = m, \qquad m_2 = 2N_b - m, \qquad m_3 = 2N_b + m, \qquad m_4 = 4N_b - m, \qquad , \ldots , \qquad \text{etc.} \tag{B2}$$

For m_2, m_4, m_6, etc.,

$$\sin(m_i \pi k/N_b) = \sin((2pN_b - m)\pi k/N_b) = -\sin(m\pi k/N_b), \tag{B3}$$

while for m_3, m_5, m_7, etc.,

$$\sin(m_i \pi k/N_b) = \sin((2pN_b + m)\pi k/N_b) = \sin(m\pi k/N_b), \tag{B4}$$

where p and q are any integers or zero. Thus the displacement contributed by the coupling group of $m_1 = m$ at the constrained point k is

$$w_{km} = \sin(m\pi k/N_b) \sum_{i=1}^{\infty} w_{mi}(-1)^{i-1} = \sin(m\pi k/N_b) W_m \tag{B5}$$

with k, $m = 1, 2, \ldots, N_b - 1$ and

$$W_m = \sum_{i=1}^{\infty} w_{mi}(-1)^{i-1}. \tag{B6}$$

Substituting equation (B5) into equation (B1) gives

$$\sum_{m=1}^{N_b-1} W_m \sin(m\pi k/N_b) = 0, \qquad k = 1, 2, \ldots, N_b - 1. \tag{B7}$$

Rewriting equation (B7) in matrix form gives

$$[\mathbf{A}]\{\mathbf{W}\} = \{\mathbf{0}\}, \tag{B8}$$

where [**A**] is a square matrix of order $N_b - 1$, with its kth row, mth column element $a_{km} = \sin(m\pi k/N_b)$. Vector $\{\mathbf{W}\}$ contains W_m, $m = 1, 2, \ldots, N_b - 1$. Clearly, if [**A**] is non-singular, the only $\{\mathbf{W}\}$ that can satisfy equation (B7) is

$$\{\mathbf{W}\} = \{\mathbf{0}\} \qquad \text{or} \qquad W_m = 0, \qquad \text{for } m = 1, 2, \ldots, N_b - 1. \tag{B9}$$

This will give the equivalent constraint conditions shown in equation (18). To prove this, we may first look at a matrix [**C**] = [**A**][**A**]. Since $|\mathbf{C}| = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2$, if $|\mathbf{C}|$ is non-zero, $|\mathbf{A}|$ must be non-zero as well and therefore [**A**] is non-singular.

The ith row and jth column element of [**C**] is

$$c_{ij} = \sum_{k=1}^{N_b-1} \sin(i\pi k/N_b) \sin(j\pi k/N_b). \tag{B10}$$

It is clear that [**C**] is symmetric and we only need to examine the elements having $i \geqslant j$. For diagonal elements, $i = j$,

$$c_{ii} = \sum_{k=1}^{N_b-1} \tfrac{1}{2}[1 - \cos(2i\pi k/N_b)]. \tag{B11}$$

From equation (A9),

$$\sum_{k=1}^{N_b-1} \cos(2i\pi k/N_b) = -1, \qquad \text{for } i = 1, 2, \ldots, N_b - 1. \tag{B12}$$

Therefore,

$$c_{ii} = N_b/2. \tag{B13}$$

For $i > j$,

$$c_{ij} = \tfrac{1}{2}\left[\sum_{k=1}^{N_b-1} \cos(i-j)\pi k/N_b - \sum_{k=1}^{N_b-1} \cos(i+j)\pi k/N_b\right]. \tag{B14}$$

Again, from equation (A9),

$$\sum_{k=1}^{N_b-1} \cos(i-j)\pi k/N_b = \begin{cases} -1 & \text{if } i-j \text{ is even,} \\ 0 & \text{if } i-j \text{ is odd,} \end{cases} \tag{B15a}$$

$$\sum_{k=1}^{N_b-1} \cos(i+j)\pi k/N_b = \begin{cases} -1 & \text{if } i-j \text{ is even,} \\ 0 & \text{if } i-j \text{ is odd,} \end{cases} \tag{B15b}$$

Thus

$$c_{ij} = 0, \qquad \text{if } i > j. \tag{B16}$$

Matrix **[C]** is diagonal, and

$$|\mathbf{C}| = (N_b/2)^{N_b-1} \neq 0, \qquad \text{if } N_b \neq 0. \tag{B17}$$

This proves that $|\mathbf{A}|$ is non-zero if $N_b \neq 0$ and the geometric constraints in equation (14) or equation (B1) are satisfied if and only if the equivalent constraints in equation (18) or equation (B9) are satisfied.