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LETTERS TO THE EDITOR

TRANSVERSE VIBRATIONS OF SIMPLY SUPPORTED RECTANGULAR PLATES WITH RECTANGULAR CUTOUTS CARRYING AN ELASTICALLY MOUNTED CONCENTRATED MASS

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1. INTRODUCTION

The problem of vibrating structural elements with elastically mounted masses is of interest in several fields of technology. If the structure is a rectangular plate or slab with an orifice which has been produced for operational reasons e.g., passage of ducts, conduits or electrical connections, an exact mathematical solution seems out of the question even in the case where the outer boundary of the plate is simply supported, in view of the impossibility of satisfying the natural boundary conditions at the free edge of the hole.

Consider the mechanical system shown in Figure 1 when it executes small amplitude, transverse vibrations. The problem is of basic interest in several fields of technology: from slabs supporting engines or motors to printed circuit boards with electronic elements attached to them. These *point masses* elastically attached to the structural element modify drastically, in general, the normal modes and natural frequencies of the structure, as shown in several studies [1–6].

The exact mathematical treatment of the problem becomes exceedingly complicated if a hole or orifice is present in the plate, the edges remaining free. An approximate yet quite comprehensive and accurate solution to the overall vibrational problem is obtained in the present paper following the approach developed in previous studies [7, 8] which essentially consistes of using co-ordinate functions which yield, in combination with variational methods, very accurate or even exact results in the case where the plate or slab is simply connected. When the slab is traversed by an orifice or hole one simply deducts the strain and kinetic energy corresponding to the non-existing portion of the structure.

2. APPROXIMATE ANALYTICAL SOLUTION

The Rayleigh-Ritz method requires minimization of the functional

$$J[W'] = U_p - T_p + U_m - T_m, \tag{1}$$

where (see Figure 1) U_p = maximum strain energy of the plate, T_p = maximum kinetic energy of the plate, U_m = maximum strain energy of the mass-spring system, and T_m = maximum strain energy of the point mass.

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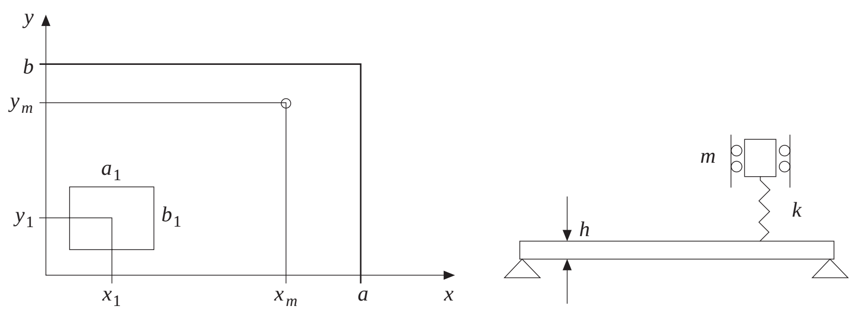

Figure 1. Vibrating system under consideration.

As has been shown elsewhere [6], each term in equation (1) can be written

$$U_p = \frac{D}{2}\int_{A_p}\left\{\left(\frac{\partial^2 W'}{\partial x'^2}+\frac{\partial^2 W'}{\partial y'^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 W'}{\partial x'^2}\frac{\partial^2 W'}{\partial y'^2}-\left(\frac{\partial^2 W'}{\partial x'\partial y'}\right)^2\right]\right\}\mathrm{d}x'\,\mathrm{d}y', \qquad (2a)$$

$$T_p = \frac{\rho h\omega^2}{2}\int_{A_p} W'^2\,\mathrm{d}x'\,\mathrm{d}y', \qquad U_m = \frac{k_m}{2}Z'^2, \qquad T_m = \frac{m\ \omega^2}{2}(Z' + W'_m)^2. \quad (2b,c,d)$$

In equations (1) and (2) above, W' = true displacement amplitude of the plate, Z' = mass displacement amplitude relative to the plate, and W'_m = plate displacement amplitude at the mass position.

By using

$$D = Eh^3/12(1-\mu^2) \qquad (3)$$

and

$$W = W'/a, \qquad x = x'/a, \qquad y = y'/b, \qquad Z = Z'/a, \qquad (4)$$

Table 1

Values of Ω_1 in the case of square plates with concentric square cutouts (Figure 2a). Comparison with finite elements results [8]

μ	a_1/a	Finite element	Variational
0	1/6	19·929	19·955
	1/3	21·657	21·680
0·3	0·1	19·463	19·517
	1/6	19·205	19·268
	0·2	19·147	19·205
	0·3	19·722	19·512
	1/3	19·772	19·819
	0·4	20·773	20·815
	0·5	23·473	23·519

TABLE 2

Values of Ω_1 in the case of rectangular plates with rectangular cutout in the middle of the rightmost border (Figure 2b). Comparison with finite elements results [8]

b/a	a_1/a	Finite element	Variational
1	0·1	19·72	19·72
1	0·2	19·52	19·54
1	0·3	19·13	19·14
2/3	0·1	32·05	32·06
2/3	1/6	31·80	31·82
2/3	0·2	31·40	31·37

equations (2) can be rendered non-dimensional. One gets

$$J[W] = \frac{Dr}{2}\int_{A_p}\left\{\left(\frac{\partial^2 W}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 W}{\partial y^2}\right)^2 - \frac{2(1-\mu)}{r^2}\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right]\right\} \mathrm{d}x\,\mathrm{d}y$$

$$- \Omega^2 \int_{A_p} W^2\,\mathrm{d}x\,\mathrm{d}y + \frac{K}{r}Z^2 - M\Omega^2(Z + W_m)^2, \tag{5}$$

where as usual, $\Omega^2 = \rho h \omega^2 a^4/D$ is the non-dimensional frequency coefficient; $M = m/M_p$, M_p being the mass of the plate without holes; and $r = b/a$.

Expressing the displacement amplitude $W(x, y)$ in terms of a double Fourier series:

$$W_a(x, y) = \sum_{n=1}^{N}\sum_{m=1}^{M} b_{mn} \sin(m\pi x)\sin(n\pi y), \tag{6}$$

and minimizing the governing functional with respect to the b_{nm}s and Z, expression (5) yields an $(M \times N + 1)$ homogeneous, linear system of equation in the b_{nm}s and Z. A secular determinant in the natural frequency coefficients of the system results from the non-triviality condition. The present study is concerned with the determination of the fundamental and first excited frequency coefficients, Ω_1 and Ω_2, in the case of plates with rectangular holes and carrying elastically mounted concentrated masses.

3. NUMERICAL RESULTS

All calculations have been performed for a simply supported rectangular plate of uniform thickness taking $\mu = 0{\cdot}30$; an exception made of a set of results presented in Table 1 where $\mu = 0$. Using the Fourier series approach a 401×401 secular determinant was

TABLE 3

Values of Ω_1 in the case of rectangular plates with rectangular cutout in the middle of the upper border (Figure 2c). Comparison with finite elements results [8]

b/a	a_1/a	Finite element	Variational
1/2	0·1	49·21	49·27
1/2	0·2	47·70	47·89
1/2	0·3	44·26	44·55
1/2	0·4	41·13	41·42
1/2	0·5	40·25	40·50

TABLE 4

Values of Ω_1 in the case of square plates with a square cutout at the rightmost border (Figure 3), when the spring-mass moves along a diagonal

a_1/a	Mass co-ordinates $(x/a, y/b)$	m/M_p	Ω_1
0·1	(0·25, 0·25)	0·1	18·69
		0·3	16·58
	(0·50, 0·50)	0·1	16·58
		0·3	13·04
	(0·75, 0·75)	0·1	18·68
		0·3	16·56
0·2	(0·25, 0·25)	0·1	18·54
		0·3	16·49
	(0·50, 0·50)	0·1	16·42
		0·3	12·91
	(0·75, 0·75)	0·1	18·44
		0·3	16·22
0·3	(0·25, 0·25)	0·1	18·18
		0·3	16·22
	(0·50, 0·50)	0·1	15·92
		0·3	12·41
	(0·75, 0·75)	0·1	17·85
		0·3	15·31

Note: the mass is rigidly attached to the plate ($k \to \infty$).

TABLE 5

Values of Ω_1 in the case of rectangular plates with a rectangular cutout at the rightmost border (Figure 3), when the spring-mass moves along diagonal

b/a	a_1/a	Mass co-ordinates $(x/a, y/b)$	m/M_p	Ω_1
2/3	0·1	(0·25, 0·25)	0·1	30·25
			0·3	26·10
		(0·50, 0·50)	0·1	26·74
			0·3	20·65
		(0·75, 0·75)	0·1	30·23
			0·3	26·06
2/3	0·3	(0·25, 0·25)	0·1	29·67
			0·3	25·76
		(0·50, 0·50)	0·1	25·97
			0·3	19·95
		(0·75, 0·75)	0·1	28·93
			0·3	23·71
2	0·1	(0·25, 0·25)	0·1	46·09
			0·3	37·77
		(0·50, 0·50)	0·1	40·57
			0·3	30·43
		(0·75, 0·75)	0·1	46·07
			0·3	37·71
2	0·3	(0·25, 0·25)	0·1	42·18
			0·3	35·79
		(0·50, 0·50)	0·1	35·79
			0·3	26·55
		(0·75, 0·75)	0·1	41·49
			0·3	34·19

Note: the mass is rigidly attached to the plate ($f_2 \to \infty$).

TABLE 6

Values of Ω_1 and Ω_2 in the case of square plates with different positions and values of mass-spring system when the cutout with $a_1/a = 0{\cdot}1$ moves along the diagonal (Figure 4).

			(a)		(b)		(c)	
Mass co-ordinates	m/M_p	Ka^2/D	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
(4A)								
$x/a = 0{\cdot}50$	0·1	1	3·149	19·58	3·149	19·69	3·149	19·62
$y/b = 0{\cdot}75$		10	9·539	20·19	9·554	20·27	9·553	20·20
		∞	17·63	41·48	17·75	41·31	17·70	41·24
	0·3	1	1·818	19·58	1·818	19·69	1·818	19·62
		10	5·539	20·08	5·546	20·17	5·546	20·09
		∞	14·72	35·30	14·77	35·11	14·74	35·02
(4B)								
$x/a = 0{\cdot}75$	0·1	1	3·153	19·56	3·153	19·66	3·153	19·59
$y/b = 0{\cdot}75$		10	9·680	19·86	9·685	19·96	9·685	19·89
		∞	18·50	40·00	18·60	39·86	18·54	39·76
	0·3	1	1·820	19·56	1·820	19·66	1·820	19·59
		10	5·606	19·80	5·609	19·90	5·609	19·83
		∞	16·41	31·68	16·41	31·60	16·38	31·50

posed for all the situations ($M = 20$, $N = 20$). Special care has been taken to manipulate such a large determinant using 80 bit floating point variables (IEEE–standard temporary reals) in order to obtain accurate results.

Table 1 illustrates the case of a square plate with a concentric square cutout. Tables 2 and 3 show fundamental frequency coefficients in the case of rectangular plates with rectangular cutouts of equal aspect ratio at the middle point of the rightmost and upper border, respectively (see Figure 2). The results are compared with the finite element determinations available in the open literature for $\mu = 0$ and $\mu = 0.30$. The agreement

TABLE 7

Values of Ω_1 and Ω_2 in the case of square plates with different positions and values of mass-spring system when the cutout with $a_1/a = 0{\cdot}3$ moves along the diagonal (Figure 4).

			(a)		(b)		(c)	
Mass co-ordinates	m/M_p	$Ka^2\ D$	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
(4*A*)								
$x/a = 0{\cdot}50$	0·1	1	3·144	19·58	3·148	18·97	3·149	18·50
$y/b = 0{\cdot}75$		10	9·378	20·42	9·523	19·58	9·525	19·08
		∞	16·92	37·56	17·11	41·26	16·83	39·90
	0·3	1	1·815	19·58	1·818	18·97	1·818	18·50
		10	5·456	20·27	5·534	19·46	5·536	18·96
		∞	13·44	33·11	14·28	35·02	14·18	33·61
(4B)								
$x/a = 0{\cdot}75$	0·1	1	3·151	19·54	3·152	18·94	3·153	18·47
$y/b = 0{\cdot}75$		10	9·612	19·94	9·674	19·24	9·674	18·76
		∞	18·15	37·31	17·95	40·21	17·60	38·45
	0·3	1	1·819	19·54	1·820	18·94	1·820	18·47
		10	5·572	19·87	5·605	19·18	5·605	18·70
		∞	15·62	30·93	15·94	31·42	15·76	30·29

TABLE 8

Values of Ω_1 and Ω_2 in the case of square plates with different positions and values of the mass-spring system when the cutout with $a_1/a = 0{\cdot}5$ moves along the diagonal (Figure 4).

Mass co-ordinates	m/M_p	Ka^2/D	(a) Ω_1	(a) Ω_2	(c) Ω_1	(c) Ω_2
(4A)						
$x/a = 0{\cdot}50$	0·1	1	3·125	23·65	3·139	17·85
$y/b = 0{\cdot}75$		10	8·912	24·85	9·424	18·67
		∞	16·54	31·81	15·47	38·75
	0·3	1	1·804	23·65	1·816	17·86
		10	5·179	24·72	5·118	18·50
		∞	11·02	30·24	13·21	35·28
(4B)						
$x/a = 0{\cdot}75$	0·1	1	3·142	23·58	3·152	17·83
$y/b = 0{\cdot}75$		10	9·381	24·25	9·636	18·22
		∞	19·48	33·09	12·71	42·06
	0·3	1	1·814	23·58	1·820	17·80
		10	5·439	24·17	5·590	18·14
		∞	14·29	30·41	16·13	32·11

between the analytical approach and the finite element results is excellent for all the situations considered (the maximum differences are of the order of 0·2%, an exception is Table 3 where the differences climb to almost 0·6%). Table 4 depicts fundamental

TABLE 9

Values of Ω_1 and Ω_2 in the case of rectangular plates with $b/a = 2/3$, when the cutout, with equal aspect ratio and $a_1/a = 0{\cdot}3$, moves along the diagonal (Figure 4)

Mass co-ordinates	m/M_p	Ka^2/D	(a) Ω_1	(a) Ω_2	(c) Ω_1	(c) Ω_2
(4A)						
$x/a = 0{\cdot}50$	0·1	1	3·154	30·99	3·157	30·05
$y/b = 0{\cdot}75$		10	9·751	31·47	9·839	30·35
		∞	26·54	62·28	27·30	58·59
	0·3	1	1·821	30·99	1·822	30·05
		10	5·636	31·43	5·685	30·33
		∞	20·57	58·90	22·83	55·70
(4B)						
$x/a = 0{\cdot}75$	0·1	1	3·157	30·96	3·158	30·03
$y/b = 0{\cdot}75$		10	9·851	31·18	9·877	30·17
		∞	28·68	52·30	28·62	51·76
	0·3	1	1·823	30·96	1·823	30·03
		10	5·691	31·16	5·705	30·16
		∞	24·16	44·84	25·23	42·68
(4C)						
$x/a = 0{\cdot}75$	0·1	1	3·154	30·99	3·156	30·04
$y/b = 0{\cdot}50$		10	9·757	31·43	9·815	30·32
		∞	26·68	50·40	27·33	49·52
	0·3	1	1·821	30·99	1·822	30·04
		10	5·640	31·40	5·671	30·30
		∞	20·73	45·54	22·45	42·90

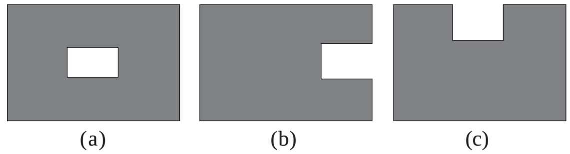

Figure 2. Vibrating simply supported plates with cutouts of the same aspect ratio.

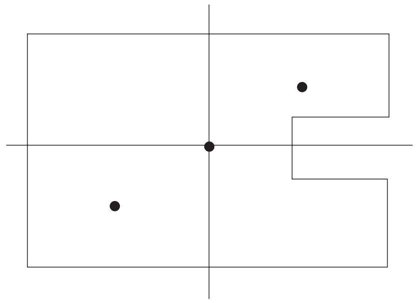

Figure 3. Vibrating simply supported rectangular plate with free, rectangular cutout of the same aspect ratio at the rightmost border when the mass-spring system moves along one of the diagonals.

frequency coefficients in the case of square plates with square cutouts at the middle point of the right border as the same mass, rigidly attached to the plate, is displaced along a diagonal, see Figure 3.

Table 5 shows values of Ω_1 in the case of rectangular plates, with cutouts of the same aspect ratio, located at the middle of the right side, as the mass is displaced along one of the diagonals. Tables 6–8 depict values of the fundamental and first excited frequency coefficients, Ω_1 and Ω_2, for the case of square plates as the center of the cut-out is displaced along a diagonal, Figure 4, and for different values of mass, spring constant and mass-spring co-ordinates. Table 9 shows values of Ω_1 and Ω_2 in the case of rectangular plates, with b/a = 2/3, when an equal aspect ratio cutout, with $a_1/a = 0{\cdot}3$, takes the following two positions along the diagonal: $x = a_1/2$ and $y = b_1/2$ and $x = a/2$ and $y = b/2$; again for different values of mass, spring constant and mass-spring system co-ordinates.

As a general conclusion one may say that the fact that the use of a double Fourier series yields results (presumably very accurate) as a large size determinantal equation is quite interesting from an academic viewpoint in view of the fact that, individually, each co-ordinate function does not satisfy the boundary conditions at the edge of the hole. However, as the size of the determinant approaches infinity, the natural boundary conditions at the hole edges tend to be satisfied [9]. The mathematical model is quite realistic, within the realm of the classical theory of vibrating plates. The same approach is valid in the case of orthotropic, simply supported rectangular plates. For other types

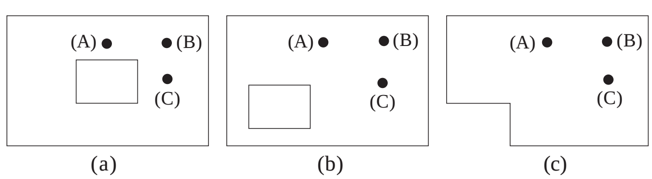

Figure 4. Mechanical system under analysis when the cutout is displaced along the diagonal. Positions of the cut out center: (a) $x_1 = a/2$, $y_1 = b/2$; (b) $x_1 = a/4$, $y_1 = b/4$; (c) $x_1 = a_1/2$, $y_1 = b_1/2$.

of boundary conditions one would express the displacement in terms of co-ordinate functions which must satisfy, at least, the essential boundary conditions and then apply the same general procedure.

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REFERENCES

1. P. A. A. LAURA, E. A. SUSEMIHL, J. L. POMBO, L. E. LUISONI and R. GELOS 1977 *Applied Acoustics* **10**, 121–145. On the dynamic behavior of structural elements carrying elastically mounted concentrated masses.
2. R. K. SINGAL and D. J. GORMAN 1992 *American Institute of Aeronautics and Astronautics Journal* **30**, 853–855. Effects of attached masses on free vibration of rigid point supported rectangular plates.
3. K. H. LOW 1993 *Journal of Sound and Vibration* **160**, 111–121. Analytical and experimental investigation on a vibrating rectangular plate with mounted weights.
4. K. H. LOW, G. B. CHAI and C. K. NG 1993 *ASME Journal of Vibration and Acoustics* **115**, 391–396. Experimental and analytical study of the frequencies of a S-C-S-C plate carrying a concentrated mass.
5. D. R. AVALOS, H. A. LARRONDO and P. A. A. LAURA 1993. *Ocean Engineering* **20**, 195–205. Vibrations of a simply supported plate carrying an elastically mounted concentrated mass.
6. D. R. AVALOS, H. A. LARRONDO and P. A. A. LAURA 1994. *Journal of Sound and Vibration* **177**, 251–258. Transverse vibrations of a circular plate carrying an elastically mounted mass.
7. P. A. A. LAURA, R. H. GUTIERREZ, L. ERCOLI, J. C. UTJES and R. CARNICER 1987 *Ocean Engineering* **14**, 285–293. Free Vibrations of rectangular plates elastically restrained against rotation with circular or square free openings.
8. P. A. A. LAURA, E. ROMANELLI and R. E. ROSSI 1997 *Journal of Sound and Vibration* **202,** 275–283. Transverse vibrations of simply supported rectangular plates with rectangular cut-outs.
9. G. S. ELSBERND and A. W. LEISSA 1970 *Developments in Theoretical and Applied Mechanics* 19–28. Free vibration of a rectangular plate clamped on three edges and free on a fourth edge.