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LONGITUDINAL VIBRATIONS OF RODS COUPLED BY A DOUBLE SPRING-MASS SYSTEM

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1. INTRODUCTION

Recently, an interesting study by Kukla *et al*. [1] was published on the problem of the natural longitudinal vibrations of two rods coupled by many translational springs where the Green function method was employed. Motivated by this publication, the present note deals with a similar mechanical system. The system here is made up of two clamped-free axially vibrating rods carrying tip masses to which a double spring-mass system is attached as a secondary system across the span. Due to employment of a boundary value problem formulation which, in our opinion, allows much more insight into the physical aspects, only one secondary system is considered here.

2. THEORY

The problem to be investigated in the present note is the natural vibration problem of the system shown in Figure 1. It consists of two clamped-free axially vibrating rods carrying tip masses as the primary system (ps) to which a double spring-mass secondary system (ss) is attached across the span. The physical properties of the ps are as follows: The length, mass per unit length, location of the spring attachment point, axial rigidity and tip mass of the ith rod are L_i, m_i, $\eta_i L_i$, $E_i A_i$ and M_i, respectively ($i = 1, 2$). The secondary system consists of two springs of stiffness k_1, k_2 and the mass M. Let one denote the longitudinal vibration displacements of the first and second rods to the left and right of the spring attachment points as $u_{11}(x, t)$, $u_{12}(x, t)$ and $u_{21}(x, t)$, $u_{22}(x, t)$ respectively as depicted in Figure 1. $z(t)$ represents the displacement of the mass M.

The equations of longitudinal motion of the four rod portions are governed by the following partial differential equations [2].

$$E_i A_i \, \partial^2 u_{ij}(x, t)/\partial x^2 = m_i \, \partial^2 u_{ij}(x, t)/\partial t^2 \quad (i, j = 1, 2). \tag{1}$$

The corresponding boundary and continuity conditions at the spring attachment points are as follows

$$u_{11}(0, t) = 0, \qquad u_{11}(\eta_1 L_1, t) = u_{12}(\eta_1 L_1, t)$$

$$E_1 A_1 u'_{12}(\eta_1 L_1, t) - E_1 A_1 u'_{11}(\eta_1 L_1, t) + k_1[z(t) - u'_{11}(\eta_1 L_1, t)] = 0,$$

$$E_1 A_1 u'_{12}(L_1, t) + M_1 \ddot{u}_{12}(L_1, t) = 0,$$

$$M\ddot{z} = -k_1[z(t) - u_{11}(\eta_1 L_1, t)] + k_2[u_{21}(\eta_2 L_2, t) - z(t)],$$

$$u_{21}(0, t) = 0, \qquad u_{21}(\eta_2 L_2, t) = u_{22}(\eta_2 L_2, t),$$

$$E_2 A_2 u'_{22}(\eta_2 L_2, t) - E_2 A_2 u'_{21}(\eta_2 L_2, t) + k_2[z(t) - u_{21}(\eta_2 L_2, t)] = 0,$$

$$E_2 A_2 u'_{22}(L_2, t) + M_2 \ddot{u}_{22}(L_2, t) = 0. \tag{2}$$

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Here dots and primes denote partial derivatives with respect to time t and position co-ordinate x, respectively. Using the standard method of separation of variables one assumes,

$$u_{ij}(x, t) = U_{ij}(x) \cos \omega t \quad (i, j = 1, 2) \tag{3}$$

where $U_{ij}(x, t)$ are the corresponding amplitude functions of the rods and ω is the unknown eigenfrequency of the combined system. Substituting these into equations (1) results in the following ordinary differential equations.

$$U''_{1j}(x) + \beta^2 U_{1j}(x) = 0, \qquad U''_{2j}(x) + \mu^2\beta^2 U_{2j}(x) = 0, \qquad (j = 1, 2). \tag{4}$$

Here, the following abbreviations are introduced

$$\beta^2 = m_1\omega^2/E_1A_1, \qquad \mu^2 = \alpha_m/\chi, \qquad \alpha_m = m_2/m_1, \qquad \chi = E_2A_2/E_1A_1. \tag{5}$$

Assuming

$$z(t) = Z \cos \omega t \tag{6}$$

and substituting (3) and (6) in (2) yield the corresponding boundary and matching conditions for amplitude functions U_{ij} and Z:

$$U_{11}(0) = 0, \qquad U_{11}(\eta_1 L_1) = U_{12}(\eta_1 L_1),$$

$$E_1A_1U'_{12}(\eta_1 L_1) - E_1A_1U'_{11}(\eta_1 L_1) + k_1[Z - U'_{11}(\eta_1 L_1)] = 0,$$

$$E_1A_1U'_{12}(L_1) - M_1\omega^2 U_{12}(L_1) = 0,$$

$$M\omega^2 Z - k_1[Z - U_{11}(\eta_1 L_1)] + k_2[U_{21}(\eta_2 L_2) - Z] = 0,$$

$$U_{21}(0) = 0, \qquad U_{21}(\eta_2 L_2) = U_{22}(\eta_2 L_2),$$

$$E_2A_2U'_{22}(\eta_2 L_2) - E_2A_2U'_{21}(\eta_2\, L_2, t) + k_2[Z - U_{21}(\eta_2 L_2)] = 0,$$

$$E_2A_2U'_{22}(L_2) - M_2\omega^2 U_{22}(L_2) = 0. \tag{7}$$

The general solutions of the ordinary differential equations (4) are simply

$$U_{1j}(x) = C_{1j} \sin \beta x + C_{2j} \cos \beta x, \qquad U_{2j}(x) = C_{3j} \sin \mu\beta x + C_{4j} \cos \mu\beta x, \quad (j = 1, 2) \tag{8}$$

where C_{1j}–C_{4j} are eight integration constants to be evaluated via conditions (7). The application of these boundary and matching conditions to the solutions (8) and the amplitude Z yields a set of nine homogeneous equations for the nine unknown constants C_{1j}–C_{4j} $(j = 1, 2)$ and Z. A non-trivial solution of this set of equations is possible only if the characteristic determinant of the coefficients vanishes. Taking into account that C_{21} and C_{41} vanish, the characteristic equation reduces to the following form

$$\begin{vmatrix} \sin \eta_1\overline{\beta} & -\sin \eta_1\overline{\beta} & -\cos \eta_1\overline{\beta} & 0 & 0 & 0 & 0 \\ \overline{\beta} \cos \eta_1\overline{\beta} + \alpha_{k1} \sin \eta_1\overline{\beta} & -\overline{\beta} \cos \eta_1\overline{\beta} & \overline{\beta} \sin \eta_1\overline{\beta} & 0 & 0 & 0 & -\alpha_{k1} \\ 0 & (\cos \overline{\beta} - \alpha_{M1}\overline{\beta} \sin \overline{\beta}) & -(\sin \overline{\beta} + \alpha_{M1}\overline{\beta} \cos \overline{\beta}) & 0 & 0 & 0 & 0 \\ \alpha_{k1} \sin \eta_1\overline{\beta} & 0 & 0 & \alpha_k \alpha_{k1} \sin \psi\overline{\beta} & 0 & 0 & \alpha_M\overline{\beta}^2 - \alpha_{k1}(1 + \alpha_k) \\ 0 & 0 & 0 & \sin \psi\overline{\beta} & -\sin \psi\overline{\beta} & -\cos \psi\overline{\beta} & 0 \\ 0 & 0 & 0 & \delta\overline{\beta} \cos \psi\overline{\beta} + \alpha_{k2} \sin \psi\overline{\beta} & -\delta\overline{\beta} \cos \psi\overline{\beta} & \delta\overline{\beta} \sin \psi\overline{\beta} & -\alpha_{k2} \\ 0 & 0 & 0 & 0 & \cos \delta\overline{\beta} - \dfrac{\alpha_{M2}}{\mu\chi}\overline{\beta} \sin \delta\overline{\beta} & -\left(\sin \delta\overline{\beta} + \dfrac{\alpha_{M2}}{\mu\chi}\overline{\beta} \cos \delta\overline{\beta}\right) & 0 \end{vmatrix} = 0 \tag{9}$$

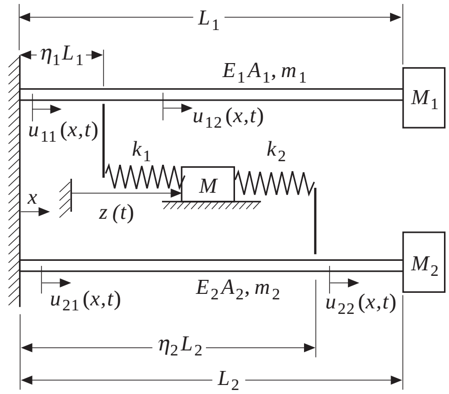

Figure 1. Two clamped-free rods with tip masses to which a double spring-mass system is attached in span.

Here, in addition to those given in (5), the following definitions are introduced:

$$\overline{\beta} = \beta L_1, \qquad \alpha_M = M/m_1L_1, \qquad \alpha_{M1} = M_1/m_1L_1, \qquad \alpha_{M2} = M_2/m_2L_2$$
$$\alpha_k = k_2/k_1, \qquad \alpha_{k1} = k_1/(E_1A_1/L_1), \qquad \alpha_{k2} = k_2/(E_2A_2/L_2),$$
$$\delta = \mu L_2/L_1, \qquad \psi = \delta\eta_2. \tag{10}$$

It is worth noting that $\overline{\beta}$ represents the dimensionless frequency parameter of the combined system.

3. EXEMPLARY NUMERICAL RESULTS

The complicated frequency equation (9) was solved by using MATHEMATICA version 2.0 for MS-DOS on a PC 386. Because there are many system parameters which can be varied, it is meaningless to tabulate the $\overline{\beta}$ values so obtained for various combinations of the many parameters. Instead, results for three examples are given in graphical form.

The results for the eigenfrequency parameters obtained with the present double spring-mass system are compared with the results of Kukla's study. For $\alpha_M = \alpha_{M1} = \alpha_{M2} = 0$, $\eta_1 = 0{\cdot}5$, $\eta_2 = 0{\cdot}6$, $\delta = 0{\cdot}66$, $L_1 = L_2 = 1m$, $\alpha_{k1} = 5$, $\alpha_k = \infty$, frequency parameters of the present system are $\overline{\beta}_1 = 1{\cdot}32684$, $\overline{\beta}_2 = 2{\cdot}322563$, $\overline{\beta}_3 = 3{\cdot}197436$, $\overline{\beta}_4 = 4{\cdot}838520$. Comparison of the results shows a very close agreement.

In all numerical evaluations made below, the values of some of the physical parameters were chosen as follows: $\alpha_L = L_2/L_1 = 1$, $\alpha_m = 1$, $\alpha_{M1} = \alpha_{M1} = 2$, $\alpha_{k1} = 1$, $\chi = 1$. These yield $\mu = \delta = 1$.

The first example aims to explore the effect of the variation of the location of the spring attachment point to the first rod on the natural frequencies of the combined system where spring stiffness ratio $\alpha_k = k_2/k_1$ is taken as a parameter. Tip mass ratio and spring attachment point on the second rod are chosen as $\alpha_M = M/m_1L_1 = 1$ and $\eta_2 = 0{\cdot}5$ respectively. The results obtained are shown in Figure 2. Common to all curves in Figure 2 (except (d)) is that for a fixed value of η_1, an increase of the parameter α_k gives rise to an increase of the eigenfrequencies of the combined system shown in Figure 1. As η_1 gets larger, i.e., the spring attachment point on the first rod approaches the tip of the rod, the fundamental frequency parameter diminishes slightly, whereas the second one increases. The other eigenfrequencies remain practically unaffected although the fourth eigenfrequency reveals a symmetric variation in a narrow band.

The second example investigates the effect of the variation of the spring attachment point to the second rod where the stiffness ratio $\alpha_k = k_2/k_1$ is taken as a parameter again, $\alpha_M = M/m_1 L_1 = 1$ and $\eta_1 = 0{\cdot}5$ are chosen. An inspection of the curves in Figure 3 indicates that in general, for a fixed value of η_2, an increase of α_k gives rise to an increase of the eigenfrequencies. However, the first and the fifth eigenfrequency curves reveal, in an interesting manner, some intersections after which the opposite behaviour is observed. In order to avoid misunderstanding during the comparison of Figures 2 and 3, it is worth noting that once α_k and one of α_{k1} and α_{k2} are chosen, the value of the other parameter is fixed according to $\alpha_{k2} = (\alpha_k \alpha_L/\chi)\alpha_{k1}$.

The next example deals with the effect of the variation of the mass M of the secondary system on the eigenfrequencies of the combined system. $\alpha_k = 1$ and $\eta_1 = \eta_2 = 0{\cdot}5$ are chosen. The mass parameter $\alpha_M = M/m_1 L$ is varied in the range 0–20. The results are

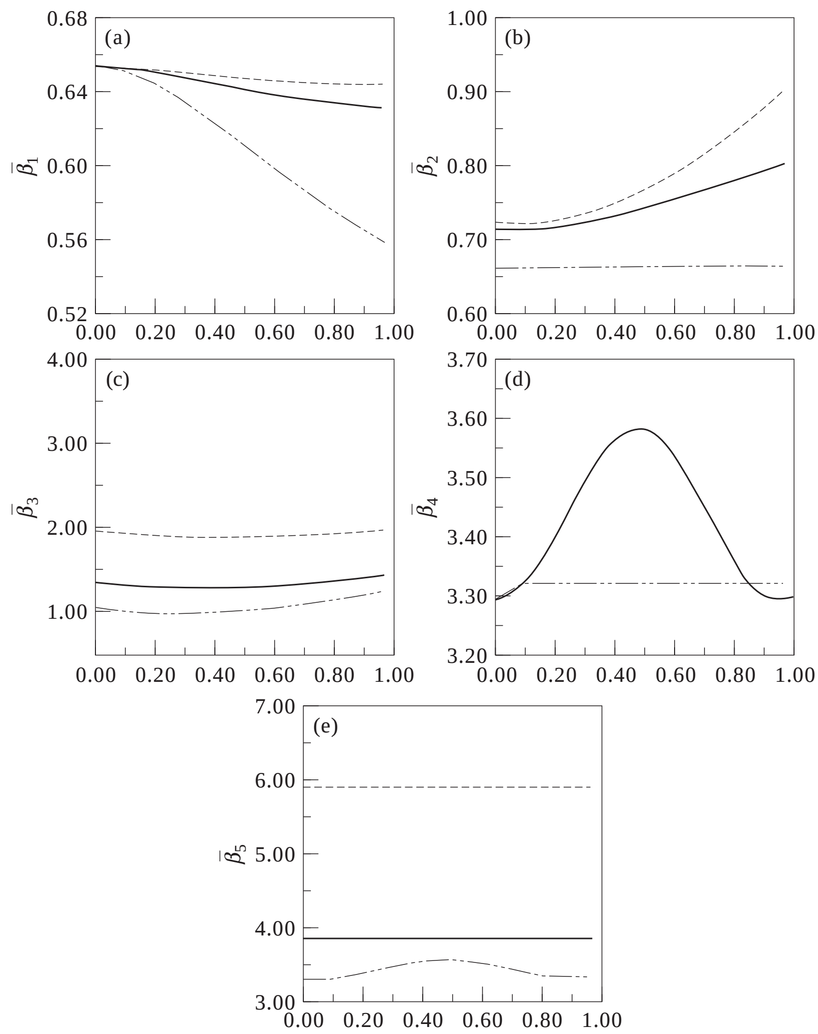

Figure 2. The first five dimensionless frequency parameters of the system shown in Figure 1 as a function of η_1. ($\alpha_{M1} = \alpha_{M2} = 2$, $\alpha_M = 1$, $\alpha_{k1} = 1$, $\eta_2 = 0{\cdot}5$: — · · —, $\alpha_k = 0{\cdot}1$; ——, $\alpha_k = 1$; – – –, $\alpha_k = 10$). (a) $\overline{\beta}_1$; (b) $\overline{\beta}_2$; (c) $\overline{\beta}_3$; (d) $\overline{\beta}_4$; (e) $\overline{\beta}_5$.

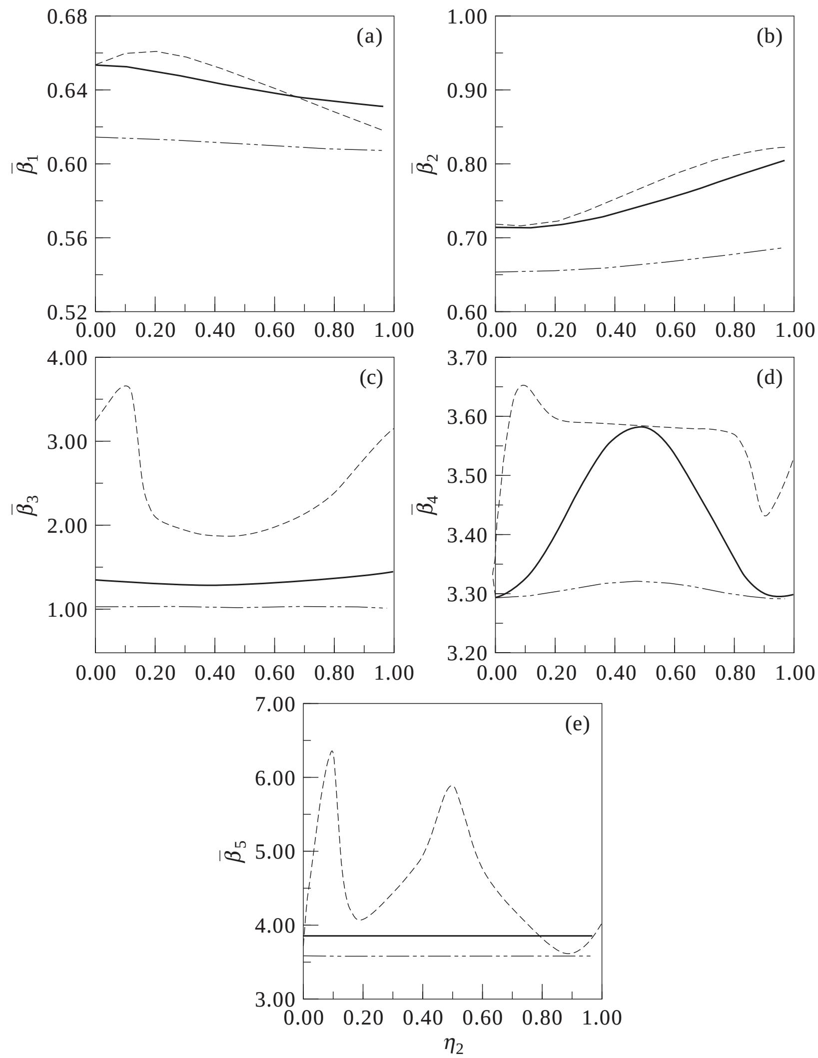

Figure 3. The first five dimensionless frequency parameters of the system shown in Figure 1 as a function of η_2 ($\alpha_{M1} = \alpha_{M2} = 2$, $\alpha_M = 1$, $\alpha_{k1} = 1$, $\eta_1 = 0\cdot5$: —··—, $\alpha_k = 0\cdot1$; ——; $\alpha_k = 1$; – – –, $\alpha_k = 10$). (a) $\bar{\beta}_1$; (b) $\bar{\beta}_2$; (c) $\bar{\beta}_3$; (d) $\bar{\beta}_4$; (e) $\bar{\beta}_5$.

shown in Figure 4. The fundamental eigenfrequency diminishes continuously as α_M gets larger whereas the second eigenfrequency is practically unaffected. After a rapid decrease in the beginning, the third and fifth eigenfrequencies remain practically constant over the whole range considered. The same tendency is observed also for the fourth eigenfrequency where the decrease in the beginning is not so pronounced.

The last example deals with the effect of the variation of the tip masses on the eigenfrequencies of the combined system, where $\alpha_{M1} = \alpha_{M2}$ is taken for the sake of simplicity, $\alpha_k = \alpha_{k1} = \alpha_M = 1$ and $\eta_1 = \eta_2 = 0\cdot5$ are chosen. The results are shown in Figure 5. The curves indicate clearly that the eigenfrequencies of the system decrease as the tip masses get larger.

Up to now, the effect of the variation of the stiffness and mass parameters of the secondary system on the eigenfrequencies of the combined system have been discussed.

This determines how the existence of the secondary system affects the frequency spectrum of the system. It is in order to report briefly one's observation on this matter. Consider the special case where the primary system consists of two entirely equal rods with equal tip masses M^* in which case the frequency equation is given by [3]

$$\alpha_M^* \bar{\beta}^* \sin \bar{\beta}^* - \cos \bar{\beta}^* = 0. \tag{11}$$

If one takes as an example $\alpha_M^* = M^*/m_1 L_1 = 2$, the first four eigenfrequency parameters are determined as $\bar{\beta}_1^* = 0{\cdot}65327$, $\bar{\beta}_2^* = 3{\cdot}29231$, $\bar{\beta}_3^* = 6{\cdot}36162$, $\bar{\beta}_4^* = 9{\cdot}47746$ where the asterisks denote the primary system parameters.

For the choice $E_1 = E_2 = E = 2{\cdot}1 \times 10^{11}$ N/m^2, $A_1 = A_2 = A = 4 \times 10^{-4}$ m^2, $L_1 = L_2 = L = 1$m, $m_1 = m_2 = m = 3{\cdot}12$ kg/m^2 the eigenfrequency of the secondary system

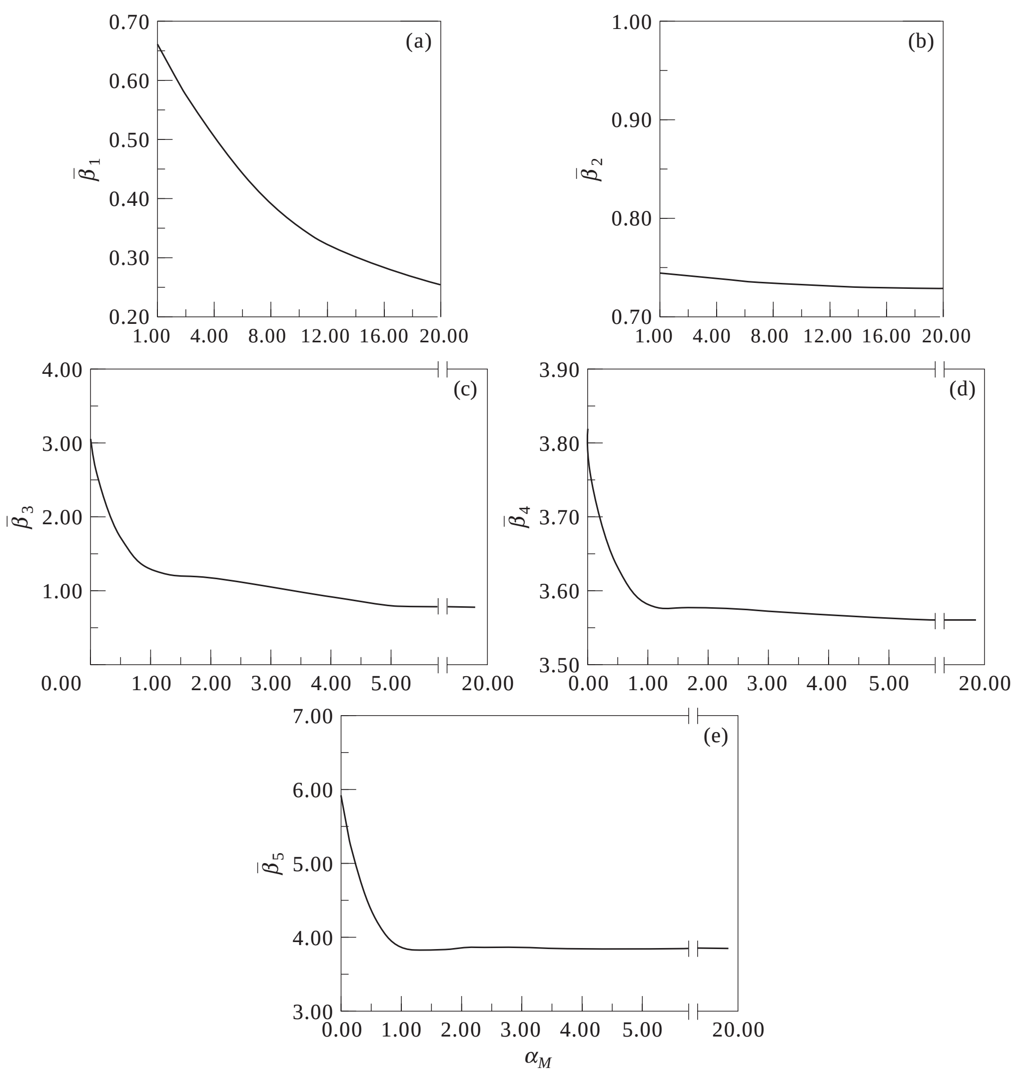

Figure 4. Effect of the variation of the mass M of the secondary system on the dimensionless frequency parameters of the system shown in Figure 1 ($\alpha_{M1} = \alpha_{M2} = 2$, $\alpha_k = \alpha_{k1} = 1$, $\eta_1 = \eta_2 = 0{\cdot}5$): (a) $\bar{\beta}_1$; (b) $\bar{\beta}_2$; (c) $\bar{\beta}_3$; (s) $\bar{\beta}_4$; (e) $\bar{\beta}_5$.

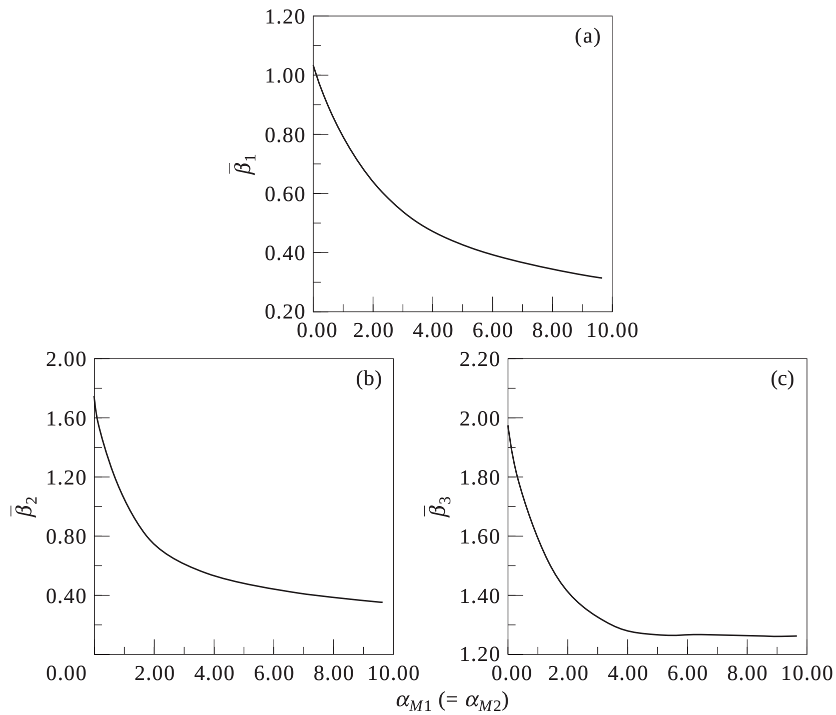

Figure 5. Effect of the variation of the masses M_1 and M_2 on the dimensionless frequency parameters of the system shown in Figure 1. ($\alpha_M = \alpha_k = \alpha_{k1} = 1$, $\eta_1 = \eta_2 = 0{\cdot}5$): (a) $\bar{\beta}_1$; (b) $\bar{\beta}_2$; (c) $\bar{\beta}_3$.

($\omega_s = \sqrt{(k_1 + k_2)/m}$) is between the first and second eigenfrequency of the primary system (eigenfrequencies of the primary system ($\omega_i = \bar{\beta}_i\sqrt{EA/mL^2}$).). Upon the attachment of the secondary system, the first eigenfrequency parameter of the combined system is lower than that of the primary system and there are two eigenfrequency parameters between the first and second; second and third eigenfrequency parameters of the primary system.

4. CONCLUSIONS

The subject of this note is the longitudinal vibration problem of a system consisting of two clamped-free rods carrying tip masses, coupled by a double spring-mass system attached to them in span. After formulating the frequency equation of the system, the effects of the variation of some system parameters upon the natural frequencies were investigated through numerical examples.

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