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DISTRIBUTED STRUCTURAL ACTUATION WITH ELECTROSTRICTORS

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In this paper a non-linear framework is developed of the dynamic behavior of a system with distributed electrostrictive coupling. A framework for the derivation of the governing equations is presented which is sufficiently general to model the dynamics of a broad range of non-linear systems. The constitutive relations for electrostrictors are developed in terms of the standard quadratic function and the hyperbolic tangent squared function of the applied electric field. Non-linear coupled equations of motion are derived by introducing the constitutive relations into Hamilton's Principle. Approximate system equations are derived using the assumed mode method. Experimental validation of the equations is accomplished by examining the static and dynamic response of a cantilevered beam actuated with a distributed electrostrictive actuator.

1. INTRODUCTION

Electrostriction is a fundamental phenomenon characterizing the response of any dielectric material subject to an applied electric field. An electric field produces expansion in the direction of the field and a contraction in the transverse directions. However, these strains are significant only in materials which feature a very large dielectric permittivity. One such example is the relaxor ferroelectric PMN–PT ($0{\cdot}9(Pb[Mg_{1/3}Nb_{2/3}]O_3$–$0{\cdot}1(PbTiO_3)$), which is used in this study.

Applications involving electrostrictors primarily use the material in stacks and rely upon the longitudinal properties of the material as opposed to the smaller transverse properties [1–7]. A large actuation potential, high set-point accuracy and low hysteresis make electrostrictors a choice material for quasi-static micro-positioning devices. Despite these advantages, few applications have implemented electrostrictors as structural actuators due to the non-linearity and temperature sensitivity of electrostrictors [8–11]. Although some control approaches have utilized electrostrictors in these non-linear regimes [12, 13], piezoceramics continue to dominate structural control applications.

In this paper an attempt is made to lower the barriers to the application of electrostrictors by creating an analysis framework which can express the approximate governing equations of a non-linear electroceramic system with distributed electrostrictive coupling. This general analysis is specifically applied to distributed electrostrictors in structural actuation and sensing applications. The paper is divided into four large sections. In section 2, variational principles and Hamilton's equation are described for coupled non-linear systems. A proof is presented which allows the simplification of Hamilton's equation when the constitutive equations can be expressed in terms of a thermodynamic formalism. In the next sections 3 and 4, the quadratic equations of motion for an electrostrictive system are derived, following the general form of Hamilton's equation.

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In section 5, hyperbolic tangent squared constitutive relationships are used as the basis for the general equations of motion. The thermodynamic formalism allows a simplification of the derivation of the equations of motion. Section 6 features an experimental investigation with an unconstrained electrostrictor and with an electrostrictor on a cantilevered beam, proving the validity and the limits of the equations of motion.

2. VARIATIONAL PRINCIPLE FOR NON-LINEAR SYSTEMS

Variation principles are a convenient way of finding the equations of motion of a system. The work done upon a system must take the form of kinetic energy, potential energy or some other form of internal energy. The governing equations of a system arise by declaring that the variations of the energies integrated along a path must be zero. In this section the use of variational principles to model the dynamics of non-linear systems is described. First, the application of Hamilton's Principle to non-linear systems is reviewed. Particular attention is given to describe the energy terms when the state variables are coupled. Then, an electric enthalpy function is introduced in order to simplify the equations.

2.1. HAMILTON'S PRINCIPLE FOR COUPLED NON-LINEAR SYSTEMS

Hamilton's Principle is a common variational principle in dynamics. The generalized form of Hamilton's Principle for a coupled electromechanical system with a quasi-static electric field is

$$\int_{t_1}^{t_2} [\delta T^* - \delta U + \delta W_e^* - \delta W_m + \delta W] \, \mathrm{d}t = 0, \tag{1}$$

where T^* is the complimentary kinetic energy, U is the potential energy, W_e^* is the complimentary electrical energy, W_m is the magnetic energy, and W represents the externally applied work.

For linear systems, the energy terms can be stated explicitly in terms of the state variables. With non-linear materials, the energy terms must be integrated as the system changes. From Crandall [14], the integral forms of the energy terms are

$$T^* = \int_V \int_0^{\tilde{\dot{u}}} \rho \dot{u}_j \, \mathrm{d}\dot{u}_j \, \mathrm{d}V, \qquad U = \int_V \int_0^{\tilde{S}} T_{ij} \, \mathrm{d}S_{ij} \, \mathrm{d}V,$$

$$W_e^* = \int_V \int_0^{\tilde{E}} D_m \, \mathrm{d}E_m \, \mathrm{d}V, \qquad W_m = \int_V \int_0^{\tilde{B}} H_m \, \mathrm{d}B_m \, \mathrm{d}V,$$

$$\delta W = \int_A f_i \delta u_i \, \mathrm{d}A - \sum_j \delta \varphi_j q_j, \tag{2}$$

where u represents the mechanical displacements, φ is the electric potential, f is the surface force and q is the externally supplied charge. S is the strain, T is the stress, E is the electric field, D is the electrical displacement, B is the magnetic flux density and H is the magnetic field. The superscript tilde ($\sim$) indicates the final state.

For active materials, additional constraints are imposed upon equation (2) in the form of constitutive relationships. In these cases, it is easier to think of the energy terms as depending on a single dummy variable, ξ, which reflects the state of the system along the arbitrarily varying path. As a result, the single state variable reflects an instant in the

progression of the system. The variation in energy still needs to be examined as the system changes. Substituting the dummy variable into the energy expressions changes the integration with respect to the state variables, S, E and B, into an integration with respect to ξ. By the chain rule, equations (2) become

$$T^* = \int_V \int_0^{\tilde{\xi}} \mathrm{d}\dot{u}_j \rho \dot{u}_j \, \mathrm{d}V, \qquad U = \int_V \int_0^{\tilde{\xi}} S'_{ij}(\xi) T_{ij}(S(\xi), E(\xi)) \, \mathrm{d}\xi \, \mathrm{d}V, \tag{3, 4}$$

$$W_e^* = \int_V \int_0^{\tilde{\xi}} E'_m(\xi) D_m(S(\xi), E(\xi)) \, \mathrm{d}\xi \, \mathrm{d}V, \qquad W_m = \int_V \int_0^{\tilde{\xi}} B'_m(S(\xi), H(\xi)) H_m(\xi) \, \mathrm{d}\xi \, \mathrm{d}V. \tag{5, 6}$$

The prime indicates a derivative with respect to ξ.

For electrically actuated active materials, such as electrostrictors, the magnetic energy terms are negligible. Substituting the potential energy and the electrical energy terms from equations (4) and (5) into Hamilton's Principle, (1), yields

$$\int_{t_1}^{t_2} \left[\delta T^* - \delta \int_V \int_0^{\xi} \frac{\partial S_{kl}}{\partial \xi} T_{kl} \, \mathrm{d}\xi \, \mathrm{d}V + \delta \int_V \int_0^{\xi} \frac{\partial E_n}{\partial \xi} D_n \, \mathrm{d}V + \delta W \right] \mathrm{d}t = 0. \tag{7}$$

The variation of each energy term needs to be examined with respect to each of the state variables. As a result, the variation is defined as

$$\delta \int_V \int_0^{\xi} (\,\cdot\,) \, \mathrm{d}\xi \, \mathrm{d}V = \int_V \int_0^{\xi} \left\{ \frac{\partial(\,\cdot\,)}{\partial E} \delta E + \frac{\partial(\,\cdot\,)}{\partial S} \delta S + \frac{\partial(\,\cdot\,)}{\partial E'} \delta E' + \frac{\partial(\,\cdot\,)}{\partial S'} \delta S' \right\} \mathrm{d}\xi \, \mathrm{d}V. \tag{8}$$

The variation is indicated by δ. Hamilton's equation states that the variation of state variables must integrate to zero. No claims are made about the variations of a derivative of a state variable, such as $\delta E'$ or $\delta S'$. As a result, the variations with respect to a derivative of a state variable, $\delta E'$ or $\delta S'$, will need to be transformed into a variation of the state variable, δE or δS, through an integration by parts with respect to ξ.

The form of Hamilton's Principle given in equations (3)–(7) places minimal restrictions on the system. There are no constraints on the constitutive relationships; the relationships need not even be smooth. Similarly, there are no restrictions about the nature of the external forcing. The only requirement is that the strain and the electric field be non-dissipative, continuous functions of ξ. Because of the generality of the framework, Hamilton's Principle will be utilized later in this section in order to derive the variational principle for an electrostrictively coupled system using two types of constitutive equations: a quadratic relationship and a hyperbolic squared relationship.

2.2. THERMODYNAMIC SIMPLIFICATION OF HAMILTON'S PRINCIPLE

The derivation of the system dynamics can be substantially simplified when the constitutive relationships are derived from a thermodynamic relationship. Thermodynamic formalisms are an expression of the First Law of Thermodynamics and instill a compatibility between the electrical and mechanical equations. There are eight ways of expressing the thermodynamic potentials, and there are numerous texts that elaborate on other expressions [13, 15]. For simplicity, let us examine the behavior of a system that can be described in terms of an electric enthalpy function, $\mathscr{H}_2$. The constitutive relationships can be defined in terms of $\mathscr{H}_2$ as

$$T_{ij} = \partial \mathscr{H}_2/\partial S_{ij} \qquad \text{and} \qquad D_m = \partial \mathscr{H}_2/\partial E_m. \tag{9}$$

Introducing the definition of electric enthalpy into Hamilton's Principle (7), yields

$$\int_{t_1}^{t_2} \left[\delta T^* - \delta \int_V \int_0^{\xi} \frac{\partial S_{kl}}{\partial \xi} \frac{\partial \mathscr{H}_2}{\partial S_{kl}} \, \mathrm{d}\xi \, \mathrm{d}V - \delta \int_V \int_0^{\xi} \frac{\partial E_n}{\partial \xi} \frac{\partial \mathscr{H}_2}{\partial E_n} \, \mathrm{d}\xi \, \mathrm{d}V + \delta W \right] \mathrm{d}t = 0. \tag{10}$$

Realizing that the integrand is a chain rule expansion of $\mathscr{H}_2$ allows the two integrals of $\mathscr{H}_2$ to be combined. Equation (10) can be expressed as

$$\int_{t_1}^{t_2} \left[\delta T^* - \delta \int_V \int_0^{\tilde{\xi}} \frac{\partial \mathscr{H}_2}{\partial \xi} \, \mathrm{d}\xi \, \mathrm{d}V + \delta W \right] \mathrm{d}t = \int_{t_1}^{t_2} \left[\delta T^* - \delta \int_V \mathscr{H}_2 \Big|_0^{\tilde{\xi}} \mathrm{d}V + \delta W \right] \mathrm{d}t = 0, \tag{11}$$

where the value at $\xi = \tilde{\xi}$ reflects the current state of the system and $\xi = 0$ reflects the null state of the system; i.e., $\mathscr{H}_2(\tilde{\xi}) = \mathscr{H}_2$ and $\mathscr{H}_2(0) = 0$.

Evaluating the enthalpy function in equation (11) produces the variational indicator of a system described by an enthalpy function as given by Tiersten [16]:

$$\int_{t_1}^{t_2} \left[\delta T^* - \delta \int_V \mathscr{H}_2 \, \mathrm{d}V + \delta W \right] \mathrm{d}t = 0. \tag{12}$$

Bringing the variation on the electric enthalpy inside the integral gives

$$\int_{t_2}^{t_1} \left[\delta T^* + \int_V \left(-\frac{\partial \mathscr{H}_2}{\partial E_n} \delta E_n - \frac{\partial \mathscr{H}_2}{\partial S_{kl}} \delta S_{kl} \right) \mathrm{d}V + \delta W \right] \mathrm{d}t = 0. \tag{13}$$

Substituting the definitions of electric enthalpy from equation (9) into equation (13) yields the final form of the variational principle for a system which can be described by an electric enthalpy function:

$$\int_{t_2}^{t_1} \left[\delta T^* + \int_V (D_n \delta E_n - T_{kl} \delta S_{kl}) \, \mathrm{d}V + \delta W \right] \mathrm{d}t = 0. \tag{14}$$

For any system where the constitutive relations can be expressed in terms of an enthalpy function, equation (14) is a form of the variational principle that is equivalent to the full form given in equation (7). In this proof, no assumptions are made about the linearity or the nature of the coupling. All that is required is that there exists an enthalpy function for the system that is differentiable with respect to S and E. Equation (14) is the standard form of Hamilton's Principle for a linear piezoceramic system [17], because such systems are derived from the thermodynamic formalism. The form of the constitutive relationships that we will use would allow the use of this thermodynamic simplification. However, not all constitutive relationships are formed from thermodynamic potentials. Often, higher order terms in constitutive relationships are discarded without re-deriving the thermodynamic potential. In brief, the thermodynamic simplification is not always desirable or feasible. As a result, in this paper we will address variations in the work and energy terms as given in equation (7) instead of always jumping to the simplifications offered by equation (14).

3. QUADRATIC CONSTITUTIVE RELATIONSHIPS

Electrostrictors demonstrate a non-linear relationship between strain and applied electric field. In particular, the strain in the material depends upon the even powers of the applied electric field. The constitutive relationship for an electrostrictor can be formulated from a thermodynamic potential, such as the Gibbs free energy or enthalpy. This expansion has been detailed in many different articles [13, 18–21]; hence, only the resulting non-linear constitutive relationships for an electrostrictor at constant temperature are presented:

$$\begin{aligned} D_m &= \varepsilon^T_{mn}E_n + \varepsilon^T_{mnop}E_nE_oE_p + 2r_{mnijkl}E_nT_{ij}T_{kl} + \cdots + \text{h.o.t.} \\ &\quad + 2m_{mnij}E_nT_{ij} + 4h_{mnopij}E_nE_oE_pT_{ij} + \cdots + \text{h.o.t.}, \\ S_{ij} &= s^E_{ijkl}T_{kl} + s^E_{ijklmn}T_{kl}T_{mn} + 2r_{mnijkl}E_mE_nT_{kl} + \cdots + \text{h.o.t.} \\ &\quad + m_{mnij}E_mE_n + h_{mnopij}E_mE_nE_oE_p + \cdots + \text{h.o.t.} \end{aligned}$$

In the literature, the higher order terms typically are dropped from the electrostrictive constitutive relationships. This yields

$$D_m = \varepsilon^T_{mn}E_n + 2r_{mnijkl}E_nT_{ij}T_{kl} + 2m_{mnij}E_nT_{ij}, \qquad S_{ij} = s^E_{ijkl}T_{kl} + 2r_{mnijkl}E_mE_nT_{kl} + m_{mnij}E_mE_n, \tag{16}$$

where D is the electrical displacement, E is the electric field, T is the stress and S is the strain. The dielectric permittivity, ε^T, indicates the charge stored in the capacitive element of the electrostrictor at constant stress. The electrostriction term, m, is a second order electromechanical coupling term. The compliance, s^E, relates stress and strain at constant (zero) electric field. The elastostriction term, r, can be thought of as either an electric field induced correction to the compliance or a stress induced correction to the electrostrictive coefficient. The electric field varying compliance, s^*, is given by

$$s^*_{ijkl} = s^E_{ijkl} + 2r_{mnijkl}E_mE_n. \tag{17}$$

The stiffness variation and importance of the elastostriction term is addressed in detail in the experimental section of this paper.

Neglecting the elastostriction term, the electrostrictive constitutive relations simplify to the standard quadratic model:

$$D_m = \varepsilon^T_{mn}E_n + 2m_{mnij}E_nT_{ij}, \qquad S_{ij} = m_{ijpq}E_pE_q + s^E_{ijkl}T_{kl}. \tag{18}$$

The quadratic model is the form most often quoted in the electrostrictive literature. The limitations of equation (18) are addressed in more detail in the experimental section of this paper.

The form of the constitutive relationships given in equation (18) are the forms most easily measured experimentally. For example, the m constants are found by applying an electric field on an unconstrained (i.e., zero stress) material. However, letting strain and electric field be the independent variables eases the development of the equations of motion using variational principles and allows for simpler and intuitive assumed modes of displacement and voltage. Rewriting the quadratic constitutive relations (18), yields

$$\begin{aligned} D_m &= \varepsilon^T_{mn}E_n - 2m_{mnij}C^E_{ijkl}m_{klpq}E_nE_pE_q + 2m_{mnij}E_nC^E_{ijkl}S_{kl}, \\ T_{ij} &= -C^E_{ijkl}m_{klpq}E_pE_q + C^E_{ijkl}S_{kl}, \end{aligned} \tag{19}$$

where $C^E_{ijkl} = (s^E_{ijkl})^{-1}$. Equation (19) could also have been found by evaluating a different thermodynamic form.

Changing from tensor notation to matrix notation yields the reduced constitutive relationship for an electrostrictor,

$$\begin{Bmatrix} \mathbf{D} \\ \mathbf{S} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}^T & 2\mathbf{m}^* \\ \mathbf{m}^{*t} & \mathbf{s}^E \end{bmatrix} \begin{Bmatrix} \mathbf{E} \\ \mathbf{T} \end{Bmatrix}, \tag{20}$$

where $\mathbf{m}^*$ varies with the electric field. The m values in the tensor notation and in the expanded matrix are material constants. Most of the electrostrictive literature reduces the fourth order tensor notation on the electrostrictive term to two indexes: $m_{mnij} = m_{mi}$. Expanding the matrices, the electrostrictive constitutive relationships, in engineering notation, are in equation 21 see page 7. Equation (21) illustrates the similarities between the constitutive relationships of electrostrictors and piezoelectrics. For example, the form of $\mathbf{m}^*$ with the electric field only in the 3-direction is similar in form to the $\mathbf{d}$ electromechanical coupling matrix for piezoceramics. However, there are some strong dissimilarities. Electrostrictors are isotropic materials when there is no applied electric field. Piezoceramics are orthotropic due to their poling. The constitutive relationships for electrostrictives can be thought of as being field dependent in this form. In addition, a factor of two is needed in the electromechanical coupling between stress and charge but this factor is not needed in the coupling between electric field and strain.

4. GENERAL QUADRATIC MODEL FOR ELECTROSTRICTIVE SYSTEMS

In this section the general equations of motion are derived for an electrostrictively coupled distributed electromechanical system. Hamilton's Principle is used in conjunction with Rayleigh–Ritz assumed modes to derive the equations of motion for the non-linear electro-elastic media.

4.1. QUADRATIC EQUATIONS OF MOTION BY THE GENERAL METHOD

The following steps provide a general framework for dealing with electromechanical systems with non-linear coupling. Although in this paper the derivation is described for an electrostrictively coupled system, the basic procedure is applicable to any non-linear electromechanical system, given that it can be described by a set of constitutive relations which are piece-wise continuous and non-dissipative.

The general system considered in this paper is represented in Figure 1. The electromechanical system is composed of an elastic body with inclusions of electrostrictive material which are electroded arbitrarily. The displacements within the elastic body and electric fields about the electrodes will be combined through the electrostrictive properties to form the electromechanical coupled equations of motion.

The general form of Hamilton's Principle was described in section 2.1. The evaluation of the variations needs to be performed for each of the terms in equation (7). The variation of the complementary kinetic energy is

$$\delta T^* = \int_V \delta \int_0^{\dot{\tilde{u}}} \mathrm{d}\dot{\mathbf{u}}^t \rho \dot{\mathbf{u}} \, \mathrm{d}V. \tag{22}$$

Bringing the variation inside of the integral gives

$$\delta T^* = \int_V \delta \dot{\mathbf{u}}^t \rho \dot{\mathbf{u}} \, \mathrm{d}V. \tag{23}$$

$$
\left\{\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ \hline S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array}\right\}
=
\left\{\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ \hline S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{13} \\ 2S_{12} \end{array}\right\}
=
\left[\begin{array}{ccc|ccc|ccc|ccc} \varepsilon_1^T & 0 & 0 & 2m_{11}E_1 & 2m_{12}E_1 & 2m_{12}E_1 & 0 & 2m_{44}E_3 & 2m_{44}E_2 \\ 0 & \varepsilon_2^T & 0 & 2m_{12}E_2 & 2m_{11}E_2 & 2m_{12}E_2 & 2m_{44}E_3 & 0 & 2m_{44}E_1 \\ 0 & 0 & \varepsilon_3^T & 2m_{12}E_3 & 2m_{12}E_3 & 2m_{11}E_3 & 2m_{44}E_2 & 2m_{44}E_1 & 0 \\ \hline m_{11}E_1 & m_{12}E_2 & m_{12}E_3 & s_{11}^E & s_{12}^E & s_{12}^E & 0 & 0 & 0 \\ m_{12}E_1 & m_{11}E_2 & m_{12}E_3 & s_{12}^E & s_{11}^E & s_{12}^E & 0 & 0 & 0 \\ m_{12}E_1 & m_{12}E_2 & m_{11}E_3 & s_{12}^E & s_{12}^E & s_{11}^E & 0 & 0 & 0 \\ 0 & m_{44}E_3 & m_{44}E_2 & 0 & 0 & 0 & s_{44}^E & 0 & 0 \\ m_{44}E_3 & 0 & m_{44}E_1 & 0 & 0 & 0 & 0 & s_{44}^E & 0 \\ m_{44}E_2 & m_{44}E_1 & 0 & 0 & 0 & 0 & 0 & 0 & s_{44}^E \end{array}\right]
\left\{\begin{array}{c} E_1 \\ E_2 \\ E_3 \\ \hline T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{array}\right\}. \tag{21}
$$

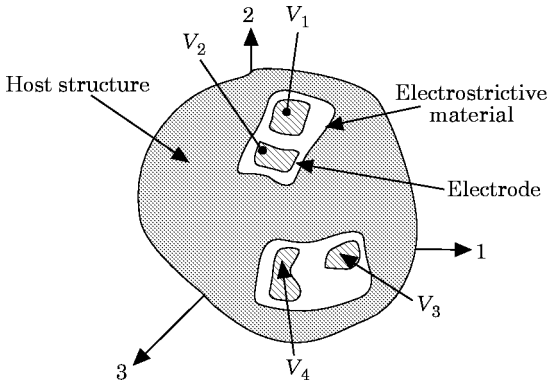

Figure 1. The electroelastic continuum geometry, illustrating inclusions of electrostrictive material which are electroded arbitrarily.

Integrating by parts yields

$$\int_{t_1}^{t_2} \delta T^* \, \mathrm{d}t = \int_V \delta \mathbf{u}^{\mathrm{t}} \rho \dot{\mathbf{u}} \Big|_{t_1}^{t_2} \mathrm{d}V - \int_{t_1}^{t_2} \int_V \delta \mathbf{u}^{\mathrm{t}} \rho \ddot{\mathbf{u}} \, \mathrm{d}V \, \mathrm{d}t, \tag{24}$$

where the first term must be zero. Hamilton's Principle allows arbitrary variation of the path between the endpoints, but requires the variation at the endpoints to be zero. The first term in equation (24) takes the difference of the variations of $\mathbf{u}$ at the endpoints t_1 and t_2, which must be zero. As a result, the variation of the kinetic energy is

$$\int_{t_1}^{t_2} \delta T^* \, \mathrm{d}t = -\int_{t_1}^{t_2} \int_V \delta \mathbf{u}^{\mathrm{t}} \rho \ddot{\mathbf{u}} \, \mathrm{d}V \, \mathrm{d}t. \tag{25}$$

The potential energy of the system, equation (4), requires more care in its evaluation than the kinetic energy terms due to the coupling between the mechanical and the electrical terms. Substituting for stress from the quadratic tensor constitutive relationship in equation (19) gives

$$U = \int_V \int_0^{\tilde{\xi}} \{ \underbrace{-C^E_{ijkl} m_{klpq} E_p E_q S'_{ij} + C^E_{ijkl} S_{kl} S'_{ij}}_{\tilde{U}} \} \, \mathrm{d}\xi \, \mathrm{d}V. \tag{26}$$

Taking the variation and requiring compatibility with the constitutive relationships yields

$$\delta U = \int_V \int_0^{\tilde{\xi}} \left\{ \frac{\partial \tilde{U}}{\partial E_n} + \frac{\partial \tilde{U}}{\partial S_{kl}} \delta S_{kl} + \frac{\partial \tilde{U}}{\partial S'_{ij}} \delta S'_{ij} \right\} \mathrm{d}\xi \, \mathrm{d}V$$

$$= \int_V \int_0^{\tilde{\xi}} \{ (-2 C^E_{ijkl} m_{klpq} E_p S'_{ij}) \delta E_q + (C^E_{ijkl} S'_{ij}) \delta S_{kl}$$

$$+ (-C^E_{ijkl} m_{klpq} E_p E_q + C^E_{ijkl} S_{kl}) \delta S'_{ij} \} \, \mathrm{d}\xi \, \mathrm{d}V. \tag{27}$$

Integrating the last term in equation (27) by parts in order to transform the variation of S' into a variation of S and cancelling the redundant strain terms yields

$$\delta U = \int_V \left\{ \left(\int_0^{\tilde{\xi}} - 2C^E_{ijkl} m_{klpq} E_p S'_{ij} \, \mathrm{d}\xi \right) \delta E_n + \left(\int_0^{\tilde{\xi}} 2C^E_{ijkl} m_{klpq} E_p E'_q \, \mathrm{d}\xi \right. \right.$$

$$\left. \left. + (-C^E_{ijkl} m_{klpq} E_p E_q + C^E_{ijkl} S_{kl}) \Big|_0^{\tilde{\xi}} \right) \delta S_{ij} \right\} \mathrm{d}V. \tag{28}$$

Evaluation of the first integration is not easy, and it will be retained in its unevaluated form since it will be cancelled by a term arising in the electrical energy expression. The second integration will be evaluated directly, since there is only one path dependent variable, E. Let $\xi = \tilde{\xi}$ represent the present state of the system and let $\xi = 0$ correspond to zero field and zero strain conditions:

$$S(\xi = \tilde{\xi}) = S, \qquad E(\xi = \tilde{\xi}) = E, \qquad S(\xi = 0) = 0, \qquad E(\xi = 0) = 0. \tag{29}$$

Equation (28) simplifies to

$$\delta U = \int_V \left\{ \left(\int_0^{\tilde{\xi}} - 2C^E_{ijkl} m_{klpq} E_p S'_{ij} \, \mathrm{d}\xi \right) \delta E_q + (C^E_{ijkl} S_{kl}) \delta S_{ij} \right\} \mathrm{d}V. \tag{30}$$

The electrical energy terms, equation (5), can be evaluated in a manner similar to the potential energy terms. Substituting for electrical displacement from the quadratic constitutive relationship (19), gives

$$W^*_e = \int_V \int_0^{\tilde{\xi}} \underbrace{\{(\varepsilon^T_{mn} - 2m_{mnij} C^E_{ijkl} m_{klpq} E_p E_q) E_n E'_m + 2m_{mnij} C^E_{ijkl} S_{kl} E_n E'_m\}}_{\tilde{W}^*_e} \, \mathrm{d}\xi \, \mathrm{d}V. \tag{31}$$

Taking the variation yields

$$\delta W^*_e = \int_V \int_0^{\tilde{\xi}} \left\{ \frac{\partial \tilde{W}^*_e}{\partial E_p} \delta E_p + \frac{\partial \tilde{W}^*_e}{\partial S_{kl}} \delta S_{kl} + \frac{\partial \tilde{W}^*_e}{\partial E'_m} \delta E'_m \right\} \mathrm{d}\xi \, \mathrm{d}V$$

$$= \int_V \int_0^{\tilde{\xi}} \{(\varepsilon^T_{mn} E'_m - 6m_{mnij} C^E_{ijkl} m_{klpq} E_p E_q E^*_m) \delta E_n + (2m_{mnij} C^E_{ijkl} E_n E'_m) \delta S_{kl}$$

$$+ (\varepsilon^T_{mn} E_n - 2m_{mnij} C^E_{klpq} m_{klpq} E_p E_q E_n + 2m_{mnij} C^E_{ijkl} E_n S_{kl}) \delta E'_m \} \, \mathrm{d}\xi \, \mathrm{d}V. \tag{32}$$

Integrating the last term in equation (32) by parts in order to transform the variation of E' into a variation on E yields.

$$\delta W^*_e = \int_V \left\{ \left(\int_0^{\tilde{\xi}} 2m_{mnij} C^E_{ijkl} E_n E'_m \, \mathrm{d}\xi \right) \delta S_{kl} \right.$$

$$+\left(\int_0^{\tilde{\xi}} -2m_{mnij}C^E_{ijkl}E_nS'_{kl}\,\mathrm{d}\xi + (\varepsilon^T_{mn}E_n - 2m_{mnij}C^E_{ijkl}m_{klpq}E_pE_qE_n + 2m_{mnij}C^E_{ijkl}E_nS_{kl})\Big|_0^{\tilde{\xi}}\right)\delta E_m\Bigg\}\,\mathrm{d}V. \tag{33}$$

Evaluating the first integral, which has an uncoupled integrand, simplifies the equation. Again, let $\xi = \tilde{\xi}$ represent the present state of the system, while $\xi = 0$ corresponds to zero field and zero strain conditions, as indicated in equation (29):

$$\delta W^*_e = \int_V \Bigg\{(m_{mnij}C^E_{ijkl}E_nE_m)\delta S_{kl} + \left(\int_0^{\tilde{\xi}} -2m_{mnij}C^E_{ijkl}E_nS'_{kl}\,\mathrm{d}\xi + (\varepsilon^T_{mn}E_n - 2m_{mnij}E_nC^E_{ijkl}m_{klpq}E_pE_q + 2m_{mnij}E_nC^E_{ijkl}S_{kl})\right)\delta E_m\Bigg\}\,\mathrm{d}V. \tag{34}$$

Each of the energy variations in equation (7) has now been evaluated in terms of the electrostrictive constitutive relationship. Substituting for the variations in kinetic energy, potential energy, electrical energy, and external work, Hamilton's equation becomes

$$\int_{t_1}^{t_2}\Bigg[\int_V -\rho\ddot{u}_j\delta u_j\,\mathrm{d}V + \int_V (-C^E_{ijkl}S_{ij} + m_{mnij}E_nC^E_{ijkl}E_m)\delta S_{kl}\,\mathrm{d}V + \int_V (\varepsilon^T_{mn}E_n - 2m_{mnij}E_nC^E_{ijkl}m_{klpq}E_pE_q + 2m_{mnij}E_nC^E_{ijkl}S_{kl})\delta E_m\,\mathrm{d}V + \int_V f_i\delta u_i - \sum_j q_j\delta\varphi_j\Bigg]\mathrm{d}t = 0, \tag{35}$$

where the unevaluated integrals in the potential energy and electrical energy terms have cancelled each other.

Comparing the terms in equation (35) and in the quadratic constitutive equation given in equation (19), Hamilton's equation for the electrostrictively coupled system can be expressed as

$$\int_{t_1}^{t_2}\Bigg[\int_V -\rho\ddot{u}_j\delta u_j\,\mathrm{d}V + \int_V -T_{kl}\delta S_{kl}\,\mathrm{d}V + \int_V D_m\delta E_m\,\mathrm{d}V + \int_V f_i\delta u_i\,\mathrm{d}V - \sum_j q_j\delta\varphi_j\Bigg]\mathrm{d}t = 0. \tag{36}$$

The derivation started by evaluating the energy along a path and concluded by stating that the energies only need to be evaluated at the endpoints. In other words, for the simplified dynamics of an electrostrictor, the variational principle can be formed directly from the quadratic constitutive relationships, as indicated in section 3.2. This is because the quadratic constitutive relationships were derived from a thermodynamic potential. The full

development of the governing equations from Hamilton's Principle was presented for the sake of generality and thoroughness.

4.2. APPROXIMATE SOLUTION METHODS

In the previous subsection the general form of Hamilton's equation was derived for an electrostrictively coupled electromechanical system. The general equations can be solved either in a finite elements routine or with a Rayleigh–Ritz formulation. Although different in form, these methods are identical in substance. In finite elements, each node introduces another degree of freedom. In Rayleigh–Ritz, representative shape functions form each degree-of-freedom. As a result, the same governing equations are used in either formulation technique. A shift between the two methods only requires a redefinition of the mechanical and electrical shape functions. In this paper terminology most often associated with the Rayleigh–Ritz assumed modes formulation will be used.

Both approximate solution methods find the strain–displacement relationship and the field–potential relationship in terms of generalized co-ordinates relating to the mechanical and electrical displacements. The strain–displacement and field–potential relationships can be introduced as

$$S_{ij} = L^u_{ijq} u_q(x, t) \qquad \text{and} \qquad E_m = L^v_m \varphi(x, t) = -\varphi_{,m}(x, t), \tag{37}$$

where u is the mechanical displacement, φ is the electric potential, L^u is the linear differential operator for the particular elasticity problem, and L^v is the gradient operator. In indicial notation, the elasticity operator for 3-D general elasticity is

$$L^u_{ijq} = \frac{1}{2}\left(\delta_{iq}\frac{\partial}{\partial r_j} + \delta_{jq}\frac{\partial}{\partial r_i}\right). \tag{38}$$

The deflections, u, of the structure can be expressed in terms of assumed deflection shape functions, $\psi^r(x)$, and the vector of time varying modal amplitudes, $r(t)$:

$$u_q(x, t) = \psi^r_{qr}(x) r_r(t) = \{\psi^r_{q1}(x) \cdots \psi^r_{qa}(x)\}\begin{Bmatrix} r_1(t) \\ \vdots \\ r_a(t) \end{Bmatrix}. \tag{39}$$

The modal displacement vectors are restricted only so that they obey the geometric boundary conditions of the problem. Similarly, the electrical potential can be expressed in terms of a modal vector and a vector of modal voltage amplitudes

$$\varphi(x, t) = \psi^v_s(x) v_s(t) = \{\psi^v_1(x) \cdots \psi^v_b(x)\}\begin{Bmatrix} v_1(t) \\ \vdots \\ v_b(t) \end{Bmatrix}, \tag{40}$$

where the only restriction on the modal field vector is that it satisfies the prescribed electrical boundary conditions. The modal field vector must obey voltage boundary conditions, such as constant across a conductor. We can simplify the algebra by combining the differential operators and the modal vectors

$$S_{ij}(x, t) = N^r_{ijr}(x) r_r(t), \quad r = 1, \cdots, a; \qquad E_m(x, t) = N^v_{ms}(x) v_s(t), \quad s = 1, \cdots, b. \tag{41}$$

4.3. APPROXIMATE SOLUTION FOR QUADRATIC SYSTEM

Substituting these modal assumptions and the notation definition into the quadratic form of the general equations of motion (35), gives

$$\int_{t_1}^{t_2}\left[\int_V \delta r_r(-\ddot{r}_s\psi^r_{js}\rho\psi^r_{jr} - r_s N^r_{ijs}C^E_{ijkl}N^r_{klr} + N^r_{klr}C^E_{ijkl}m_{mnij}N^v_{ns}N^v_{mp}v_s v_p)\,\mathrm{d}V$$

$$+\int_V \delta v_r(N^v_{nt}\varepsilon^T_{mn}N^v_{mr}v_t - 2N^v_{mr}N^v_{nt}m_{mnij}C^E_{ijkl}m_{klpq}N^v_{pu}N^v_{qv}v_t v^u v_v$$

$$+2v_t N^v_{mr}m_{mnij}N^v_{nt}C^E_{ijkl}N^r_{klu}r_u)\,\mathrm{d}V + \int_V \delta r_r\psi^r_{rs}f_s\,\mathrm{d}V + \sum \delta v_r\psi^v_s q_s\Bigg]\mathrm{d}t = 0. \tag{42}$$

Allowing arbitrary variations of r and v, two coupled equations in the generalized co-ordinates are obtained. These will be called the actuator and the sensor equations of the non-linear electro-elastic system. The actuator equation describes the dynamics of the mechanical system and the sensor equation describes the dynamics of the electrical system:

$$M_{sr}\ddot{r}_s + K_{sr}r_s - \Theta_{rsp}v_s v_p = B^f_r, \qquad \text{actuator equation;}$$

$$2\Theta_{utr}r_u v_t - G_{rtuv}v_t v_u v_v + Q_{rt}v_t = B^q_{rs}q_s, \qquad \text{sensor equation.} \tag{43}$$

The system mass is defined so that it includes inertial components from the electrostrictive actuator and from the host structure:

$$M_{sr} = \int_V \psi^r_{js}\rho\psi^r_{jr}\,\mathrm{d}V. \tag{44}$$

The stiffness of the system is defined so that both the electrostrictor's and the structure's stiffness are included:

$$K_{sr} = \int_V N^r_{ijs}C^E_{ijkl}N^r_{klr}\,\mathrm{d}V. \tag{45}$$

The electromechanical coupling term is defined as

$$\Theta_{rsp} = \int_V N^r_{klr}C^E_{ijkl}m_{mnij}N^v_{ns}N^v_{mp}\,\mathrm{d}V, \tag{46}$$

the electrostrictive capacitance is

$$Q_{rt} = \int_V N^v_{nt}\varepsilon^T_{mn}N^v_{mr}\,\mathrm{d}V, \tag{47}$$

and the higher order charge storage is

$$G_{rstq} = \int_V 2N^v_{mr}m_{mnij}C^E_{ijkl}m_{klpq}N^v_{pu}N^v_{qv}N^v_{nt}\,\mathrm{d}V. \tag{48}$$

The forcing matrices are defined in terms of the modal vectors evaluated at the points at which the external work is supplied. The effect of the distributed forcing is found by integrating the force over the mode and the applied charge is evaluated at the rth electrode:

$$B_r^f = \int_V \psi_{rs}^r f_s \, \mathrm{d}V, \qquad B_{rs}^q = \psi_s^v(x_r). \tag{49}$$

The actuator and sensor equation as given in equation (43) contain the generalized dynamics of an electrostrictively coupled electromechanical system. In the actuator equation, the M, K, and B^f terms are the standard terms that arise from approximate techniques to analyze continuous systems. The M and K terms are influenced not only by the host structure but also by the actuators. The actuator also makes its presence felt in the Θ electromechanical coupling term. The quadratic behavior of the electrostrictor is indicated by the quadratic voltage terms. The B^f term indicates the effect that the forcing has upon each deflection mode of the structure.

In the sensor equation, the Q and B^q terms are the standard terms that arise from placing charge on a capacitor. The Q term is directly related to the dielectric permittivity of an electrostrictor. Due to the large relative dielectric permittivity of electrostrictors, the Q term tends to dominate the sensor equation. At large actuation fields, the higher-order charge storage term, G, dominates. The electromechanical coupling provided by the Θ term is most significant when the electrostrictor is used as a sensor. The B^q term indicates the location of the actuator being supplied with charge.

4.4. MATRIX REPRESENTATION OF QUADRATIC SYSTEM EQUATIONS

In the most general form, the constitutive relationships for electrostrictors and the governing equations for an electrostrictively coupled electromechanical system are best described in index notation. However, the index notation is rather cumbersome. This simplification of the quadratic equations for an electrostrictively coupled electromechanical system assumes that there is only one prescribed voltage source (i.e., v is scalar). In this case, equation (43) reduces to

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} - \Theta^* v = \mathbf{B}^f, \qquad \text{actuator equation;} \\ &2\Theta^{*\mathrm{t}}\mathbf{r} - G^* v + Q v = B^q q, \qquad \text{sensor equation;} \end{aligned} \tag{50}$$

where the Θ^* matrix and the G^* term include voltage dependencies. The matrix form of the system mass, the system stiffness and the electrostrictive capacitance remain essentially unchanged from the form given in equations (44), (45) and (47). The mass matrix is

$$\mathbf{M} = \int_V \psi^{r\mathrm{t}} \rho \psi^r \, \mathrm{d}V. \tag{51}$$

Similarly, the stiffness matrix is

$$\mathbf{K} = \int_V \mathbf{N}^{r\mathrm{t}} \mathbf{C}^E \mathbf{N}^r \, \mathrm{d}V. \tag{52}$$

The electrostrictive capacitance matrix is

$$Q = \int_V \mathbf{N}^{v\mathrm{t}} \varepsilon^T \mathbf{N}^v \, \mathrm{d}V \tag{53}$$

and influence matrices are

$$\mathbf{B}^f = \int_A \psi^{r^t} \mathbf{f} \, dA, \qquad B^q = \psi^{v^t}(\mathbf{x}). \tag{54}$$

The electrostrictive term, m, necessitates a rearrangement of the electromechanical coupling term and the higher order charge storage term. The electromechanical coupling matrix becomes

$$\Theta^* = \int_V \mathbf{N}^{r^t} \mathbf{C}^{a^E} \mathbf{m}^{*^t} \mathbf{N}^v \, dV, \tag{55}$$

and the higher order charge storage matrix is

$$G^* = \int_V 2\mathbf{N}^{v^t} \mathbf{m}^* \mathbf{C}^{a^E} \mathbf{m}^{*^t} \mathbf{N}^v \, dV, \tag{56}$$

where $\mathbf{m}^*$ is implicitly defined in equations (20) and (21).

The matrix forms of the terms used in equations (51)–(56) are the expansions of the tensors previously defined in this paper. The matrix form of the electric field varying electrostriction term, the dielectric coupling and the stiffness terms were given in equation (21). The mechanical displacement shape function is a matrix of the deformation shapes of all of the shape functions. The mechanical displacement is given by the product of the shape function with their time varying amplitudes:

$$\mathbf{u}(\mathbf{x}, t) = \psi^r(\mathbf{x})\mathbf{r}(t) = \begin{bmatrix} \psi^r_{11}(\mathbf{x}) & \psi^r_{12}(\mathbf{x}) & \dots \\ \psi^r_{21}(\mathbf{x}) & \psi^r_{22}(\mathbf{x}) & \dots \\ \psi^r_{31}(\mathbf{x}) & \psi^r_{32}(\mathbf{x}) & \dots \end{bmatrix} \begin{Bmatrix} r_1(t) \\ r_2(t) \\ \vdots \end{Bmatrix}. \tag{57}$$

The strains in the structure are product of the linear differential operator for the particular elasticity problem and the vector of mechanical displacements:

$$\mathbf{S}(\mathbf{x}, t) = \mathbf{N}^r(\mathbf{x})\mathbf{r}(t) = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{1}{2}\frac{\partial}{\partial z} & \frac{1}{2}\frac{\partial}{\partial y} \\ \frac{1}{2}\frac{\partial}{\partial z} & 0 & \frac{1}{2}\frac{\partial}{\partial x} \\ \frac{1}{2}\frac{\partial}{\partial y} & \frac{1}{2}\frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \psi^r_{11}(\mathbf{x}) & \psi^r_{12}(\mathbf{x}) & \dots \\ \psi^r_{21}(\mathbf{x}) & \psi^r_{22}(\mathbf{x}) & \dots \\ \psi^r_{31}(\mathbf{x}) & \psi^r_{32}(\mathbf{x}) & \dots \end{bmatrix} \begin{Bmatrix} r_1(t) \\ r_2(t) \\ \vdots \end{Bmatrix}. \tag{58}$$

The electrical potentials correspond to the voltages at the electrodes:

$$\varphi(\mathbf{x}, t) = \psi^v(\mathbf{x})v(t). \tag{59}$$

The electrical field comes from the variation of the electrical potential,

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{N}^v(\mathbf{x})v(t) = -\nabla\psi^v v, \tag{60}$$

where ∇ is the gradient operator.

The forms of the actuator and sensor equations as given in equation (50) represent a simplification when only one actuator is present on the structure. The general form of the actuator and sensor equations, (43), can also be readily translated to matrix form for other actuation scenarios, such when as the electric field is present in only one direction or the active material is used only as an actuator.

5. HYPERBOLIC TANGENT FORMULATION

The high field behavior of electrostrictors is best described in terms of a hyperbolic tangent squared formulation instead of the more traditional quadratic formulation. The hyperbolic tangent formulation captures the high field saturation found in electrostrictors. Additionally, the ability to incorporate the hyperbolic tangent formulation illustrates the generality of the non-linear analysis framework presented in this paper. In this section the hyperbolic tangent constitutive relationships are presented, the general equations of motion are derived, the general equations are placed in an assumed modes formulation. By assuming that the voltage is an applied time function, the general equations of motion are represented as state space equations.

5.1. HYPERBOLIC TANGENT CONSTITUTIVE RELATIONS

A very general form of the electrostrictive constitutive relationships was given in equation (15). This relationship was simplified to the form given in equation (16) by neglecting higher order terms. As an alternate simplification, the higher order terms can be combined into hyperbolic tangent formulation. This is an empirical simplification that is introduced because it provides good correlation with the experimental strain data across a wide range of electric field levels.

The hyperbolic tangent form of the constitutive relations are derived from a thermodynamic potential which is composed of hyperbolic tangent terms [9, 11, 13]. Electric field direction terms are chosen to be the lowest order arrangements which allows the material to be isotropic. The enthalpy formulation, $\mathscr{H}$ (different from the *electric* enthalpy function, $\mathscr{H}_2$), yields electric field and stress as the dependent variables:

$$\mathscr{H} = -\tfrac{1}{2}\varepsilon_{mn}E_mE_n - \tfrac{1}{2}s_{ijkl}T_{ij}T_{kl} - \frac{1}{k^2}\,r_{mnijkl}T_{ij}T_{kl}\tanh^2(k|E|)\,\frac{E_mE_n}{|E|^2} - \frac{1}{k^2}\,m_{mnij}T_{ij}\tanh^2(k|E|)\,\frac{E_mE_n}{|E|^2}. \tag{61}$$

The constitutive relations are found by evaluating the partial derivatives of $\mathscr{H}$:

$$(\partial\mathscr{H}/\partial E_m)^T = -D_m \qquad \text{and} \qquad (\partial\mathscr{H}/\partial T_{ij})^E = -S_{ij}. \tag{62}$$

The hyperbolic model for the electromechanical coupling of an electrostrictor becomes

$$D_m = \varepsilon^T_{mn}E_n + \frac{2}{k}\,m_{mnij}T_{ij}\frac{\sinh(k|E|)}{\cosh^3(k|E|)}\,\frac{E_n}{|E|} + \frac{2}{k}\,r_{mnijkl}T_{ij}T_{kl}\frac{\sinh(k|E|)}{\cosh^3(k|E|)}\,\frac{E_n}{|E|},$$

$$S_{ij} = s^E_{ijkl}T_{kl} + \frac{1}{k^2}\,m_{mnij}\tanh^2(k|E|)\,\frac{E_mE_n}{|E|^2} + \frac{2}{k^2}\,r_{mnijkl}T_{kl}\tanh^2(k|E|)\,\frac{E_mE_n}{|E|^2}. \tag{63}$$

The inherent one-dimensional behavior of the isotropic electrostrictor is extended to three dimensions with the directional unit vectors $E_n/|E|$ and $E_mE_n/|E|^2$. These equations

represent the non-linear electrostrictive constitutive relationships at constant temperature. This form of the constitutive equations will be called the hyperbolic model. The material parameters are measured most easily through the constitutive relationship expressed in equation (63). Note that the hyperbolic model expressed in equation (63) is similar to the model developed by Hom and Shankar [11] if electric field is substituted for the electrical polarization.

The hyperbolic form of the constitutive relationships given in equation (63) needs to be reformulated so that strain is an independent variable, in order to ease the development of the general equations of motion:

$$
\begin{aligned}
T_{kl} &= C^*_{ijkl}S_{ij} - \frac{1}{k^2}\, C^*_{ijkl}m_{mnij}\tanh^2(k|E|)\,\frac{E_mE_n}{|E|^2}\,,\\
D_m &= \varepsilon^T_{mn}E_m + \frac{2}{k}\, m_{mnij}C^*_{tuij}S_{tu}\,\frac{\sinh(k|E|)}{\cosh^3(k|E|)}\,\frac{E_n}{|E|}\\
&\quad -\frac{2}{k^3}\, m_{mnij}m_{vwtu}C^*_{tuij}\,\frac{\sinh^3(k|E|)}{\cosh^5(k|E|)}\,\frac{E_nE_vE_w}{|E|^3}\\
&\quad +\frac{2}{k}\, r_{mnijkl}C^*_{tuij}C^*_{pqkl}\,\frac{\sinh(k|E|)}{\cosh^3(k|E|)}\,\frac{E_n}{|E|}\\
&\quad \times\left(S_{pq}S_{tu} - \frac{2}{k^2}\, m_{vwtu}S_{pq}\tanh^2(k|E|)\,\frac{E_vE_w}{|E|^2}\right.\\
&\quad \left.+\frac{1}{k^4}\, m_{rspq}m_{vwtu}\tanh^4(k|E|)\,\frac{E_rE_sE_vE_w}{|E|^4}\right).
\end{aligned}
\tag{64}
$$

The superscript asterisk on the stiffness matrix indicates that electric field terms are buried within. The variable stiffness matrix is defined as

$$
C^*_{ijkl} = \left(s^E_{ijkl} + \frac{2}{k^2}\, r_{mnijkl}\tanh^2(k|E|)\,\frac{E_mE_n}{|E|^2}\right)^{-1}. \tag{65}
$$

The constitutive relationships are rearranged to give equation (64) results in terms that are of higher order than those truncated to create the original enthalpy formulation. As a result, equations (64) and (65) are simplified. Care is taken that the constitutive relationships can still be expressed in terms of a thermodynamic formalism.

The hyperbolic tangent function in the denominator needs to be brought into the numerator. This transformation is done using a Taylor series expansion:

$$
C^*_{ijkl} = C^*_{ijkl}|_{E=0} + E_m\left.\frac{\partial C^*_{ijkl}}{\partial E_m}\right|_{E=0} + \text{h.o.t.} = \frac{1}{s^E_{ijkl}} - 2\,\frac{r_{mnijkl}}{(s^E_{ijkl})^2}\,E_mE_n + \text{h.o.t.} \tag{66}
$$

Dropping the higher order terms, the field varying stiffness becomes

$$
C^*_{ijkl} = \frac{1}{s^E_{ijkl}} - 2\,\frac{r_{mnijkl}}{(s^E_{ijkl})^2}\,E_mE_n = C^E_{ijkl} - \tilde{C}_{ijklmn}E_mE_n. \tag{67}
$$

The simplified stiffness term is substituted into the electrical displacement term for the hyperbolic model, which was given in equation (62). Neglecting any terms that include any combination of strain and electric field that is higher than third order, the constitutive relationships become

$$T_{kl} = C^E_{ijkl}S_{ij} - \tilde{C}_{ijklmn}E_mE_nS_{ij} - \frac{1}{k^2}\,m_{mnij}C^E_{ijkl}\tanh^2(k|E|)\,\frac{E_{\mathrm{m}}\mathrm{E_n}}{|\mathrm{E}|^2},$$

$$D_m = \varepsilon^T_{mn}E_n + \frac{2}{k}\,m_{mnij}C^E_{ijkl}S_{kl}\frac{\sinh(k|E|)}{\cosh^3(k|E|)}\frac{E_n}{|E|} - \frac{2}{k^3}\,m_{mnij}m_{vwtu}C^E_{tuij}$$

$$\times\frac{\sinh^3(k|E|)}{\cosh^5(k|E|)}\frac{E_nE_vE_w}{|E|^3} + \tilde{C}_{ijklmn}E_nS_{ij}S_{kl}. \tag{68}$$

This is an equivalent form, but not equal form, to the hyperbolic model presented in equation (63). The form given in equation (68) is that most easily incorporated into the dynamic system model, while the form given in equation (63) is the form in which the material coefficients are most easily measured.

The form of the hyperbolic constitutive relations given in equation (68) can be derived from a thermodynamic potential. The elastic enthalpy function, $\mathscr{H}_1$, yields the constitutive relations of the form expressed in equation (68):

$$\mathscr{H}_1 = -\tfrac{1}{2}\varepsilon_{mn}E_mE_n + \tfrac{1}{2}C^E_{ijkl}S_{ij}S_{kl} - \tfrac{1}{2}\tilde{C}_{ijklmn}E_mE_nS_{ij}S_{kl}$$

$$-\frac{1}{k^2}\,m_{mnij}C^E_{ijkl}S_{kl}\tanh^2(k|E|)\,\frac{E_mE_n}{|E|^2}$$

$$+\frac{1}{2k^4}\,m_{mnij}m_{vwtu}C^E_{tuij}\tanh^4(k|E|)\,\frac{E_mE_nE_vE_w}{|E|^4}. \tag{69}$$

The constitutive relationships can be found by differentiating the elastic enthalpy function:

$$(\partial\mathscr{H}_1/\partial E_m)^{\mathscr{T}} = +D_m \quad \text{and} \quad (\partial\mathscr{H}_1/\partial S_{ij})^E = T_{ij}. \tag{70}$$

The hyperbolic model could also have been described in terms of electrical polarization instead of electric field [9, 11]. An electrical polarization expression of the constitutive behavior is slightly simpler than the electric field expression and, hence, will make the general equations of motion simpler. However, an electrical polarization expansion requires prescribing the charges on the electrodes instead of the voltage on the electrodes. Since the electric field is most often the prescribed state variable, the hyperbolic model for this paper has been expressed in terms of electric field instead of electrical polarization.

5.2. GENERAL EQUATION FOR HYPERBOLIC TANGENT SYSTEM BY GENERAL METHOD

The equations of motion for an electrostrictor can be found for the hyperbolic constitutive relationships. The hyperbolic model given in equation (68) was derived from thermodynamic formalism. It was proven in section 2.2. that the final equation of motion can be written directly if a thermodynamic formalism exists. Thus, instead of repeating the steps described performed for the quadratic system, the constitutive relationships are substituted into Hamilton's expression, equation (14). The resulting governing equation is succinctly expressed by substituting equations (68) into the simplified form of the variational principle, equation (14):

$$\int_{t_1}^{t_2}\left[\int_V -\rho\ddot{u}_j\delta u_j\,\mathrm{d}V + \int_V\left(C^E_{ijkl}s_{ij} - \tilde{C}_{ijklmn}E_mE_nS_{ij}\right.\right.$$

$$\left.\left.-\frac{1}{k^2}\,m_{mnij}C^E_{ijkl}\tanh^2(k|E|)\frac{E_mE_n}{|E|^2}\right)\delta S_{kl}\,\mathrm{d}V\right.$$

$$+\int_V \left(\varepsilon^T_{mn} E_n + \frac{2}{k} m_{mnij} C^E_{ijkl} S_{kl} \frac{\sinh(k|E|)}{\cosh^3(k|E|)} \frac{E_n}{|E|} \right.$$

$$- \frac{2}{k^3} m_{mnij} m_{vwtu} C^E_{tuij} \frac{\sinh^3(k|E|)}{\cosh^5(k|E|)} \frac{E_n E_v E_w}{|E|^3}$$

$$\left. + \tilde{C}_{ijklmn} E_n S_{ij} S_{kl} \right) \delta E_m \, \mathrm{d}V + \int_V f_i \delta u_i \, \mathrm{d}V - \sum_j q_j \delta \varphi_j \Bigg] \mathrm{d}t = 0. \tag{71}$$

5.3. APPROXIMATE SOLUTION FOR HYPERBOLIC TANGENT SYSTEM

The hyperbolic system of equations can be expressed in the simplified Rayleigh–Ritz formulation through the modal assumptions given at the beginning of section 4.2. Substituting the modal assumptions and allowing arbitrary variations of displacements and voltage, two coupled equations of motion are obtained in the generalized co-ordinates. For the hyperbolic model, the actuator and sensor equations become

$$M_{ps}\ddot{r}_p + K_{ps} r_p - \tilde{K}_{pqrs} r_p v_q v_r - \Theta_{pqs} F^*_p F^*_q = B^f_s, \qquad \text{actuator equation;}$$

$$Q_{rs} v_r + 2\Theta_{psr} \beta^* F^*_p r_r - G_{pqrs} \beta^* F^*_p F^*_q F^*_r + R_{prvs} v_p r_r r_v = B^q_s, \qquad \text{sensor equation.} \tag{72}$$

Neither the mass matrix nor the electrostrictive capacitance change with the addition of higher order material non-linearities:

$$M_{sr} = \int_V \psi^r_{js} \rho \psi^r_{jr} \, \mathrm{d}V, \qquad Q_{rt} = \int_V N^v_{nt} \varepsilon^T_{mn} N^v_{mr} \, \mathrm{d}V, \tag{73}$$

The stiffness of the system has been divided into two terms. The zero field stiffness is the same as that given in equation (45),

$$K_{sr} = \int_V N^r_{ijs} C^E_{ijkl} N^r_{klr} \, \mathrm{d}V. \tag{74}$$

The addition of the elastostriction term introduces a stiffness term that is electric field dependent. In this case, the stiffness correction is

$$\tilde{K}_{pqrs} = \int_V \tilde{C}_{ijklmn} N^r_{ijp} N^r_{kls} N^v_{mp} N^v_{nq} \, \mathrm{d}V. \tag{75}$$

The electromechanical coupling term changes from the quadratic to the hyperbolic tangent formulation:

$$\Theta_{pqs} = \int_V m_{mnij} C^E_{ijkl} N^r_{kls} N^v_{mp} N^v_{nq} \, \mathrm{d}V. \tag{76}$$

The higher order charge storage becomes

$$G_{pqrs} = \int_V 2 m_{mnij} m_{vwtu} C^E_{tuij} N^v_{np} N^v_{vq} N^v_{wr} N^v_{ms} \, \mathrm{d}V. \tag{77}$$

The inclusion of the elastostriction term introduces a elastostriction charge storage term

$$R_{prvs} = \int_V \tilde{C}_{ijklmn} N^r_{ijr} N^r_{klv} N^v_{np} N^v_{ms} \, \mathrm{d}V. \tag{78}$$

The hyperbolic electric field dependence has been incorporated into two terms,

$$F^*_p = \int_V \frac{1}{k} \tanh\left(k\sqrt{N^v_{ij} v_j N^v_{ik} v_k}\right) \frac{v_p}{\sqrt{N^v_{mn} v_n N^v_{mq} v_q}} \, \mathrm{d}V \tag{79}$$

and

$$\beta^* = \int_V \operatorname{sech}^2\left(k\sqrt{N^v_{ij} v_j N^v_{ik} v_k}\right) \mathrm{d}V. \tag{80}$$

5.4. STATE SPACE REPRESENTATION OF HYPERBOLIC TANGENT SYSTEM

The general equations of motion which involve hyperbolic trigonometry can be placed in a state space representation using matrices instead of tensors. For this simplification, it is assumed that the elastostriction term is zero. Also, it is assumed that the voltages are prescribed functions of time. In other words, it is assumed that the active materials are actuators. However, there are no restrictions on the number of active elements. As a result, equation (72) can be described as

$$\begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{u} \end{Bmatrix}, \tag{81}$$

where

$$\mathbf{x} = \begin{Bmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{Bmatrix}, \qquad \mathbf{u} = \begin{Bmatrix} \mathbf{f} \\ \left(\frac{1}{k} \tanh\left(k\mathbf{N}^v\mathbf{v}\right)\right)^2 \end{Bmatrix}, \tag{82}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}^d \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{B}^f & \mathbf{M}^{-1}\Theta \end{bmatrix}. \tag{83}$$

The mass matrix, $\mathbf{M}$, the stiffness matrix, $\mathbf{K}$, and the influence matrix, $\mathbf{B}^f$, remain unchanged from the forms given in equations (51), (52) and (54). The electromechanical coupling matrix is now defined as

$$\Theta = \int_V \mathbf{N}^{r\prime} \mathbf{C}^E \mathbf{m}^t \, \mathrm{d}V. \tag{84}$$

The output equations depend on which variables are desired. Let us assume that the strain values at some discrete location are desired, $y = S(x_i)$: then the other parts of equation (81) become

$$\mathbf{C} = [\mathbf{N}^r(x_i) \quad \mathbf{0}], \qquad \mathbf{D} = [\mathbf{0} \quad \mathbf{0}]. \tag{85}$$

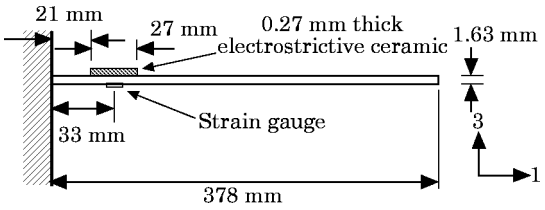

Figure 2. The cantilevered beam test article.

6. EXPERIMENTS

Experiments were conducted to test the validity of the analytical models of the electrostrictively coupled electromechanical system. In these tests, the electrostrictor was driven by a prescribed voltage and both the strain and the induced charge were measured.

6.1. EXPERIMENTAL SET-UP

The experiments were conducted on a cantilevered aluminum beam with a surface bonded electrostrictive wafer, as indicated in Figure 2. The beam was 37·8 cm long, 2·7 cm wide and 0·16 cm thick. An electrostrictor was bonded to one side of the beam 2·1 cm from the base and extending 2·7 cm. The electrostrictor was a 0·027 cm thick 0·9 PMN – 0·1 PT ceramic wafer, manufactured by AVX. The surfaces were electroded with aluminum and the transverse properties of the wafers (i.e., the m_{12} values) were utilized. The wafer was bonded to one side of the beam. Transverse motion of the wafer induced not only a moment, but also stretching in the beam. The maximum relative dielectric of this electrostrictor occurred at a temperature of 40°C. All of the experiments were conducted with the wafer temperature at 28°C. As a result, the experimental data will feature larger hysteresis than the classic curves of an electrostrictor and will feature larger strains [12, 18]. The basic material properties of the beam and of the electrostrictor are given in Table 1. The damping ratio of the beam was measured with a ring-down test, the dielectric permittivity was measured with an Omega multimeter, and the m_{3311}, k, r_{331111} and stiffness constants were obtained through experimentation.

TABLE 1

Material properties of beam and actuator

Electrostrictors	Stiffness	$C^{a^E}_{1111} = 120$ GPa
	Poisson ratio	$\nu^a = 0{\cdot}38$ [22]
	Dielectric constant	$\varepsilon^T_{33} = 17\,000\varepsilon_0$
	m constant	$m_{3311} = 6{\cdot}6\text{e} - 16\ \text{m}^2/\text{V}^2$
	k constant	$k = 1{\cdot}6\text{e} - 6$ V/m
	r constant	$r_{331111} = 3{\cdot}25\text{e} - 24\ \text{m}^2/(\text{V}^2\ \text{Pa})$
	Thermal expansion	$\alpha < 1\text{e} - 6/°\text{C}$ [22]
	Fracture toughness	$K_{1c} = 0{\cdot}9\ \text{MPa}\sqrt{\text{m}}$ [23]
	Ultimate strength	$\sigma_u = 24$ MPa [2]
Beam	Stiffness	$C^s_{11} = 68$ GPA
	Poisson ratio	$\nu^s = 0{\cdot}38$
	Damping ratio	
	First mode	$\zeta = 0{\cdot}6\%$
	Second mode	$\zeta = 0{\cdot}4\%$
	Third mode	$\zeta = 0{\cdot}3\%$

The electrostrictive wafer was excited with a Kepco model BOP 1000M voltage amplifier which was fed a sinusoidal signal from a Phillips model PM 5191 function generator. The modal displacements were measured with a strain gauge located on the reverse side of the beam from the electrostrictive wafer, 3·3 cm from the base. Data was collected on a Macintosh Quadra 950 using LabView software, sampling at 4 kHz. The electrical displacement was measured by integrating the current applied to the actuator.

The strain data was windowed in the frequency domain with an 11-point Hamming window and then averaged in the time domain over four cycles. A similar process was performed on the voltage data in order to eliminate the possible effects of added phase.

6.2. MODEL COMPARISON AND SHAPE FUNCTIONS

The experimental results are compared with the models developed in section 4. Both the quadratic model given in equation (50) and the hyperbolic model given in equation (81) were used. The values used in the simulations of the experimental data are taken from values presented in the literature or measured for this paper. No iteration was attempted to enhance the correlation between the experiments and the model.

For all of the model simulations, the driving voltage was a single frequency sinusoid and the beam was initially at rest. The model used seven assumed deflection shape functions, $\psi_r(x)$. Five of these deflection modes reflected the exact mode shape for a cantilevered beam. The other two mode shapes represented the beam profile if there was a static electric field on the electrostrictive wafer; one mode represented bending while the other mode represented extension. These "static" mode shapes help better to represent any residual strain energy due to the discontinuous stiffness at the electrostrictor [24].

The exact mode shapes for a cantilevered beam are [25]

$$\psi^r_{3r}(x) = (\cosh \lambda_r x - \cos \lambda_r x) - \alpha_r(\sinh \lambda_r x - \sin \lambda_r x), \tag{86}$$

where λ and α are unique for each mode of the structure. The first "static" mode shape represents the bending induced by the electrostrictive wafer. This mode has constant curvature around the actuator and zero curvature elsewhere. If a_1 and a_2 represent the endpoints of the actuator, then the "static" bending mode is given by

$$\psi^r_{3r} = \begin{cases} 0, & \mathrm{x} < \mathrm{a}_1 \\ \dfrac{(x - a_1)^2}{a_2 - a_1}, & a_1 < x < a_2, \\ a_2 - a_1 + 2(x - a_2), & a_2 < x. \end{cases} \tag{87}$$

The "static" mode shape associated with extension exhibits constant slope around the actuator and zero slope elsewhere:

$$\psi^r_{3r} = \begin{cases} 0, & \mathrm{x} < \mathrm{a}_1, \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 < x < a_2, \\ 1, & a_2 < x. \end{cases} \tag{88}$$

The process of applying Rayleigh–Ritz analysis to a beam with surface mounted actuators is explained in detail in other sources [12, 13, 17].

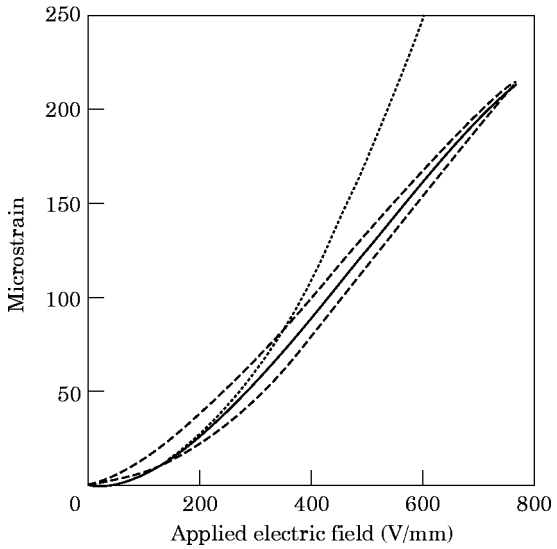

Figure 3. The unconstrained transverse expansion of an electrostrictive wafer. Curve fitting the hyperbolic model, equation (63), to the data gives the electrostrictive constant, $m_{3322} = 6{\cdot}6\text{e} - 16\ \text{m}^2/\text{V}^2$ and the relaxation parameter, $k = 1{\cdot}6\text{e} - 6\ \text{V/m}$. The same m value was used in the quadratic model and gives good low field correlation. - - - , Experiment; ——, hyperbolic model; · · · , quadratic model.

6.3. FREE WAFER TESTS

The electrostrictive effect is characterized by placing a slowly varying electric field across an unconstrained sample. The resulting curve gives the unadulterated effect of the electromechanical coupling. In this paper two models were derived to describe the electromechanical coupling: a quadratic model, equation (18), and a hyperbolic model, equation (63). Removing all but the direct electrostriction terms for the induced strain and requiring the electric field to be only in the 3-direction, the electromechanical couplings are

$$S_{ij} = m_{33ij} E_3 E_3, \quad \text{quadratic model;} \quad S_{ij} = \frac{1}{k^2} m_{33ij} \tanh^2 (kE_3), \quad \text{hyperbolic model.} \quad (89)$$

Notice that at low field levels, the hyperbolic model reduces to the form of the quadratic model.

The material parameters were obtained by fitting the hyperbolic model to the experimental data with a unconstrained non-linear optimization. Experimental data was from electric fields ranging from DC to 800 V/mm. The resulting material values at 28°C are $m_{3311} = 6{\cdot}6\text{e} - 16\ \text{m}^2/\text{V}^2$ for the transverse electrostrictive constant and $k = 1{\cdot}6\text{e} - 6\ \text{V/m}$ for the relaxation parameter. The electrostrictive constant scales the magnitude of the strain and the relaxation parameter dictates when the strain tends to "bend over" or saturate with increasing electric field. Since the quadratic model is a low field simplification of the hyperbolic model, the same electrostrictive constant is used in both models.

As indicated in Figure 3, the hyperbolic model provides a close correlation between the experiment and the model through out the range of electric field levels. The quadratic model forms a good approximation of the experiment in the low and moderate electric field regions. Above 300 V/mm, the quadratic model will over predict the response.

6.4. STIFFNESS VARIATION

Elastostriction describes the field-based correction to the induced strain. As indicated in section 2, the elastostriction term can be interpreted as an electric field based correction to the compliance of the material. In particular, from equations (17) and (63), the compliance of an electrostrictor can be expressed in terms of an electric field in only the 3-direction as

$$s^*_{ijkl} = s^E_{ijkl} + 2r_{\ 33ijkl}E_3^2, \qquad \text{quadratic model;}$$

$$s^*_{ijkl} = s^E_{ijkl} + \frac{2}{k^2}\, r_{33ijkl} \tanh^2 (kE_3), \qquad \text{hyperbolic model.} \tag{90}$$

The first term in the equations represents the compliance at zero electric field and the second term is the elastostriction correction. In this section we seek to define the magnitude of the elastostriction term. Notice that the hyperbolic model reduces to the quadratic model at low field levels, and thus the same value will be used in both models.

Measuring the elastostriction term is difficult. There are two possible techniques for measuring stiffness as a function of electric field: measuring the actuation potential or measuring the natural frequency. By far the simplest technique would be to measure the actuation potential. The actuation potential gives the amount of deformation that the electrostrictor can produce when its expansion is constrained. Thus, the deformation of an electrostrictor sandwiched between layers of metal would change based on the relative thickness and stiffness of the components. This would be an easy method from which to find the actuator stiffness if the material non-linearity was an electric field-based non-linearity. However, *if* the non-linearity of an electrostrictor was a strain based non-linearity, then the reduced strain from the constraint would give erroneous results. For example, measuring the induced strain of a piezoceramic would give anomalous results, since the non-linearity of a piezoceramic is a function of the strain of the material, not the electric field [26]. In other words, the non-linearity of a piezoceramic is a function of the deformations of the microstructure. Measuring the stiffness by noting changes in the natural frequency is valid regardless of whether the non-linearity is electric field based or strain based.

The variation in the transverse compliance was measured by noting the natural frequency of a cantilevered beam with an electrostrictor bonded to it. The natural frequency of the system will vary as a DC electric field is applied to the electrostrictor. The first natural frequency was experimentally measured by applying a small sinusoidal signal, 0·5 V/mm, in addition to a large DC electric field. In order to translate the natural frequencies into an actuator stiffness, equation (81) was used to calculate the natural frequency as a function of the material stiffness. The stiffness at different electric fields was obtained by matching the natural frequency versus electric field from the experiment with the natural frequency versus stiffness from the simulation. The material stiffness as well as the model predictions are shown in Figure 4. The effective transverse stiffness decreased by roughly 20% as the electric field increased from 0 to 1300 V/mm. The elastostriction factor, r, was curve fitted from the hyperbolic model given in equation (90). The calculated elastostriction constant, r_{331111}, equals $3{\cdot}25\text{e} - 24\ \text{m}^2/(\text{V}^2\ \text{Pa})$. The relaxation parameter, k, is the same as that which was calculated from Figure 3. The hyperbolic model provides solid prediction across the range of applied electric fields, while the quadratic model will under predict the stiffness.

The stiffness variation holds strong implications when calculating the induced strain. The strain resulting from a constrained electrostrictor is a function of the material stiffness.

For the case of an electrostrictor sandwiched between two pieces of metal, the one-dimensional approximation of the induced transverse strain is given by

$$S_{constrained} = \frac{C^a t^a}{C^a t^a + C^s t^s} S_{free}. \tag{91}$$

In other words, the induced strain is proportional to the unconstrained motion of the wafer. The proportionality is given by the stiffness ratios and thickness ratios of the actuator and the host structure.

The stiffness variation can be examined in detail for the case of a 0·27 mm thick electrostrictive wafer bonded between two pieces of 25 mm wide, 0·50 mm thick aluminum. The induced strain was measured with a strain gauge mounted on the aluminum. The results are given in Figure 5. The induced strain was predicted with equation (91) and with the free strain given in Figure 3. If a constant stiffness is assumed, then the predicted strain will exceed the measured strain. If the variable stiffness given in Figure 4 is included, then the predicted and measured strain agree.

It is also indicated in Figure 5 that the *non-linearity of an electrostrictor is an electric field based non-linearity*. Since equation (91) accurately predicts the constrained dynamics of an electrostrictor, the non-linearity of an electrostrictor must be an electric field based non-linearity instead of a strain based non-linearity. This simplifies the treatment of electrostrictors in structural actuation [13].

6.5. QUASI-STATIC RESPONSE

The quasi-static dynamics of the cantilevered beam system are illustrated in Figure 6. The experiment found strain as a function of applied electric field across the electrostrictive wafer. The voltage was a sinusoid at 0·1 Hz. The theoretical strain was found by numerically evaluating the quadratic actuator equation given by equation (50) and the hyperbolic actuator equation given by equation (81).

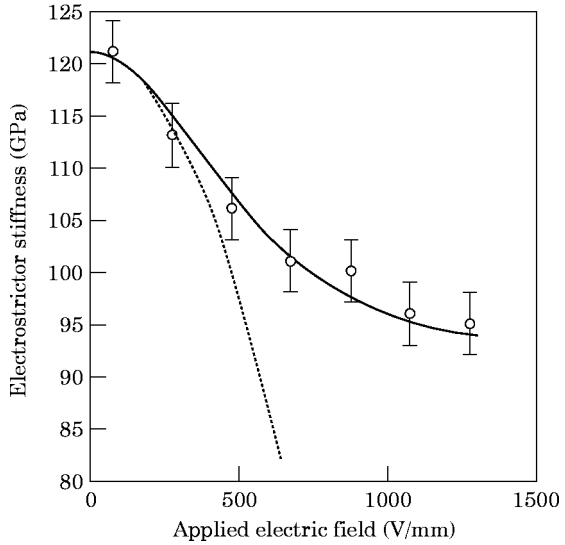

Figure 4. The transverse stiffness variation as a function of the electric field in 0·9 PMN – 0·1 PT. The hyperbolic model reflects the best fit of equation (67). This gives an $r_{331111} = 3{\cdot}25e - 24\ m^2/(V2\,Pa)$. The quadratic model uses the same elastostriction value. ○, Experiment; ——, hyperbolic model; · · ·, quadratic model.

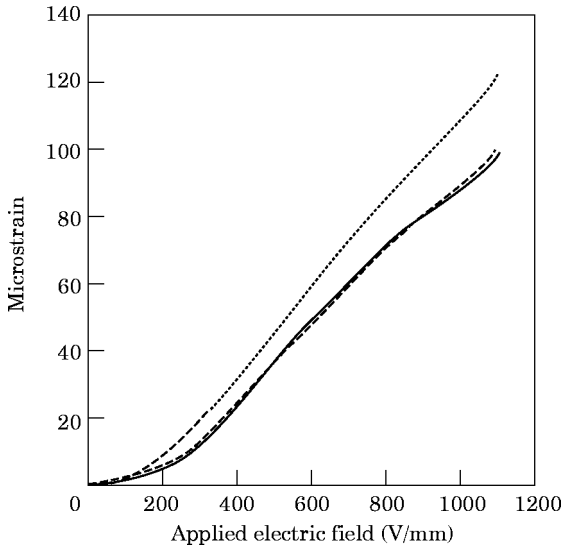

Figure 5. The effect of the stiffness variation on the induced strain on an electrostrictor sandwiched between two pieces of aluminum. The models use the fact that the electrostrictor exhibits an electric field-based non-linearity. - - - , Experiment; ——, hyperbolic model; . . . , quadratic model.

At low fields, good agreement is achieved with both model and experiments. The quadratic model tends to break down above 300 V/mm, because the initial simplifying approximations are no longer valid at the higher field levels. The hyperbolic model provides excellent agreement across the entire range of applied electric field levels.

6.6. CHARGE VARIATION

The sensor equation of an electrostrictively coupled system can be verified by examining the charge variation on the electrostrictive wafer. In this case, a 6·00 Hz sinusoidal voltage

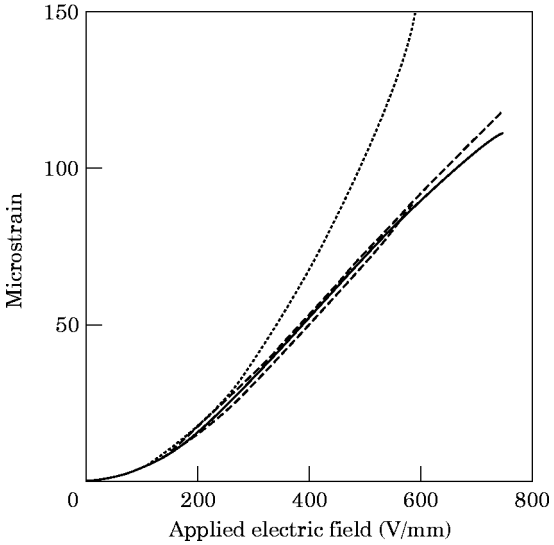

Figure 6. Quasi-static deflections of the cantilevered beam. The hyperbolic model used equation (81) to predict the response. The quadratic model used equation (50). Curve identifications as in Figure 5.

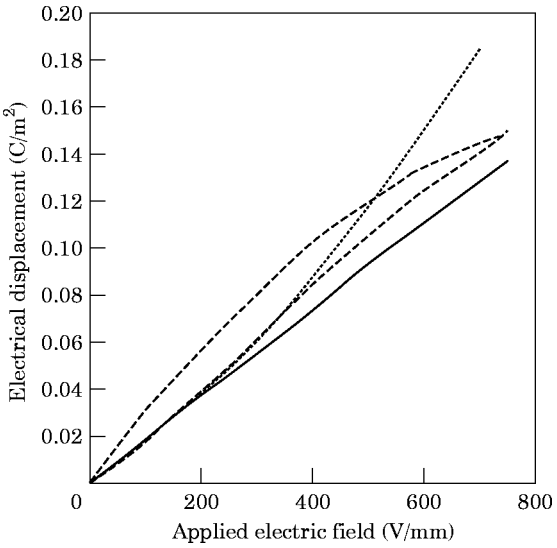

Figure 7. Charge variations on the electrostrictor during 6 Hz dynamic actuation. The hyperbolic model used the state-space form of the sensor equation as given by equation (81) and the quadratic model used equation (50) to calculate the response. The discrepancy between the modelled and experimental result could result from a variation in the dielectric permittivity. Curve identifications as in Figure 5.

signal was supplied to the wafer. The charge on the electrostrictor was determined by integrating the current applied to the electrostrictor. The current supplied to the electrostrictor was measured with a sensing resistor in series with the electrostrictor. The integrating circuit featured a single pole integrator with a corner at 0·1 Hz. The charge in the simulation was calculated through the sensor relation in the general equations of motion.

As shown in Figure 7, the quadratic model overpredicts the charge at large electric field levels. The hyperbolic model under predicts the charge at all levels, but more closely approximates the shape of the charge–field curve. The mismatch between experimental and modelled performance in the low field region suggests that the dielectric permittivity of the electrostrictor might have been larger than that which was used in the model. The dielectric is sensitive to changes in temperature, frequency and electric field. The dielectric was calculated from the capacitance, measured using an Omega multimeter. The multimeter measures the capacitance at 390 Hz. The capacitance of electrostrictors increases as the frequency decreases. The capacitance could increase by 20% in the frequency range between the 390 Hz measurement and the 6 Hz experiment [27]. A 20% increase in the capacitance would provide a very close correlation between the experiment and the model in Figure 7.

6.7. FREQUENCY RESPONSE

For a linear system, the frequency response can be expressed as a transfer function. Unfortunately, when referring to a non-linear system, a transfer function loses its meaning. As a result, the r.m.s. strain was calculated as a function of the excitation frequency. The low electrical bias case, 75 V/mm, is presented in Figure 8 and the high electrical bias case of 400 V/mm is shown in Figure 9. In both cases, a 75 V/mm sinusoidal signal provided the excitation.

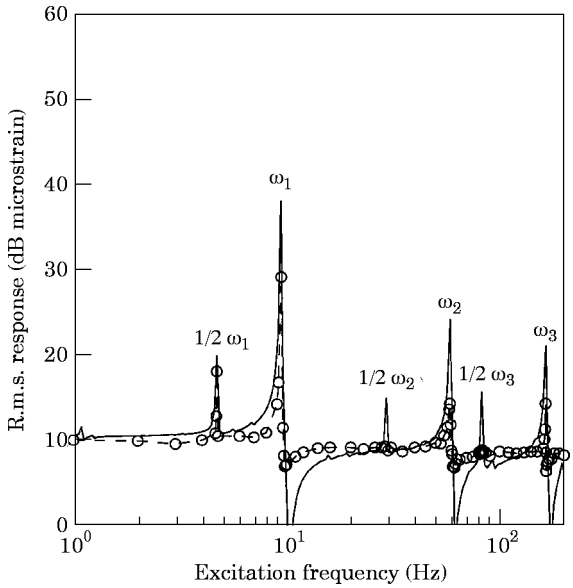

Figure 8. The frequency response of the cantilevered beam. There is a low DC bias of 75 V/mm. Superharmonic resonances are prevalent due to the quadratic non-linearity inherent to electrostrictors. The quadratic model and the hyperbolic model yield identical results at this low electric field level. - - ○ - -, Experiment; ——, model.

In Figure 8, the first, second and third modes of the beam are visible, and are represented by peaks in both the modelled and experimental data. There are also large responses at one half of these natural frequencies. A non-linear system, such as an electrostrictor, is

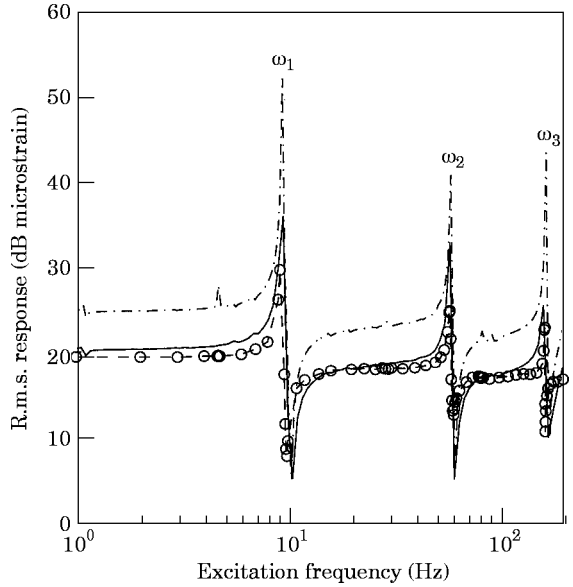

Figure 9. The frequency response of the cantilevered beam. There is a large DC bias of 400 V/mm. The superharmonic resonances are virtually eliminated because the high bias effectively linearizes the system. - - ○ - -, Experiment; ——, hyperbolic model; - · - · - , quadratic model. Figure 9: Frequency response of the cantilevered beam. There is a large DC bias of 400 V/mm. The superharmonic resonances are virtually eliminated because the high bias effectively linearizes the system.

not constrained to respond at the same frequency as the excitation. An excitation at half of the natural frequency induces a superharmonic resonance at the natural frequency due to the quadratic non-linearity inherent in electrostrictors. The hyperbolic model and the quadratic model provide identical results at this field level. Additionally, the modelled and experimental results agree closely over the range of frequencies.

The superharmonic resonances are less visible in the experiment and in the simulation at the high electric bias field, as seen in Figure 9. By going to the large bias levels, the non-linear system has essentially been linearized. The hyperbolic model again provides close agreement over the range of frequencies. The quadratic model overpredicts the response at high electric field levels, because the 400 V/mm DC electric bias is beyond the valid range of the reduced material constitutive relationships.

7. CONCLUSIONS

In this paper, theory and experiments describing the use of electrostrictive elements for structural actuation have been presented. The electromechanical behavior was presented in context of its constitutive relationships. The general form of the non-linear constitutive relationships for an electrostrictor was placed in terms of quadratic relationships and in terms of hyperbolic tangent squared relationships. The constitutive relations were introduced into a generalized form of Hamilton's Principle. Introducing an assumed mode formulation allowed these equations to be succinctly expressed in terms of an actuator equation and a sensor equation. The actuator equation describes the dynamics of the mechanical system, the sensor equation describes the dynamics of the electrical system and these equations are coupled by the electromechanical terms. The equations of motion in assumed modes form could be utilized in a finite elements solution scheme or in a Rayleigh–Ritz solution, since the general forms of the two techniques are similar. Although the derivation was applied to the case of an electrostrictor, the derivation is far more general and forms a framework around which the global dynamics of many coupled non-linear systems can be derived.

The general model for an electrostrictively coupled system was applied to the special case of an electrostrictive wafer (0·9 PMN – 0·1 PT) mounted on a cantilevered beam. The dynamics of the electrostrictively coupled system were predicted based upon the separate dynamics of the cantilevered beam and of the electrostrictor. There was solid agreement between the predicted and experimentally measured strain and electrical displacement over a wide range of frequencies and voltages. The quadratic non-linearity of the electrostrictor can lead to a superharmonic resonance, in which case an excitation at half of the natural frequencies leads to a resonance at the natural frequency. The superharmonics greatly decreased as the bias voltage was increased.

This investigation did not address the temperature sensitivity of PMN–PT. The stiffness variation with voltage was modelled, but was not included in the beam experiments. However, the formalism by which these factors could be included has been established in this paper.

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APPENDIX: NOMENCLATURE

a number of assumed structural displacement modes
b number of assumed electrical modes (i.e., the number of electrical surfaces)
B^f generalized co-ordinate conversion matrix for forces; $[a \times 6]$ matrix
B^q generalized co-ordinate conversion matrix for charges at electrodes; $[b \times 3]$ matrix
$\mathbf{C}$ stiffness matrix; $[6 \times 6]$ matrix
$\mathbf{C}^d$ damping matrix; $[6 \times 6]$ matrix
$\mathbf{D}$ electrical displacement vector; $[3 \times 1]$ vector
$\mathbf{E}$ electric field vector; $[3 \times 1]$ vector
ε dielectric of the electrostrictor; $[3 \times 3]$ matrix
$\mathbf{f}$ vector of applied forces; $[6 \times 1]$ vector
F^* hyperbolic field term; tensor
$\mathbf{G}$ higher order electrostrictive charge storage matrix; $[b \times b]$ matrix
H enthalpy of the system; scalar
$\mathbf{I}$ identity matrix; $[a \times a]$ matrix
k hyperbolic relaxation factor; scalar
$\mathbf{K}$ stiffness matrix; $[a \times a]$ matrix
$\mathbf{L}^u$ linear differential operator for elasticity; $[6 \times 6]$
$\mathbf{L}^v$ differential operator for electricity; $[3 \times 1]$ vector
$\mathbf{M}$ mass matrix; $[a \times a]$ matrix
m electrostrictive material constants relating electric field with strain; scalar
$\mathbf{m}^*$ electrostrictive coupling term—this includes a electric field term; $[3 \times 6]$ matrix
$\mathbf{N}^r$ modal vector of mechanical strains; $[6 \times a]$ matrix
$\mathbf{N}^v$ modal vector of electric fields; $[3 \times b]$
v Poisson ratio
ψ^r modal vector of mechanical displacements; $[6 \times a]$ matrix
ψ^v modal vector of voltages; $[3 \times b]$ matrix
φ electrical potential; scalar
$\mathbf{q}$ applied electrode charges; $[3 \times 1]$ vector
$\mathbf{Q}$ electrostrictive capacitance matrix; $[b \times b]$ matrix
Θ electromechanical coupling matrix; $[a \times b]$ matrix
$\tilde{\Theta}$ electric field correction to the electromechanical coupling term; tensor
r modal amplitudes of physical displacement; $[a \times 1]$ vector—or elastostrictive coupling constants; tensor
R elastostriction coupling term; tensor
ρ density; scalar
$\mathbf{s}$ compliance matrix; $[6 \times 6]$ matrix
$\mathbf{S}$ strain vector; $[6 \times 1]$ vector
t time or ply thickness; scalar
$\mathbf{T}$ stress vector: $[6 \times 1]$ vector
T^* complimentary kinetic energy; scalar
$\mathbf{u}$ vector of mechanical displacements; $[6 \times 1]$ matrix
U potential energy; scalar
$\mathbf{v}$ vector of generalized electric coordinates; $[b \times 1]$ vector
V volume; scalar
W_e^* complimentary electrical energy; scalar
W_m magnetic energy; scalar
W externally applied work; scalar
ω natural frequency; scalar
ζ damping ratio; scalar
ξ dummy variable indicating the state of the system along an arbitrary path; scalar
$\tilde{\xi}$ value of the dummy variable at the end of the path and reflecting the current state of the system; scalar

Superscripts

a pertains to the actuator
E value measured at constant electric field
s pertains to the structure
t transpose
T value measured at constant stress
-1 inverse
$(\,)'$ derivative with respect to ξ
$*$ includes an electric field term

Subscripts

$i, j, k, l, m, n, p, q, r, s, t, u$ tensor indices