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VIBRATION OF RECTANGULAR PLATES BY REDUCTION OF THE PLATE PARTIAL DIFFERENTIAL EQUATION INTO SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Exact solutions for the vibration modes of thin uniform isotropic rectangular plates exist only when at least one pair of opposite sides is simply supported. For other support conditions, approximate methods such as the Rayleigh-Ritz method can be used to obtain the approximate vibratory modes in terms of shape functions which satisfy at least the geometric boundary conditions. Several studies on the vibration of rectangular plates with various types of boundary conditions are reported by Leissa [1, 2].

Kantorovich and Krylov [3] proposed a variational method to determine an approximate closed form solution of a partial differential equation by reducing it to an ordinary differential equation. In this method, the approximate vibration mode is assumed as the product of an unknown function is one spatial variable and a known function of both variables, and the plate partial differential equation is reduced to an ordinary differential equation in the unknown function. Kerr [4] observed that this ordinary differential equation is amenable to numerical solution when the approximate solution is restricted to a separable function in the spatial variables. Jones and Milne [5] used such an extension of the Kantorovich method alternatively in the principal directions to obtain a convergent separable solution for the vibration mode of clamped rectangular plates. Bhat, Singh and Mundkur [6] obtained expressions for the averaged natural boundary conditions and used the extended Kantorovich method successively in alternating principal directions until convergence, to obtain the natural frequencies and sets of plate characteristic functions for rectangular plates with different boundary conditions. In references [5, 6] an approximate natural frequency value is also used in the algorithm for the determination of the unknown function from the known function.

In the present study, the extended Kantorovich approach is modified to determine the optimum separable solutions of the partial differential equation describing the plate vibrations. In the modification, the variational method is used to reduce the partial differential equation into two simultaneous ordinary differential equations which are solved exactly in terms of four unknown modal parameters. Four algebraic equations necessary for the solution of these unknown parameters are then derived from the ordinary differential equations and their boundary conditions. The theoretical and numerical aspects pertaining to the application of the generalized Kantorovich approach is illustrated for the special case of clamped rectangular plates. The corresponding natural frequencies and plate characteristic functions, which are very good approximations for the plate mode shapes, are presented and discussed.

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2. ANALYSIS FOR PLATES WITH CLASSICAL SUPPORT CONDITIONS

The dynamic problem of plate vibration can be treated as an equivalent static problem where the plate inertia is represented by a load intensity function, $q = m\omega^2 w$. Under classical support conditions, the virtual work done by the edge forces and moments corresponding to an arbitrary constraint-true virtual displacement vanishes. The principle of virtual work gives the equation governing the vibration mode of the rectangular plate, shown in Figure 1, as

$$\delta(U) = \int_0^b \int_0^a m\omega^2 w \delta(w)\, \mathrm{d}x\, \mathrm{d}y, \tag{1}$$

where

$$\delta(U) = \delta\left(\frac{1}{2} D \int_0^b \int_0^a [\{w_{,xx} + w_{,yy}\}^2 - 2(1-\nu)\{w_{,xx} w_{,yy} - w_{,xy}^2\}]\, \mathrm{d}x\, \mathrm{d}y\right). \tag{2}$$

Here the subscript $_{,x}$, etc., in equation (2) denotes derivatives with respect to x, etc., and the symbol $\delta()$ indicates the variation in the variable () for constraint-true virtual displacement $\delta(w)$ of the plate mode. Repeated integration by parts reduces equation (2) to

$$\begin{aligned}\delta(U) = {} & \int_0^b \int_0^a D\{w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\}\delta(w)\, \mathrm{d}x\, \mathrm{d}y + \int_0^b [D\{w_{,xx} + \nu w_{,yy}\}\delta(w_{,x})]_{x=0}^{x=a}\, \mathrm{d}y \\ & + \int_0^a [D\{w_{,yy} + \nu w_{,xx}\}\delta(w_{,y})]_{y=0}^{y=b}\, \mathrm{d}x - \int_0^b [D\{w_{,xxx} + (2-\nu) w_{,xyy}\}\delta(w)]_{x=0}^{x=a}\, \mathrm{d}y \\ & - \int_0^a [D\{w_{,yyy} + (2-\nu) w_{,xxy}\}\delta(w)]_{y=0}^{y=b}\, \mathrm{d}x + [[2(1-\nu) D w_{,xy} \delta(w)]_{x=0}^{x=a}]_{y=0}^{y=a}.\end{aligned} \tag{3}$$

Here, the notation $[f(x)]_{x=x_1}^{x=x_2}$ denotes $f(x_2) - f(x_1)$. Substitution of equation (3) into equation (1) yields the variational form of the plate vibration equation in non-dimensional form as

$$\int_0^b \int_0^a \left[\frac{1}{\alpha^2} w_{,\overline{xxxx}} + 2w_{,\overline{xxyy}} + \alpha^2 w_{,\overline{yyyy}} - \Omega^2 w\right]\delta(w)\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}$$

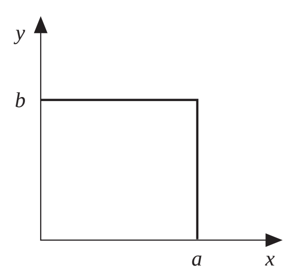

Figure 1. Rectangular plate.

$$+ \int_0^1 \left[\left\{ \frac{1}{\alpha^2} w_{,\bar{x}\bar{x}} + \nu w_{,\bar{y}\bar{y}} \right\} \delta(w_{,\bar{x}}) \right]_{\bar{x}=0}^{\bar{x}=1} \mathrm{d}\bar{y} + \int_0^1 [\{\alpha^2 w_{,\bar{y}\bar{y}} + \nu w_{,\bar{x}\bar{x}}\} \delta(w_{,\bar{y}})]_{\bar{y}=0}^{\bar{y}=1} \mathrm{d}\bar{x}$$

$$- \int_0^1 \left[\left\{ \frac{1}{\alpha^2} w_{,\bar{x}\bar{x}\bar{x}} + (2-\nu) w_{,\bar{x}\bar{y}\bar{y}} \right\} \delta(w) \right]_{\bar{x}=0}^{\bar{x}=1} \mathrm{d}\bar{y} - \int_0^1 [\{\alpha^2 w_{,\bar{y}\bar{y}\bar{y}}$$

$$+ (2-\nu) w_{,\bar{x}\bar{x}\bar{y}}\} \delta(w)]_{\bar{y}=0}^{\bar{y}=1} \mathrm{d}\bar{x} + [[2(1-\nu) w_{,\bar{x}\bar{y}} \delta(w)]_{\bar{x}=0}^{\bar{x}=1}]_{\bar{y}=0}^{\bar{y}=1} = 0. \qquad (4)$$

In thin plate theory, the shear force expressions are modified to include the effect of edge twisting moments. The last term in equation (4) represents additional corner force components which also are necessary for the accurate representation of the edge twisting moments. It is noted that this additional term vanishes for plates with a pair of supported opposite edges.

The separation of variables method has been successfully used to obtain the exact plate vibration modes for the special case of plates with a pair of simply supported opposite edges. In general, the exact vibration modes of plates with classical support conditions is not necessarily a separable function of the spatial variables. However, equation (4) can be used to obtain the best separable solution for the plate vibration mode. Substitution of $w(\bar{x}, \bar{y}) = X(\bar{x})Y(\bar{y})$, $\delta(w) = Y\delta(X) + X\delta(Y)$, $\delta(w_{,\bar{x}}) = Y\delta(X^{\mathrm{I}}) + X^{\mathrm{I}}\delta(Y)$ and $\delta(w_{,\bar{y}}) = Y^{\mathrm{I}}\delta(X) + X\delta(Y^{\mathrm{I}})$ into equation (4) yields

$$\int_0^1 \left[\frac{1}{\alpha^2} B^{(00)} X^{\mathrm{IV}} + 2\{B^{(02)} - (1-\nu) b^{(01)}\} X^{\mathrm{II}} - \{\Omega^2 B^{(00)} - \alpha^2 (B^{(04)} + b^{(12)}\right.$$

$$\left. - b^{(03)})\} X \right] \delta(X) \, \mathrm{d}\bar{x} + \int_0^1 \left[\alpha^2 A^{(00)} Y^{\mathrm{IV}} + 2\{A^{02} - (1-\nu) a^{(01)}\} Y^{\mathrm{II}} \right.$$

$$\left. - \left\{ \Omega^2 A^{(00)} - \frac{1}{\alpha^2} (A^{(04)} + a^{(12)} - a^{(03)}) \right\} Y \right] \delta(Y) \, \mathrm{d}\bar{y}$$

$$+ [\bar{M}_x \delta(X^{\mathrm{I}})]_{\bar{x}=0}^{\bar{x}=1} + [\bar{M}_y \delta(Y^{\mathrm{I}})]_{\bar{y}=0}^{\bar{y}=1} - [\bar{V}_x \delta(X)]_{\bar{x}=0}^{\bar{x}=1} - [\bar{V}_y \delta(Y)]_{\bar{y}=0}^{\bar{y}=1} = 0, \qquad (5)$$

where

$$A^{(mn)} = \int_0^1 \frac{\mathrm{d}^m X}{\mathrm{d}\bar{x}^m} \frac{\mathrm{d}^n X}{\mathrm{d}\bar{x}^n} \mathrm{d}\bar{x}, \qquad B^{(mn)} = \int_0^1 \frac{\mathrm{d}^m Y}{\mathrm{d}\bar{y}^m} \frac{\mathrm{d}^n Y}{\mathrm{d}\bar{y}^n} \mathrm{d}\bar{y}, \qquad a^{(mn)} = \left[\frac{\mathrm{d}^m X}{\mathrm{d}\bar{x}^m} \frac{\mathrm{d}^n X}{\mathrm{d}\bar{x}^n} \right]_{\bar{x}=0}^{\bar{x}=1}, \quad (6\text{–}8)$$

$$b^{(mn)} = \left[\frac{\mathrm{d}^m Y}{\mathrm{d}\bar{y}^m} \frac{\mathrm{d}^n Y}{\mathrm{d}\bar{y}^n} \right]_{\bar{y}=0}^{\bar{y}=1}, \qquad \bar{M}_x = (1/\alpha^2)[B^{(00)} X^{\mathrm{II}} + \nu B^{(02)} X],$$

$$\bar{M}_y = \alpha^2 A^{(00)} Y^{\mathrm{II}} + \nu A^{(02)} Y, \qquad (9\text{–}11)$$

$$\bar{V}_x = (1/\alpha^2)[B^{(00)} X^{\mathrm{III}} + \{(2-\nu) B^{(02)} - 2(1-\nu) b^{(01)}\} X^{\mathrm{I}}],$$

$$\bar{V}_y = \alpha^2 A^{(00)} Y^{\mathrm{III}} + \{(2-\nu) A^{(02)} - 2(1-\nu) a^{(01)}\} Y^{\mathrm{I}}. \qquad (12, 13)$$

Here, the superscript above the functions X and Y in equations (10–13) denotes the order of the derivative of the function with respect to its variable. Since equation (5) is valid for arbitrary variations $\delta(X)$ and $\delta(Y)$ which satisfy the plate boundary conditions, the plate vibration equation reduces to

$$X^{\mathrm{IV}} + \frac{2\alpha^2\{B^{(02)} - (1-\nu)b^{(01)}\}}{B^{(00)}} X^{\mathrm{II}} - \alpha^2 \left\{ \Omega^2 - \frac{\alpha^2(B^{(04)} + b^{(12)} - b^{(03)})}{B^{(00)}} \right\} X = 0, \qquad (14a)$$

$$Y^{\mathrm{IV}} + \frac{2\{A^{(02)} - (1-\nu)a^{(01)}\}}{\alpha^2 A^{(00)}} Y^{\mathrm{II}} - \frac{1}{\alpha^2} \left\{ \Omega^2 - \frac{(A^{(04)} + a^{(12)} - a^{(03)})}{\alpha^2 A^{(00)}} \right\} Y = 0. \qquad (14b)$$

Equation (14a) is a fourth order differential equation in X with constant coefficients. The coefficients of equation (14a) are related to the function Y through equations (7) and (9). A similar observation can be made about equation (14b). Thus, the plate vibration equation reduces to a pair of coupled simultaneous ordinary differential equations.

The boundary conditions for equations (14a) at $\bar{x} = 0$ or $\bar{x} = 1$ can be expressed from equations (5), (10) and (12) as (1) clamped, $X = 0$ and $X^{\mathrm{I}} = 0$; (2) simply supported, $X = 0$ and $\bar{M}_x = 0$; (3) free, $\bar{M}_x = 0$ and $\bar{V}_x = 0$. Using equation (10), the boundary conditions for the simply supported edge at $\bar{x} = 0$ or $\bar{x} = 1$ can be further simplified to $X = 0$ and $X^{\mathrm{II}} = 0$. While the expression for $\bar{M}_x$ in equation (10) agrees with that reported in reference [6], the expression for $\bar{V}_x$ in equation (12) has an additional term which can be traced to the corner forces term in equation (4). The boundary conditions for equation (14b) at $\bar{y} = 0$ or $\bar{y} = 1$ can be expressed in a similar manner.

In references [5, 6], an approximate Ω is used in addition to an assumed $Y(\bar{y})$ to determine $X(\bar{x})$ from equation (14a). Since the differential equation and its four boundary conditions are homogeneous, the complimentary function $X(\bar{x})$ with such an assumed Ω cannot satisfy all four boundary conditions. In fact, $X(\bar{x})$ and Ω corresponding to an assumed $Y(\bar{y})$ can be computed using the method described below.

Equation (14a) can be rewritten in terms of the roots $\pm p_2$ and $\pm \mathrm{j}p_1$ of its auxiliary equation as

$$X^{\mathrm{IV}} - (p_2^2 - p_1^2)X^{\mathrm{II}} - p_2^2 p_1^2 X = 0. \qquad (15)$$

Comparison of coefficients of equations (14a) and (15) yields

$$p_2^2 - p_1^2 = -[2\alpha^2\{B^{(02)} - (1-\nu)b^{(01)}\}]/B^{(00)},$$
$$\Omega^2 = (1/\alpha^2)p_2^2 p_1^2 + \{\alpha^2(B^{(04)} + b^{(12)} - b^{(03)})\}/B^{(00)}. \qquad (16, 17)$$

Imposition of the four boundary conditions on the complimentary function of equation (16) gives a relationship between the parameters p_1 and p_2 as

$$P(p_1, p_2) = 0. \qquad (18)$$

The parameters p_1 and p_2 can be determined from the algebraic equations (16) and (18) and the unknown functions which satisfy equation (14a). The boundary conditions can also be expressed within an arbitrary constant multiple. The corresponding Ω can be determined from equation (17). It must be noted that the roots of the auxiliary equation for any other presumed Ω will not satisfy equation (18).

Using equation (14b) and its boundary conditions, the following equations can be obtained in a similar manner:

$$q_2^2 - q_1^2 = -(2\{A^{(02)} - (1-\nu)a^{(01)}\})/\alpha^2 A^{(00)},$$
$$\Omega^2 = \alpha^2 q_2^2 q_1^2 + [(A^{(04)} + a^{(12)} - a^{(03)})]/\alpha^2 A^{(00)}, \qquad Q(q_1, q_2) = 0. \qquad (19\text{–}21)$$

Here $\pm jq_1$ and q_2 are the roots of the auxiliary equation associated with equation (14b). From equations (6) and (8), it can be observed that the right side of equation (19) is an implicit function of the modal parameters p_1 and p_2. Further, the parameters (q_1, q_2) corresponding to an assumed (p_1, p_2) can be determined from equations (6), (8), (19) and (21). Thus, the iterative numerical approach can be used to solve the four simultaneous nonlinear algebraic equations (16), (18), (19) and (21) for the modal parameters. The Rayleigh frequency associated with the separable vibration mode can be determined from equations (17) and (20).

3. ILLUSTRATION USING CLAMPED PLATES

The solution of equation (14a) for clamped edge conditions is either symmetric or antisymmetric about $\bar{x} = 1/2$. Imposing a zero edge displacement condition, the solutions of equation (14a) can be expressed as

$$X(\bar{x}) = \left\{\frac{\cosh(\bar{x}-1/2)p_2}{\cosh p_2/2} - \frac{\cos(\bar{x}-1/2)p_1}{\cos p_1/2}\right\}, \qquad \text{symmetric;}$$

$$= \left\{\frac{\sinh(\bar{x}-1/2)p_2}{\sinh p_2/2} - \frac{\sin(\bar{x}-1/2)p_1}{\sin p_1/2}\right\}, \qquad \text{antisymmetric.} \tag{22}$$

Incorporating the condition $X^{\mathrm{I}}(0) = 0$ in equation (22), the relationship between p_1 and p_2 can be expressed as

$$P(p_1, p_2) = 0, \tag{23}$$

where

$$P(p_1, p_2) = p_2 \tanh p_2/2 + p_1 \tan p_1/2, \qquad \text{symmetric;}$$

$$= (1/p_2)\tanh p_2/2 - (1/p_1)\tan p_1/2, \qquad \text{antisymmetric.} \tag{24}$$

Substitution of equation (22) into equations (6) and (8) and simplification using equations (23, 24) gives

$$a^{(01)} = 0, \qquad A^{(02)} = \tfrac{1}{2}(p_2^2 - p_1^2) + A_*, \qquad A^{(00)} = 1 - [(p_2^2 - p_1^2)/2p_2^2p_1^2]A_*, \tag{25–27}$$

where

$$A_* = 1 - \{1 - p_2 \tanh p_2/2\}^2, \qquad \text{symmetric;}$$

$$= 1 - \{1 - p_2/\tanh p_2/2\}^2, \qquad \text{antisymmetric.} \tag{28}$$

Equation (19) can be simplified using equations (25)–(28) as

$$q_2^2 - q_1^2 = (2/\alpha^2)A(p_1, p_2), \tag{29}$$

where

$$A(p_1, p_2) = \{A_* + (1/2)(p_2^2 - p_1^2)\}/\{[(p_2^2 - p_1^2)/2p_2^2p_1^2]A_* - 1\}. \tag{30}$$

A similar analysis on the solution $Y(\bar{y})$ of equation (14b) gives results similar to equations (23) and (29) in the form

$$Q(q_1, q_2) = 0, \qquad p_2^2 - p_1^2 = 2\alpha^2 B(q_1, q_2). \tag{31, 32}$$

For the present case where all four edges of the plate are clamped, the following results are valid:

$$Q(q_1, q_2) = P(q_1, q_2), \qquad B(q_1, q_2) = A(q_1, q_2). \tag{33, 34}$$

Thus, the functions Q and B can be reconstructed from equations (23), (24), (28) and (30) by replacing p_1 and p_2 with q_1 and q_2. The four unknowns p_1, p_2, q_1 and q_2 can be determined from equations (23), (29), (31) and (32), and the frequency associated with the approximate vibration mode can be calculated from equations (17) and (20).

4. NUMERICAL ANALYSIS

The numerical calculations for the determination of the unknowns p_1, p_2, q_1 and q_2 can be simplified by sub-dividing equations (23), (29), (31) and (32) into two groups of two equations each. Equations (23) and (32) can be used to determine p_1 and p_2 from the assumed values of q_1 and q_2. From p_1 and p_2 thus obtained, a better approximation for q_1 and q_2 can be determined using equations (29) and (31). Thus, equations (23), (32) and (29) and (31) can be applied alternatively to compute the unknowns to the desired accuracy from an assumed first approximation.

Equations (24) and (30) show that if (p_1, p_2) is a solution of equations (23) and (32) then $(-p_1, p_2)$, $(p_1, -p_2)$ and $(-p_1, p_2)$ are also solutions of these equations. Further, equation (22) indicates that the sign of p_1 or p_2 does not have any influence on the expressions for the vibration mode and the associated frequency. Thus, without loss of generality, the solutions for the unknowns p_1, p_2, q_1 and q_2 can be assumed to be positive.

When $B = 0$, the solutions of equations (23) and (32) satisfy the condition $p_1 = p_2$ corresponding to a beam mode in the x direction. For a small value of B, this x-beam mode solution can be used as a first approximation in an iterative numerical scheme to obtain the accurate solution (p_1, p_2). The solution (p_1, p_2) for this small value of B can be used to obtain the solution for a larger value of B. Thus, the dependence of p_1 (or p_2) on B is multi-valued with a branch corresponding to each of the x-beam modes. Similarly, the dependence of q_1 (or q_2) on A is also multi-valued with a branch associated with each y-beam mode. Thus, the mth x-beam mode parameters (p_1, p_2) and the nth y-beam mode parameters (q_1, q_2) can be used to generate a solution of equations (23), (29), (31) and (32) and this solution can be labeled as the (m, n)th separable solution. From the computational point of view, it is advantageous to use the (p_1, p_2) parameters of $(m+1, n)$th solution and the (q_1, q_2) parameters of $(m, n+1)$th solution as the first approximation to determine the $(m+1, n+1)$th solution parameters.

Iterative schemes for the determination of (p_1, p_2) from equations (23) and (32) and (q_1, q_2) from equations (29) and (31) are similar. The steps for the computation of (p_1, p_2) from equations (23) and (32) are given below:

(1) Equation (23) can be considered as an implicit definition of p_2 as a function of p_1. Thus, the Newton-Raphson method is used to determine p_2 corresponding to an assumed p_1. The derivative $(\mathrm{d}p_2/\mathrm{d}p_1)$ is then evaluated from $-\{(\partial P/\partial p_1)/(\partial P/\partial p_2)\}$.

(2) Since p_2 is a known function of p_1, equation (32) can be regarded as an equation in the single variable p_1. Thus, the Newton-Raphson method is used to improve the assumed value of p_1 for step (1).

(3) By repeating steps (1) and (2), (p_1, p_2) can be determined to the desired accuracy. This scheme is reasonably quick and is used to obtain (p_1, p_2) to six significant figure accuracy.

TABLE 1

SS mode parameters, $a/b = 1{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
1	1	4·31207	6·52611	4·31207	6·52611	35·99896
1	3	3·63208	14·51211	10·91003	11·88584	131·90213
3	1	10·91003	11·88584	3·63208	14·51211	131·90213
3	3	10·50880	17·45161	10·50880	17·45161	220·05865
1	5	3·43522	23·22999	17·24404	17·85340	309·03784
5	1	17·24404	17·85340	3·43522	23·22999	309·03784
3	5	10·19664	25·09587	17·02905	21·91703	393·35577
5	3	17·02905	21·91703	10·19664	25·09587	393·35577
5	5	16·77691	28·34210	16·77691	28·34210	562·17816
1	7	3·34988	32·05025	23·54339	23·98423	565·45203
7	1	23·54339	23·98423	3·34988	32·05025	565·45203
3	7	10·00713	33·39119	23·41529	27·12809	648·02057
7	3	23·41529	27·12809	10·00713	33·39119	648·02057
5	7	16·57370	35·86621	23·23189	32·51634	813·74701
7	5	23·23189	32·51634	16·57370	35·86621	813·74701

A second scheme for the determination of (p_1, p_2) from equations (23) and (32) can be formulated by exchanging the roles of p_1 and p_2 in steps (1) and (2) of the earlier scheme. For an assumed p_2, equation (23) is satisfied by several values of p_1. Thus, this second scheme is useful only when the assumed first approximation is reasonably accurate. This second scheme is used to refine the results obtained from the first scheme to twelve significant figure accuracy, especially when m or n is large. The complete numerical scheme to determine p_1, p_2, q_1 and q_2 from equations (23), (29), (31) and (32) can be summarized in the following steps.

(1) Assume a first approximation for (p_1, p_2, q_1, q_2). The assumed solution must satisfy equations (23) and (31).

(2) Compute B using equations (28), (30) and (34). Solve for (p_1, p_2) from equations (23) and (32) using previously enunciated steps.

TABLE 2

SA mode parameters, $a/b = 1{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
1	2	3·85829	10·30263	7·69024	9·06295	73·40536
3	2	10·71560	14·23073	7·13560	15·71577	165·02304
1	4	3·50981	18·84785	14·08521	14·83675	210·52634
3	4	10·33396	21·14452	13·79507	19·55599	296·36633
5	2	17·15200	19·47662	6·83936	23·95715	340·59042
1	6	3·38539	27·63413	20·39557	20·90734	427·35699
5	4	16·89924	24·93426	13·50673	26·57787	467·29092
3	6	10·09026	29·19721	20·23176	24·45830	510·64716
7	2	23·49144	25·20896	6·68834	32·56338	596·36670
5	6	16·66803	32·01364	20·01603	30·33620	677·74500
7	4	23·32573	29·60673	13·30234	34·50278	720·48639
1	8	3·32332	36·47358	26·68914	27·07617	723·30811
3	8	9·94113	37·64561	26·58665	29·89135	805·35010
7	6	23·14005	35·75067	19·82320	37·45086	927·70618
9	2	29·80062	31·15732	6·60015	41·29623	931·50360

TABLE 3

AS mode parameters, $a/b = 1{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
2	1	7·69024	9·06295	3·85829	10·30263	73·40535
2	3	7·13560	15·71577	10·71560	14·23073	165·02303
4	1	14·08521	14·83675	3·50981	18·84785	210·52634
4	3	13·79507	19·55599	10·33396	21·14452	296·36632
2	5	6·83936	23·95715	17·15200	19·47662	340·59043
6	1	20·39557	20·90734	3·38539	27·63413	427·35700
4	5	13·50673	26·57787	16·89924	24·93426	467·29093
6	3	20·23176	24·45830	10·09026	29·19721	510·64717
2	7	6·68834	32·56338	23·49144	25·20896	596·36670
6	5	20·01603	30·33620	16·66803	32·01364	677·74498
4	7	13·30234	34·50278	23·32573	29·60673	720·48641
8	1	26·68914	27·07617	3·32332	36·47359	723·30810
8	3	26·58664	29·89135	9·94113	37·64562	805·35007
6	7	19·82320	37·45086	23·14004	35·75067	927·70618
2	9	6·60015	41·29622	29·80063	31·15732	931·50358

(3) Compute A using equations (28) and (30). Solve for (q_1, q_2) from equations (29) and (31).

(4) Repeat steps (2) and (3) until convergence.

5. RESULTS AND DISCUSSION

Computation of the modal parameters corresponding to the four mode categories of clamped rectangular plates were carried out on a VAX 6410 digital computer in quadruple precision for two aspect ratios. Ten beam-mode parameters each in the x and y directions were used as a first approximation to generate the plate mode parameters. The parameters p_1, p_2, q_1 and q_2 were computed to twelve significant figure accuracy. For each of the eight cases, the 10×10 plate modes were arranged in ascending frequency order. The

TABLE 4

AA mode parameters, $a/b = 1{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
2	2	7·38664	12·00118	7·38664	12·00118	108·23591
2	4	6·96057	19·75927	13·95485	16·76274	242·66710
4	2	13·95485	16·76274	6·96057	19·75927	242·66710
4	4	13·64092	22·89778	13·64092	22·89778	371·37587
2	6	6·75266	28·23656	20·32765	22·30505	458·53113
6	2	20·32765	22·30505	6·75266	28·23656	458·53113
4	6	13·39470	30·47086	20·12394	27·18754	583·74861
6	4	20·12394	27·18754	13·39470	30·47086	583·74861
2	8	6·63903	36·91992	26·64822	28·16470	754·03538
8	2	26·64822	28·16470	6·63903	36·91992	754·03538
6	6	19·91481	33·78547	19·91481	33·78547	792·46216
8	4	26·51179	32·15353	13·22613	38·62890	877·32942
4	8	13·22613	38·62890	26·51179	32·15353	877·32942
6	8	19·74186	41·27373	26·34828	37·87819	1083·30163
8	6	26·34828	37·87819	19·74186	41·27373	1083·30163

TABLE 5

SS mode parameters, $a/b = 2{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
1	1	3·83571	10·60528	4·60406	5·18945	49·16178
3	1	10·69992	14·44429	4·07433	8·09844	89·55766
5	1	17·14487	19·60894	3·74278	12·07412	174·52268
1	3	3·37906	28·32502	10·97416	11·21198	246·49784
3	3	10·07653	29·81874	10·82729	12·82192	284·75485
7	1	23·48800	25·29246	3·57233	16·32985	301·10984
5	3	16·65426	32·53217	10·62167	15·57750	363·58347
9	1	29·79877	31·21323	3·47457	20·67663	467·97195
7	3	23·12911	36·16556	10·42863	19·00240	484·23238
1	5	3·28391	46·07195	17·27020	17·41859	601·88407
3	5	9·83761	46·98052	17·20625	18·50137	640·28348
9	3	29·53526	40·46887	10·27136	22·79871	646·50249
11	1	36·09669	37·25675	3·41223	25·06186	674·66763
5	5	16·35568	48·72444	17·09809	20·50421	717·98453
7	5	22·83003	51·19961	16·97115	23·19854	836·00797

modal parameters of the first fifteen modes in each of the eight cases are shown in Tables 1 to 8.

The present approach reduces the determination of the plate characteristic function to the solution of four non-linear algebraic equations in four unknown parameters. In the numerical solution, these four equations are treated as two sets of two equations and these equation sets are solved successively using a chosen first approximation. In contrast to the numerical schemes reported in references [5, 6], the present scheme does not assume an Ω in each stage of the computation. The solutions of p_1 and p_2 corresponding to an assumed Ω in equation (14a) need not satisfy the four homogenous boundary conditions for the function X. Thus, the computed (p_1, p_2) must be corrected to satisfy equation (23). Such an approach is relatively cumbersome and may not converge to the desired solution.

TABLE 6

SA mode parameters, $a/b = 2{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
1	2	3·49667	19·48790	7·81087	8·15039	127·96933
3	2	10·31428	21·64137	7·55343	10·24827	166·56249
5	2	16·88523	25·29339	7·26745	13·55532	247·39098
7	2	23·31698	29·86327	7·05303	17·40919	370·71005
1	4	3·31962	37·19528	14·12431	14·30710	404·45883
3	4	9·93189	38·32722	14·03109	15·60436	442·75280
5	4	16·48167	40·45787	13·88396	17·93395	520·76024
9	2	29·68145	34·98558	6·90433	21·51153	535·37883
7	4	22·96418	43·41691	13·72563	20·96665	639·73489
11	2	36·01245	40·44707	6·79977	25·73784	740·58636
9	4	29·39119	47·04384	13·58051	24·44667	800·04401
1	6	3·26011	54·95205	20·41425	20·53912	838·77830
3	6	9·77207	55·70959	20·36783	21·46674	877·28642
5	6	16·26213	57·18091	20·28585	23·21588	954·90152
13	2	42·32565	46·12478	6·72368	30·03255	985·89276

TABLE 7

AS mode parameters, $a/b = 2{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
2	1	7·36490	12·26256	4·33332	6·41160	63·66675
4	1	13·94426	16·93155	3·88015	10·02630	126·67985
6	1	20·32277	22·40947	3·64460	14·18444	232·73244
2	3	6·74154	28·89947	10·91433	11·83934	260·70688
4	3	13·38030	31·04270	10·72610	14·09011	318·97338
8	1	26·64572	28·23251	3·51744	18·49644	379·54424
6	3	19·90233	34·25081	10·52123	17·22901	418·67292
8	3	26·33888	38·24685	10·34532	20·86650	560·20152
10	1	32·94877	34·22378	3·44026	22·86600	566·35451
2	5	6·56416	46·41598	17·24532	17·83190	616·24084
4	5	13·10191	47·75433	17·15605	19·40184	674·15047
10	3	32·72116	42·80948	10·20609	24·78267	743·05843
6	5	19·59835	49·87735	17·03555	21·77898	771·90762
12	1	39·24317	40·30693	3·38894	27·26242	792·89531
8	5	26·05128	52·67801	16·90700	24·73851	910·33816

The present approach is found to be satisfactory for the determination of several hundred vibration modes. At high frequency, the ratio of the inner-nodal distance of the vibration mode to the plate thickness may not be large enough to ensure the validity of the thin plate theory. Under such conditions a higher order plate theory must be used for the vibration analysis and the thin plate results can be used as the first approximation in the numerical calculations. Evaluation of response of plates to vibratory excitations requires quite accurate values for natural frequency and mode shapes. Even the best of finite element methods cannot provide accurate values for higher natural frequencies and mode shapes and hence response evaluation using them is prone to error. The present method is an analytical method providing quite accurate results even for higher modes and can be used to verify the results from approximate or numerical techniques. Further, it must be noted that the approximate separable solutions of the plate vibration equation

TABLE 8

AA mode parameters, $a/b = 2{\cdot}0$

i	j	p_1	p_2	q_1	q_2	Ω
2	2	6·94114	20·33248	7·69950	8·98902	142·16255
4	2	13·62376	23·32185	7·40363	11·80291	201·60630
6	2	20·11282	27·49086	7·15069	15·43855	303·81229
2	4	6·63213	37·62612	14·08744	14·80601	418·74574
8	2	26·50491	32·37144	6·97190	19·43947	447·93733
4	4	13·21549	39·27838	13·96126	16·66154	476·69391
6	4	19·73078	41·84426	13·80419	19·38071	575·08446
10	2	32·84989	37·68315	6·84768	23·61362	632·94919
8	4	26·18368	45·15625	13·65064	22·66293	714·72761
2	6	6·51810	55·23803	20·39637	20·89156	853·19171
12	2	39·17067	43·26428	6·75888	27·87877	858·24524
10	4	32·58854	49·06267	13·51580	26·29999	895·64650
4	6	13·02045	56·35994	20·33033	22·24817	911·15510
6	6	19·49641	58·16360	20·23641	24·34847	1008·62498
12	4	38·95911	53·43354	13·40265	30·16293	1117·52724

TABLE 9

Comparison of $\Omega = \omega ab(m/D)^{1/2}$ for square clamped plate

	Modes				
	(1, 1)	(1, 2)/(2, 1)	(2, 2)	(1, 3)/(3, 1)	(2, 3)/(3, 2)
Reference [2]†	35·992	73·413	108·27	131·64	–
Reference [5]	35·999	73·405	108·236	131·902	165·023
Reference [6]	35·999	73·405	108·236	131·902	165·023
Reference [7]†	35·986	73·395	108·218	131·779	–
Present	35·999	73·405	108·236	131·902	165·023

† Beam characteristics functions used in the Rayleigh-Ritz method.
‡ Characteristic orthogonal polynomials used in the Rayleigh-Ritz method.

satisfy the exact geometric boundary conditions of the plate. A more accurate solution of the plate vibration equation can be obtained by using these plate characteristic functions as shape functions in the Rayleigh-Ritz method. Since these plate characteristic functions are reasonable approximations of the plate vibration modes, they are a better choice for the shape functions than those reported in the literature [2, 7]. Even though the zeros of the plate characteristic functions fall along straight lines parallel to the edges, the linear combination of these functions can produce curved nodal lines also.

A comparison of the present results with published data for the case of a clamped square plate is given in Table 9. The present method which does not assume an Ω for each iteration is more suitable for the determination of a large number of the higher vibration modes of plates with the other combinations of the classical boundary conditions. The Rayleigh-Ritz method involves the numerical solution of a higher order matrix eigenvalue problem, which is prone to numerical difficulties, unless the deflection shapes are chosen judiciously. The plate characteristic functions developed here can be conveniently used as deflection shapes in the Rayleigh-Ritz method in order to improve the results still further.

6. CONCLUSIONS

The optimum separable solutions of the plate vibration equation are obtained by reducing the plate vibration equation into simultaneous ordinary differential equations. Imposition of the boundary conditions on the reduced equations results in four non-linear algebraic equations. The numerical solution scheme for these equations is described. The results for the plate characteristic function parameters are presented for clamped rectangular plates. The present method yielded accurate results for the parameters of several hundred natural frequencies and plate characteristic functions.

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APPENDIX: NOTATION

a	Plate length in x direction
$a^{(mn)}$	Defined in equations (8) and (25)
A	Defined in equations (28) and (30)
$A^{(mn)}$	Defined in equation (6)
b	Plate width in y direction
$b^{(mn)}$	Defined in equation (9)
B	Defined in equations (28), (30) and (34)
$B^{(mn)}$	Defined in equation (7)
D	Flexural rigidity of plate
m	Mass per unit area of plate
p_1, p_2	Mode parameters in x direction
P	Defined in equations (18), (23) and (24)
q_1, q_2	Mode parameters in y direction
Q	Defined in equations (21), (23), (24) and (33)
U	Strain energy
w	Plate displacement
x, y	Cartesian co-ordinates
$\bar{x}, \bar{y}$	Defined by $\bar{x} = x/a$, $\bar{y} = y/b$
X, Y	Functions in $w \approx X(\bar{x})Y(\bar{y})$
α	Plate aspect ratio, $\alpha = a/b$
$\delta()$	Variation in ()
ω	Plate frequency
Ω	Defined by $\Omega = \omega ab(m/D)^{1/2}$