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ACCURATE FREE VIBRATION ANALYSIS OF SHEAR-DEFORMABLE PLATES WITH TORSIONAL ELASTIC EDGE SUPPORT

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The modified Superposition–Galerkin method is utilized to solve the free-vibration problem of shear deformable plates resting on uniform elastic foundations. Lateral displacement of the plate is forbidden at the boundaries. Due to elasticity in the support, edge rotation is opposed by bending moments proportional to the degree of rotation. Results of a study conducted on thick Mindlin plates are presented. The analytical procedure is found to be much more efficient than that involved in the traditional superposition method and good convergence is encountered.

1. INTRODUCTION

It is well known that actual restraint along the edges of rectangular plates often deviates from that provided by classical edge conditions traditionally denoted as simply supported, clamped and free. Nevertheless, such edge conditions are fairly easy to formulate mathematically and are therefore usually assumed to apply when investigating the static and dynamic behaviour of rectangular plates. In many cases computed results are found to be satisfactory for design purposes. There are problems however, where such simplified modelling of plate edge conditions leads to significant error. The purpose of this paper is to examine one such family of problems related to rectangular plate free vibration.

It is generally agreed that a certain amount of elasticity exists in plate edge supports. This elasticity manifests itself in two ways. First, rotation of the plate edge about an axis running along that edge, at the plate mid-surface, may no longer be zero, as it is considered to be in clamped edge conditions. Rather, it may be opposed by a moment proportional to the degree of rotation. One calls this torsional elastic edge support. Furthermore, lateral displacement along the edge of the plate may not be completely forbidden but may be opposed by forces proportional to the displacement. One calls this lateral elastic edge support. In a series of recent papers the author has conducted a free vibration analysis of thin rectangular plates with various combinations of uniform elastic support distributions along the edges [1]. Results of an even more recent study where the stiffness of the elastic edge restraint are allowed to have any arbitrary distribution along the edges have also been published [2].

All of the above results are restricted in application in that they apply to thin isotropic plates only, i.e., plates where the effects of deformation due to transverse shear forces may be neglected. It is well known that in the analysis of thick (Mindlin) plates, or even thin composite plates where resistance to transverse shear force induced deformation is relatively weak, the effects of this latter deformation cannot be neglected in analysing plate

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behaviour. This was well recognized by Reissner and Mindlin and, in fact, Mindlin subsequently published his well known formulation of the differential equations and boundary conditions generally accepted as being applicable in the study of transverse shear deformable plates [3]. He also took into account the effects of rotary inertia on plate free vibration.

In this paper a free vibration analysis is conducted of thick shear-deformable plates with rotary elastic edge support utilizing the Mindlin theory. Lateral displacement along the plate edges is forbidden. Rotary elastic stiffness is considered to be uniform along each edge.

Later it will become obvious to the reader that the same analytical approach can be exploited to handle problems where not all of the plate edges have the torsional elastic support described here. One could, for example, have two adjacent edges with elastic support, while the other two have classical simple support. Alternatively, the latter two edges could be free, or given clamped edge support.

2. MATHEMATICAL PROCEDURES

The superposition method has been widely used by the author and his colleagues in obtaining analytical type solutions for the free vibration frequencies and mode shapes of thin rectangular plates with various types of boundary conditions. More recently it has been successfully applied to the problem of analysing shear-deformable thick plates and composite plates with classical type boundary conditions [4–6].

In every case it has been found to work very well, however, the amount of work involved in exploiting it for the analysis of this latter family of plates has turned out to be excessive. For this reason the author developed, what is referred to as the Superposition–Galerkin method. This is essentially a modified superposition method which gives equally accurate results with a vastly reduced work requirement. This modified method is described in considerable detail in reference [7] and has been utilized to analyse composite plates with classical edge support in reference [8]. The analytical procedure utilized here follows very closely that described in the above reference [7] so only a brief description will be provided for the sake of completeness, with emphasis on that portion of the analysis which differs from that presented earlier.

The free vibration of a thick rectangular plate with uniform rotary elastic support along all four edges is chosen for analysis. This plate is shown schematically on the left side of Figure 1. An analysis is achieved by superimposing the four rectangular plate forced vibration problems (building blocks) shown on the right side of the figure. Each building block is driven by a distributed harmonic bending moment acting along one edge. A solution is obtained by constraining driving coefficients appearing in these building block solutions in such a way that their combined assembly satisfies the prescribed boundary conditions of the plate problem of interest.

2.1. COMPUTATION OF FIRST BUILDING BLOCK RESPONSE

The three governing differential equations as presented by Mindlin are written in dimensionless form as [4],

$$\frac{\partial^2 W}{\partial \xi^2}\frac{1}{\phi^2}\frac{\partial^2 W}{\partial \eta^2}+\frac{\partial \Psi_\xi}{\partial \xi}+\frac{1}{\phi}\frac{\partial \Psi_\eta}{\partial \eta}+\frac{\lambda^4 \phi_h^2}{v_3}W=0 \tag{1}$$

$$\frac{\partial^2 \Psi_\xi}{\partial \xi^2}+\frac{v_1}{\phi^2}\frac{\partial^2 \Psi_\xi}{\partial \eta^2}+\frac{v_2}{\phi}\frac{\partial^2 \Psi_\eta}{\partial \xi \partial \eta}-\frac{v_3}{\phi_h^2}\left(\Psi_\xi+\frac{\partial W}{\partial \xi}\right)+\frac{\lambda^4 \phi_h^2}{12}\Psi_\xi=0 \tag{2}$$

and

$$\frac{\partial^2 \Psi_\eta}{\partial \xi^2} + \frac{1}{\phi^2 v_1}\frac{\partial^2 \Psi_\eta}{\partial \eta^2} + \frac{v_2}{\phi v_1}\frac{\partial^2 \Psi_\xi}{\partial \xi \partial \eta} - \frac{v_3}{\phi_h^2 v_1}\left(\Psi_\eta + \frac{1}{\phi}\frac{\partial W}{\partial \eta}\right) + \frac{\lambda^4 \phi_h^2}{12 v_1}\Psi_\eta = 0. \tag{3}$$

A list of symbols appears in the Appendix. Expressions for dimensionless shear forces, moments, etc., are given as

$$Q_\xi = \Psi_\xi + \frac{\partial W}{\partial \xi}, \qquad Q_\eta = \Psi_\eta + \frac{1}{\phi}\frac{\partial W}{\partial \eta}, \qquad M_\xi = \frac{\partial \Psi_\xi}{\partial \xi} + \frac{v}{\phi}\frac{\partial \Psi_\eta}{\partial \eta},$$

$$M_\eta = \frac{\partial \Psi_\eta}{\partial \eta} + v\phi\frac{\partial \Psi_\xi}{\partial \xi}, \qquad M_{\xi\eta} = \frac{\partial \Psi_\eta}{\partial \xi} + \frac{1}{\phi}\frac{\partial \Psi_\xi}{\partial \eta}. \tag{4}$$

The first building block has simple support along the non-driven edges. Lateral displacement is forbidden along the driven edge. The amplitude of the driving moment acting along this latter edge has a spatial distribution given in series form as

$$M_\eta = \sum_{m=1,2}^{KK} E_m \sin m\pi\xi. \tag{5}$$

In view of the prescribed boundary conditions it follows that plate lateral displacement, cross-section rotations etc., may be expressed as

$$W(\xi,\eta) = \sum_{m=1}^{KK} X_m(\eta)\sin m\pi\xi, \qquad \Psi_\xi(\xi,\eta) = \sum_{m=1}^{KK} Y_m(\eta)\cos m\pi\xi,$$

$$\Psi_\eta(\xi,\eta) = \sum_{m=1}^{KK} Z_m(\eta)\sin m\pi\xi. \tag{6}$$

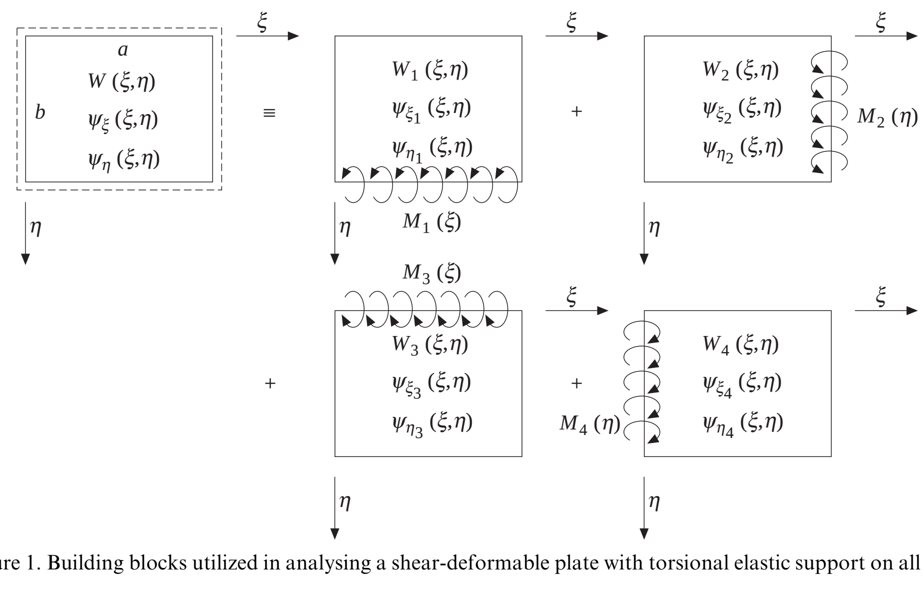

Figure 1. Building blocks utilized in analysing a shear-deformable plate with torsional elastic support on all edges.

The Galerkin method is now employed to compute the response of the first building block to any Fourier component, let one say the mth component, of the driving moment distribution. Toward this end the functions $X_m(\eta)$, $Y_m(\eta)$, etc., of equations (6) are expanded in appropriate series form. The Galerkin method requires that each term in these series must satisfy all prescribed boundary conditions along the edges $\eta = 0$ and $\eta = 1$. It is shown in reference [8] that these conditions are satisfied by the following expansions for these functions:

$$X_m(\eta) = \sum_{i=1,2}^{K} E_i \sin \mathrm{i}\pi\eta, \qquad Y_m(\eta) = \sum_{j=1,2}^{K} E_j \sin \mathrm{j}\pi\eta, \tag{7, 8}$$

$$Z_m(\eta) = \sum_{l=1,2}^{K} E_l \cos (l-1)\pi\eta + E_m \frac{\eta^2}{2}. \tag{9}$$

Equations (7)–(9) employ standard Fourier series. It is easily shown that addition of the quadratic term to equation (9) permits the moment condition to be satisfied along the driving edge. Following the Galerkin procedure, the above series are differentiated term-by-term, as required, and substituted into the governing differential equations. The resulting three quantities obtained, designated as Q_1, Q_2, and Q_3, are each expanded in an appropriate Fourier series of K terms, with each coefficient in these new Fourier expansions set equal to zero. This is in keeping with the Galerkin procedure and results in a set of three K non-homogenous algebraic equations relating the unknown coefficients E_i, E_j, etc. Solution of this set of equations provides the response of the first building block to any Fourier driving component. Sine series have been used in the first two expansions above with a cosine series used in the third so that advantage may be taken of function orthogonality. With a solution for the first building block available it will be obvious that solutions for the other three building blocks may be extracted from this first building block. The remaining three building blocks differ from the first only in that they are driven along different edges.

2.2. GENERATION OF THE EIGENVALUE MATRIX

We look at the problem of the plate with uniform rotational elastic restraint. We require four elastic stiffness constants to characterize the boundaries. These constants are denoted as k_1, k_2, k_3, and k_4, where subscript 1 pertains to the edge, $\eta = 1$, with the subscripts 2, 3, etc., pertaining to the other edges as we move counterclockwise around the plate. It follows immediately that the boundary conditions which must be satisfied by the superimposed building blocks can be written in dimensionless form as:

$$\partial\Psi_\eta/\partial\eta + K_{1R}\Psi_\eta = 0|_{\eta=1}, \qquad \partial\Psi_\xi/\partial\xi + K_{2R}\Psi_\xi = 0|_{\xi=1}, \tag{10, 11}$$

$$\partial\Psi_\eta/\partial\eta - K_{3R}\Psi_\eta = 0|_{\eta=0}, \qquad \partial\Psi_\xi/\partial\xi + K_{4R}\Psi_\xi = 0|_{\xi=0}. \tag{12, 13}$$

Here, the dimensionless torosional elastic coefficients are $K_{1R} = k_1 b/D$, $K_{2R} = k_2 a/D$, $K_{3b} = k_3 b/D$ and $K_{4R} = k_4 a/D$.

Well established procedures are followed in generating the eigenvalue matrix elements. One considers the four building blocks to be superimposed, one-upon-the-other, and begins by expanding the contribution of each building block to the left side of equation (10) in series form. Here it is advantageous to use a sine series. It is required that each net coefficient in this new series should equal zero. Using KK terms in this series, KK homogenous algebraic equations relating the four KK unknown Fourier coefficients used in representing the driving moments are generated. An identical procedure is followed to enforce the moment-edge rotation equilibrium conditions along the other three edges. This gives rise to a total of four KK homogenenous algebraic equations relating the four KK unknown coefficients. The coefficient matrix of this set of equations forms the present Eigenvalue matrix. Eigenvalues are those values of the parameter λ^2 which cause the determinant of the matrix to vanish. With eigenvalues established, mode shapes associated with the eigenvalues are obtained after setting one of the non-zero Fourier driving coefficients equal to unity and solving for the others. It is discussed in reference [6] how problems of this type may permit one half of the eigenvalues matrix to be extracted from the other. Caution must be used so that correct changes of sign are introduced, when required, on building up the second half of the matrix.

3. PRESENTATION OF RESULTS

It will be immediately appreciated by the reader that it is not possible to present enough computed results to cover all possible design needs. For that reason data will be presented for first mode free vibration of square plates only and for two specific values of thickness-to-edge length ratios, ϕ_h. Furthermore, equal elastic stiffness ratios will be used on all edges with elastic support.

In Figure 2 first mode eigenvalues are plotted against the rotational elastic coefficient, K_{IR}, for a plate with equal elastic support along all four edges. Values of ϕ_h, utilized are 0·01, corresponding to a fairly thin plate, and 0·10, corresponding to a much thicker plate. All results are based on the thick plate theory. In all such problems there are frequency limits which one must approach as the torsional coefficient approaches values of zero or infinity. Here, these frequency limits are, of course, those of the simply supported plate, and the fully clamped plate, respectively. All limits were computed utilizing the computational procedure described in reference [7]. Convergence studies have shown that

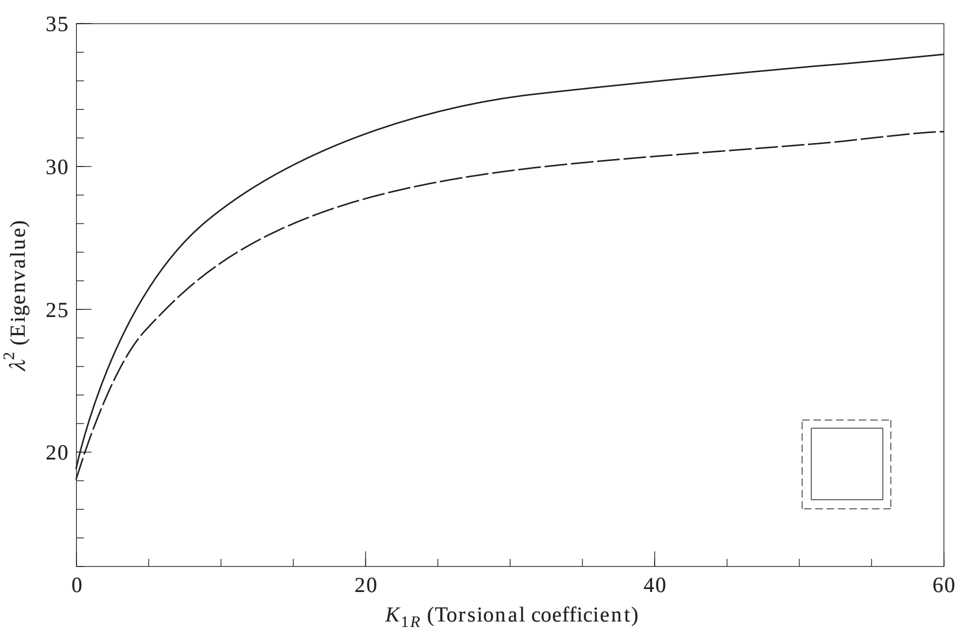

Figure 2. Eigenvalues versus torsional coefficient K_{1R} for square plate with uniform elastic support on all four edges (lower range of K_{IR}). ϕ_h: ——— 0·01; — — — 0·10.

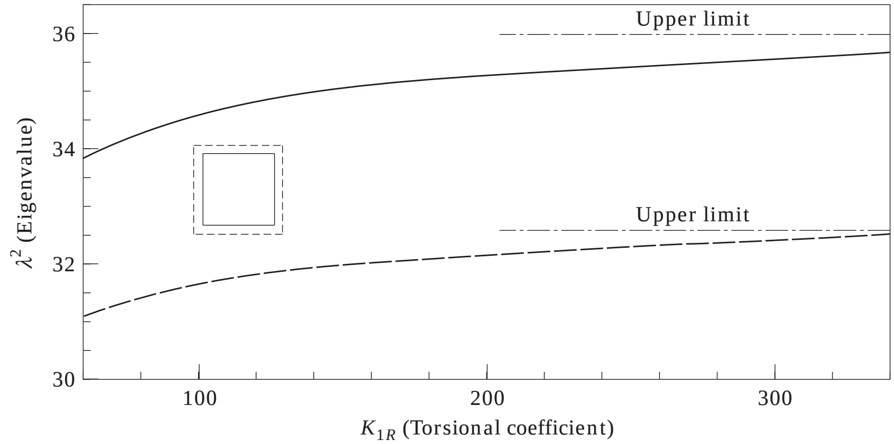

Figure 3. Eigenvalues versus torsional coefficient K_{1R} for square plate with uniform elastic support on all four edges (upper range of K_{IR}). Key as Figure 2.

values of KK = 7, and K = 23, give four digit accuracy in computed eigenvalues. These values for series summations have been used in all computations conducted for studies reported here. The reader will appreciate that increasing the quantity K gives a better series representation of the true building block solution. Increasing the quantity KK improves the degree to which the boundary conditions are satisfied. Both of these parameters should be adjusted upward if higher convergence is required.

Lower limits for all eigenvalue curves presented here are 19·73, for a value of $\phi_h = 0{\cdot}01$, and 19·05, for $\phi_h = 0{\cdot}10$. The lower limit based on thin plate theory is, of course, $2\pi^2$ or 19·74. As expected, Eigenvalue limits computed here are slightly below this with much greater difference encountered for the thicker plate. It will be recalled that in classical thin plate theory the plate transverse shear stiffness is essentially considered to be infinite and effects of rotary inertia are neglected.

The curves of Figure 2 are seen to rise rapidly at first, indicating that free vibration frequencies are very sensitive to torosional coefficients in the range $0{\cdot}0 < K_{1R} < 20$. From there on the plate fundamental frequency is much less sensitive to such change. In

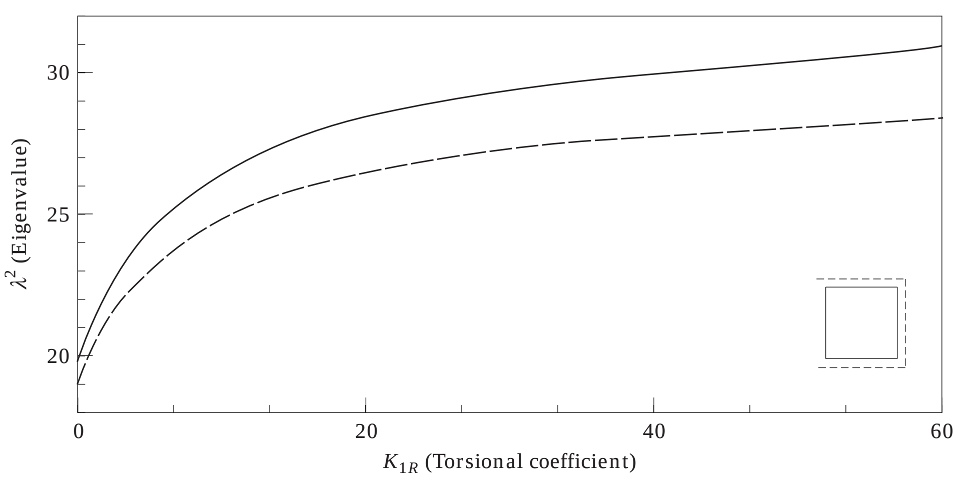

Figure 4. Eigenvalues versus torsional coefficient K_{1R} for square plate with uniform elastic support on all three edges (lower range). Key as Figure 2.

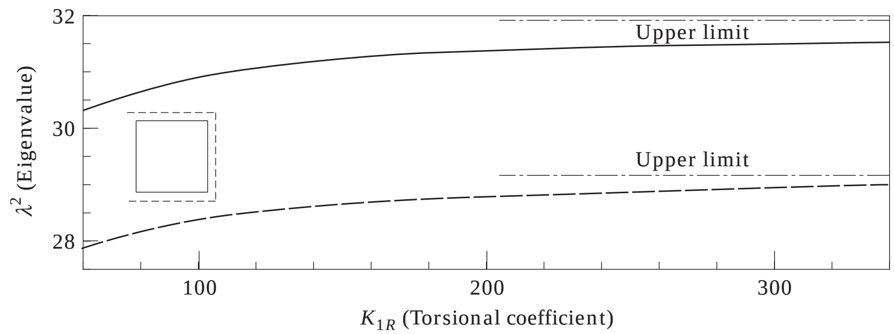

Figure 5. Eigenvalues versus torsional coefficient K_{1R} for square plate with uniform elastic support on three edges (upper range). Key as Figure 2.

Figure 3 the same curves continue on with a higher range of edge stiffness. It is seen that both curves approach their computed upper limits as the edge stiffness becomes arbitrarily large. The upper curves in the figures are of less immediate interest as they closely follow the expected trajectory based on thin plate theory. They serve a useful purpose, however, in that they allow one to gauge the magnitude of the error that would be encountered if one failed to use thick plate theory in computing fundamental frequencies for the thick shear-deformable plate.

It will be appreciated now that one can easily compute eigenvalues for plates with less than four elastically supported edges, as discussed earlier. One need only delete from the eigenvalue matrix the contribution associated with the non-elastically supported edge. Such edges will have simple support. It will be apparent that if these latter edges are to be clamped or free the analyst must choose building blocks with slightly different boundary conditions.

In Figures 4 and 5 computed results are presented for first mode vibration of a square plate with elastic support along three edges. Similar behaviour is observed, with different limits approached at the upper ends of the curves, of course. In Figure 6 results are presented for a square plate with two adjacent edges given elastic support. It is found that for this plate, and the following one, the approach toward known upper limits is considerably more rapid as the torsional coefficient is increased. For this reason results are presented for a range of K_{IR} from 0·0–60·0, only. The final data, presented in Figure 7,

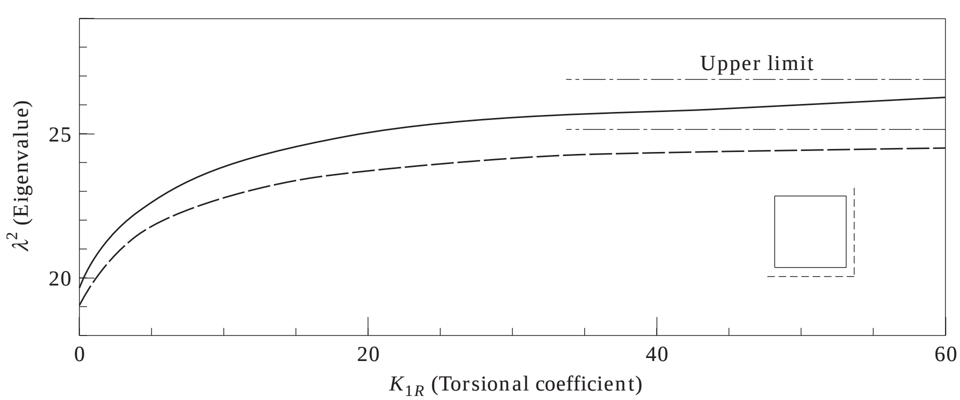

Figure 6. Eigenvalues versus torsional coefficient K_{1R} for square plate with uniform elastic support on two adjacent edges. Key as Figure 2.

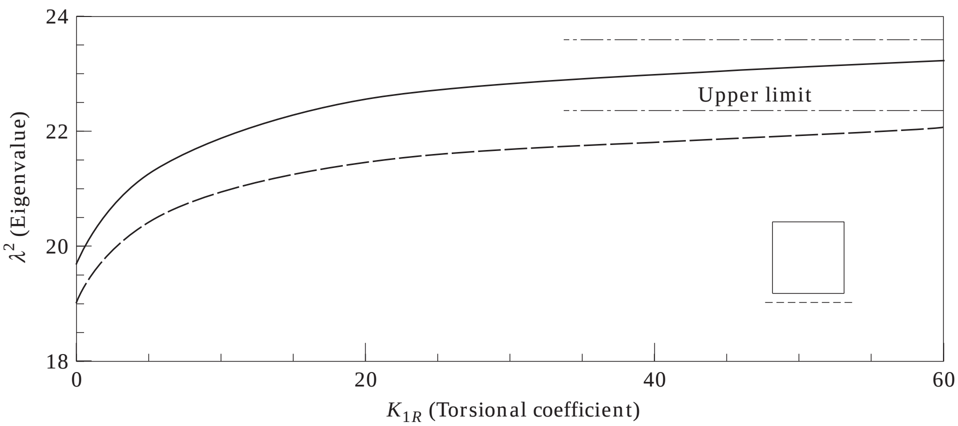

Figure 7. Eigenvalues versus torsional coefficient K_{1R} for square plate with uniform elastic support on one edge only. Key as Figure 2.

pertains to a square plate with elastic support along one edge only. The reader will appreciate that a superposition of building blocks is really not required for this problem. Only a solution to the first building block is required. However, by deleting the contributions of the other three building blocks to the eigenvalue matrix as described earlier, accurate eigenvalues are immediately available for this latter problem. These results are presented in the interest of completeness.

It is recognized that the main function of the curves presented so far is to give the reader an insight with regard to how eigenvalues may be expected to vary with edge rotational stiffness for the shear-deformable plates discussed here. Some designers or researchers may wish to have available numerical results against which they can compare their computed results, whether based on the computational procedure described here, or some other procedure. For this reason eigenvalues computed here are presented in digital form in Table 1. Intervals in the parameter K_{1R} are selected between one and fifteen with a view to giving approximately equal differences in tabulated eigenvalues as the stiffness is increased. As indicated earlier, these eigenvalues are considered to have four digit accuracy.

TABLE 1

Computed eigenvalues for first mode free vibration of square shear-deformable plates with rotational elastic edge support. Equal stiffness assigned to all elastically supported edges

	K_{1R} (Torsional coefficient)							
ϕ_h	1	2	3	4	5	7·5	10·0	15·0
	Four edges—equal elastic support							
0·01	21·49	22·89	24·02	24·97	25·78	27·34	28·48	30·04
0·10	20·65	21·90	22·90	23·72	24·41	25·74	26·69	27·96
	Three edges—equal elastic support							
0·01	21·06	22·11	22·96	23·68	24·28	25·46	26·31	27·47
0·10	20·26	21·20	21·95	22·57	23·09	24·09	24·80	25·75
	Two adjacent edges—equal elastic support							
0·01	20·61	21·30	21·30	22·31	22·69	23·42	23·94	24·63
0·10	19·85	20·47	20·96	21·36	21·69	22·32	22·76	23·33
	One edge with elastic support							
0·01	20·18	20·53	20·82	21·06	21·26	21·65	21·94	22·31
0·10	19·46	19·78	20·03	20·04	20·42	20·75	20·99	21·30

3. SUMMARY AND CONCLUSIONS

The Superposition–Galerkin method exploited here to obtain accurate eigenvalues for shear-deformable plates with torsional elastic edge support functions extremely well and is easy to employ. This simplification comes about largely because complicated sixth order ordinary differential equations, as discussed in reference [7], need not be contended with, as is the case in the traditional superposition method. Also, satisfaction of the boundary conditions is much easier to achieve. While the present study is devoted to shear-deformable thick isotropic plates it will be apparent that the same mathematical proceedure can be exploited to study the vibration behaviour of shear-deformable composite plates. As seen earlier, other families of boundary conditions can be easily handled with the computational procedure described here. To the author's knowledge the work reported here represents a significant step forward in the ability to compute free vibration behaviour of shear-deformable elastically supported rectangular plates.

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APPENDIX: LIST OF SYMBOLS

a,b	plate edge lengths
D	plate flexural rigidity
k_1, k_2, k_3, k_4	torosional elastic stiffness constants
K_{R1}, K_{R2}, K_{R3}, K_{R4}	dimensionless torsional elastic coefficients
K	number of terms used in Calerkin expansions
KK	number of terms used in representing plate lateral displacement and cross-sectional rotations
M_ξ, M_η	dimensionless bending moments associated with ξ, and η directions, respectively
$M_{\xi\eta}$	dimensionless twisting moment
Q_ξ, Q_η	dimensionless shear forces associated with ξ and η directions, respectively
w	plate lateral displacement divided by edge length a
κ^2	shear correction factor $= 0{\cdot}8601$
ν	Poisson ratio for plate material
ν_1	$(1 - \nu)/2$
ν_2	$(1 + \nu)/2$
ν_3	$6\kappa^2(1 - \nu)$
ϕ	plate aspect ratio b/a

ϕ_h	ratio of plate thickness to edge length a
ξ, η	co-ordinate distances divided by edge lengths a, and b, respectively
Ψ_ξ, Ψ_η	Plate cross-sectional rotations associated with ξ and η directions, respectively
λ^2	eigenvalue $= \omega a^2\sqrt{\rho/D}$
ω	circular frequency of plate vibration
ρ	mass of plate per unit area