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# NATURE OF STATIONARITY OF THE NATURAL FREQUENCIES AT THE NATURAL MODES IN THE RAYLEIGH–RITZ METHOD

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A variational formulation of the Rayleigh–Ritz method to obtain approximate natural frequencies and natural modes is presented. The stationarity of the natural frequencies with respect to the arbitrary coefficients in the linear combination of the assumed deflection shapes, and also at the natural modes is investigated. It is concluded that the natural frequencies are stationary and need not always be minimum, with respect to the arbitrary coefficients; however, they are minimum with respect to the natural modes. This may provide a means of checking the accuracy of the computed natural frequencies obtained by using energy techniques such as the Rayleigh–Ritz, Galerkin, and finite element methods.



## 1. INTRODUCTION

In the Rayleigh–Ritz method, the vibration deflection shapes are assumed to be in the form of a linear combination of functions which satisfy at least the geometrical boundary conditions of the vibrating structure. The maximum potential and kinetic energies formulated using these deflection shapes are used to obtain an expression for the natural frequency in the form of the Rayleigh quotient. At this point, the coefficients of the linear combination of terms are adjusted so that the frequency is made stationary at the natural modes, in conformity with Rayleigh's principle. This results in as many linear equations as the number of arbitrary coefficients considered in the linear combination in the form of an eigenvalue problem. The solution of this eigenvalue problem provides the natural frequencies and the associated natural modes.

The Rayleigh–Ritz method assumes a harmonic motion in free vibration and employs the condition that maximum kinetic energy is equal to maximum potential energy, which however occur a quarter period apart from each. The Rayleigh–Ritz method has been shown to be a form of variational method using Hamilton's principle in a generalized co-ordinate function space [1–3]. The Rayleigh–Ritz method must consider a complete set of assumed deflection functions that satisfy at least the geometrical boundary conditions. Theoretically, it can compute exact natural frequencies and the associated natural modes if the assumed deflection functions form a complete set. Further, as rightly pointed out in some recent books on vibration such as by Rao [4], the process only makes the computed natural frequencies stationary with respect to the coefficients in the linear combination of the assumed deflection functions, and not "minimum" with respect to them, as described in some vibration text books [5, 6]. The requirement of the assumed deflection shapes to

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satisfy "at least the geometrical boundary conditions" is not explained often, which becomes clear when the variational formulation is examined.

## 2. SOME RESULTS FROM VARIATIONAL FORMULATION

Assuming that all the applied forces are derivable from a potential function $V$, one can write [2]

$$\int_{t_1}^{t_2} \delta(T - V)\, \mathrm{d}t = 0, \tag{1}$$

where $T$ is the kinetic energy, $t_1$ and $t_2$ are the times at which configurations of the system are specified and $\delta$ is the variation. The present discussion is illustrated using a one-dimensional continuous system such as an elastic beam, which in no way compromises the generality of the principles involved. The kinetic and potential energy expressions of the beam are given by

$$T = \tfrac{1}{2}\int_0^L m(x)\dot{w}^2(x, t)\, \mathrm{d}x, \qquad V = \tfrac{1}{2}\int_0^L EI(x)[w''(x, t)]^2\, \mathrm{d}x, \tag{2, 3}$$

where $m(x)$ is the mass per unit length, $EI(x)$ is the bending rigidity, $L$ is the length of the beam and $w(x, t)$ is the beam deflection as a function of the spatial co-ordinate $x$ and the time $t$. Expressing the deflection $w(x, t)$ in separable form in space and time domains, one can write

$$w(x, t) = X(x)T(t). \tag{4}$$

Substituting equation (4) into equations (2) and (3), the variation in equation (1) can be written as

$$\int_{t_1}^{t_2} \left[\int_0^L m\dot{w}\delta\dot{w}\, \mathrm{d}x - \int_0^L EIw''\delta w''\, \mathrm{d}x\right] \mathrm{d}t = 0, \tag{5}$$

where mass $m$ and bending rigidity $EI$ are assumed constant.

The variation in $w(x, t)$ can be expressed as

$$\delta w(x, t) = \delta X(x) \cdot T(t) + X(x) \cdot \delta T(t). \tag{6}$$

Substituting equation (6) into equation (5) results in

$$\int_{t_1}^{t_2} \left\{\left[\int_0^L mX\dot{T}^2\delta X\, \mathrm{d}x - \int_0^L EIX''T^2\delta X''\, \mathrm{d}x\right]\right.$$
$$\left. + \left[\int_0^L mX^2\dot{T}\delta\dot{T}\, \mathrm{d}x - \int_0^L EIX''^2T\delta T\, \mathrm{d}x\right]\right\} \mathrm{d}t = 0. \tag{7}$$

The terms containing $\delta\dot{T}$ and $\delta X''$ are integrated by parts using the conditions that the system configurations are defined and do not allow variations at $t_1$ and $t_2$. Further, it is also noted that the variations are consistent with the boundary conditions at $x = 0$ and $L$. This means that the variations in the deflection must not violate the geometrical boundary conditions at $x = 0$ and $L$. Consequently, equation (7) can be obtained in the form

$$\int_{t_1}^{t_2} \left\{ \left[ \int_0^L mX\dot{T}^2 \, \mathrm{d}x - EIX''''T^2 \, \mathrm{d}x \right] \delta X - \left[ \int_0^L mX^2 \ddot{T} \, \mathrm{d}x + \int_0^L EIX''^2 T \, \mathrm{d}x \right] \delta T \right\} \mathrm{d}t = 0. \quad (8)$$

Interchanging the order of integration in the first half of equation (8) one obtains

$$\int_0^L \left\{ mX \left( \int_{t_1}^{t_2} \dot{T}^2 \, \mathrm{d}t \right) - EIX'''' \left( \int_{t_1}^{t_2} T^2 \, \mathrm{d}t \right) \right\} \delta X \, \mathrm{d}x$$

$$- \int_{t_1}^{t_2} \left\{ \ddot{T} \left( \int_0^L mX^2 \, \mathrm{d}x \right) + T \left( \int_0^L EIX''^2 \, \mathrm{d}x \right) \right\} \delta T \, \mathrm{d}t = 0. \quad (9)$$

Since the variations $\delta X$ and $\delta T$ are quite arbitrary, their coefficient terms must independently be zero. Consequently, one obtains two differential equations from equation (9):

$$\ddot{T} \left( \int_0^L mX^2 \, \mathrm{dx} \right) + T \left( \int_0^L EIX''^2 \, \mathrm{d}x \right) = 0, \qquad mX \left( \int_{t_1}^{t_2} \dot{T}^2 \, \mathrm{d}t \right) - EIX'''' \left( \int_{t_1}^{t_2} T^2 \, \mathrm{d}t \right) = 0.$$

(10, 11)

Equation (10) has a simple harmonic solution of the type $T(t) = A \cos \omega t + B \sin \omega t$, where

$$\omega^2 = \left[ EI \int_0^L [X''(x)]^2 \, \mathrm{d}x \right] \Big/ \left[ m \int_0^L X^2(x) \, \mathrm{d}x \right]. \quad (12)$$

In view of the harmonic nature of $T(t)$, one has $\int \dot{T}^2 \, \mathrm{d}t = \omega^2 \int T^2 \, \mathrm{d}t$ when the integration is carried over a very long time or over a period. The function $X(x)$ is obtained by solving the boundary value problem presented by equation (11), having an infinite number of solutions of the type $\phi_i(x)$ which are the natural modes with the associated natural frequencies $\omega_i$. The complete solution for $w(x, t)$ can be written as

$$w(x, t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t), \quad (13)$$

where $q_i(t) = A_i \cos \omega_i t + B_i \sin \omega_i t$, the infinite number of terms for $T(t)$ corresponding to $\omega_i$. When it is difficult to solve the boundary value problem posed in equation (11), an approximate solution is obtained using the Rayleigh–Ritz method where the solution of the type

$$X(x) = \sum_{i=1}^{n} Q_i f_i(x) \quad (14)$$

is assumed. The function $X(x)$ is expressed in terms of the generalized co-ordinates $Q_i$ in the function space defined by $f_i(x)$. The functions $f_i(x)$ must satisfy at least the geometrical boundary conditions, which is in conformity with the requirement for the generalized co-ordinate functions to have their variations to be consistent with the geometrical constraints. As is well known, neither $f_i(x)$, nor the eigenvectors obtained by the analysis using a truncated series in equation (14) provide the exact natural modes. In the usual Rayleigh–Ritz formulation, the maximum kinetic and potential energies obtained using the assumed deflection shapes in equation (14) are equated, i.e., $U_{max} = T_{max} = T^*_{max}\omega^2$, and the natural frequencies are obtained in the form

$$\omega^2 = [U_{max}(Q_1, Q_2, Q_3, \ldots)]/[T^*_{max}(Q_1, Q_2, Q_3, \ldots)]. \tag{15}$$

Equation (15) can also be seen as a direct consequence of equation (12) following the variational formulation. The natural frequencies are obtained from equation (15) by the condition

$$\partial\omega^2/\partial Q_j = 0, \qquad j = 1, 2, \ldots, n, \tag{16}$$

which results in an eigenvalue problem of the type

$$[\mathbf{K}]\{\mathbf{Q}\} - \omega^2[\mathbf{M}]\{\mathbf{Q}\} = \mathbf{0}. \tag{17}$$

The eigenvalues of this problem will provide exact natural frequencies only if $f_i(x) \equiv \phi_i(x)$. In such a case, matrices [**K**] and [**M**] are purely diagonal and one has

$$\partial^2\omega^2/\partial Q_j^2 = K_{jj} - \omega^2 M_{jj} = 0 \tag{18}$$

at the squared natural frequencies $\omega^2 = \omega_i^2$, indicating that the $\omega_i^2$ are not stationary with respect to coefficients $Q_j$ at all. However, if $f_i(x) \neq \phi_i(x)$, then the second variation of $\omega^2$ with respect to $Q_j$ will be positive if $\omega^2$ is a minimum with respect to $Q_j$ and will be negative if $\omega^2$ is maximum with respect to $Q_j$. Hence $\omega_i^2$ is a minimum if the matrix $[\mathbf{K} - \omega_i^2\mathbf{M}]$ is positive definite, while it corresponds to a maximum if $[\mathbf{K} - \omega_i^2\mathbf{M}]$ is not positive definite. This is illustrated using the example of a cantilever beam. The deflection shape is assumed in the form

$$X(x) = \sum_{j=1}^{4} Q_j x^{1+j} \tag{19}$$

using four terms, where each term satisfies the geometrical boundary conditions at $x = 0$. One obtains

$$[\mathbf{K}] = \begin{bmatrix} 4 & 6 & 8 & 10 \\ 6 & 12 & 18 & 24 \\ 8 & 18 & 28{\cdot}8 & 40 \\ 10 & 24 & 40 & 400/7 \end{bmatrix}, \qquad [\mathbf{M}] = \begin{bmatrix} 1/5 & 1/6 & 1/7 & 1/8 \\ 1/6 & 1/7 & 1/8 & 1/9 \\ 1/7 & 1/8 & 1/9 & 1/10 \\ 1/8 & 1/9 & 1/10 & 1/11 \end{bmatrix}. \tag{20}$$

The eigenvalues are obtained as

$$p_1, p_2, p_3, p_4 = (EI/ml^4)\ (3{\cdot}516,\ 22{\cdot}158,\ 63{\cdot}347,\ 281{\cdot}596)$$

and the coefficients $Q_i$ at each of the eigenvalues are

$$\{\mathbf{Q}\}^{(1)} = \begin{Bmatrix} 0{\cdot}913 \\ -0{\cdot}400 \\ -0{\cdot}052 \\ 0{\cdot}059 \end{Bmatrix}, \qquad \{\mathbf{Q}\}^{(2)} = \begin{Bmatrix} 0{\cdot}373 \\ -0{\cdot}797 \\ 0{\cdot}469 \\ -0{\cdot}074 \end{Bmatrix}, \qquad \{\mathbf{Q}\}^{(3)} = \begin{Bmatrix} -0{\cdot}154 \\ 0{\cdot}602 \\ -0{\cdot}732 \\ 0{\cdot}279 \end{Bmatrix},$$

$$\{\mathbf{Q}\}^{(4)} = \begin{Bmatrix} -0{\cdot}097 \\ 0{\cdot}492 \\ -0{\cdot}775 \\ 0{\cdot}384 \end{Bmatrix}.$$

One has $[\mathrm{K} - \omega_1^2\mathrm{M}]$ positive definite, while $[\mathrm{K} - \omega_i^2\mathrm{M}]$ for $i = 2, 3, 4$ are not positive definite.

Now a natural frequency at a given stationary point with respect to the $j$th co-ordinate direction (arbitrary coefficient in the Rayleigh–Ritz method) is examined. One can do that by examining whether $(\mathrm{K}_{jj} - \omega_i^2\mathrm{M})$ is greater or less than zero. This is illustrated in Figures 1(a)–1(p) where variation of $\omega_i^2$ with $\mathrm{Q}_j$ are plotted at each of the modes, while keeping $\mathrm{Q}_k$, $k \neq j$, constant. It can be seen that $\partial\omega^2/\partial\mathrm{Q}_j = 0$ corresponds to a minimum whenever $\mathrm{K}_{jj} - \omega_i^2\mathrm{M}_{jj} > 0$. As an example note that $\mathrm{K}_{11} = 4$ and $\mathrm{M}_{11} = 1/5$, and for $i = 1$ one has $[4 - (1/5)\omega_1^2 > 0]$. Hence $\partial\omega_1^2/\partial\mathrm{Q}_1 = 0$ corresponds to a minimum for $\omega_1^2$ with respect to $\mathrm{Q}_1$ in the neighbourhood of the first natural mode. Similarly one can verify that $\mathrm{K}_{jj} - \omega_1^2\mathrm{M}_{jj} > 0$ for $j = 2, 3, 4$. Hence $\omega_1^2$ is a minimum with respect to $\mathrm{Q}_j$, $j = 2, 3, 4$ also. As can be seen from Figure 1(h), $\omega_2^2$ is a minimum with respect to $\mathrm{Q}_4$ since $\mathrm{K}_{44} - \omega_2^2\mathrm{M}_{44} > 0$. Further, $\mathrm{K}_{jj} - \omega_2^2\mathrm{M}_{jj} < 0$ for $j = 2, 3, 4$. Hence $\partial\omega_2^2/\partial\mathrm{Q}_j = 0$ corresponds to maximum values of $\omega_2^2$, which can be verified from Figures 1(e)–1(g). Also, the $\omega_i^2$ for $i = 3, 4$, are maximum with respect to $\mathrm{Q}_j, j = 2, 3, 4$ in the neighbourhood of the corresponding natural modes.

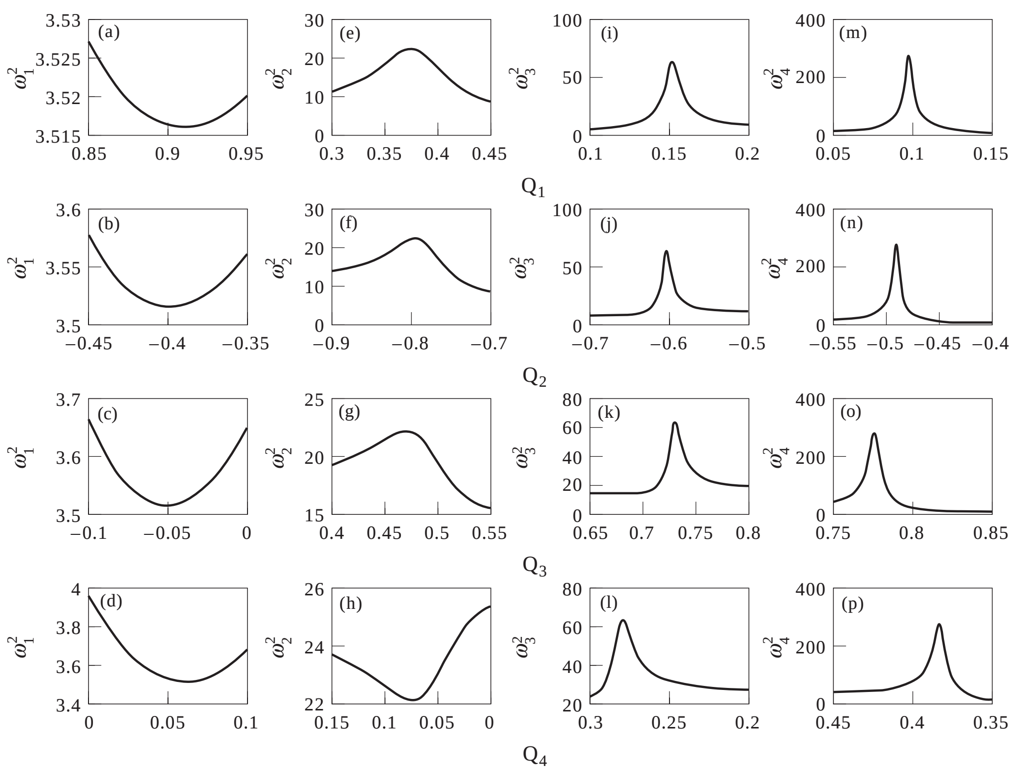


Figure 1. Frequency variation with $Q_j$ in the neighbourhood of natural modes: $\omega_1^2$ as function of (a) $Q_1$, (b) $Q_2$, (c) $Q_3$, (d) $Q_4$; $\omega_2^2$ as function of (e) $Q_1$, (f) $Q_2$, (g) $Q_3$, (h) $Q_4$; $\omega_3^2$ as function of (i) $Q_1$, (j) $Q_2$, (k) $Q_3$, (l) $Q_4$; $\omega_4^2$ as function of (m) $Q_1$, (n) $Q_2$, (o) $Q_3$, (p) $Q_4$.

The above condition holds good even if the structure is unconstrained and the "fundamental frequency" corresponds to the rigid body mode and the natural frequency is zero, the case discussed in Badrakhan [7]. In his example of a free–free beam, he uses the deflection function of

$$y = y_0 \sin(\pi x) - h. \tag{21}$$

This function was chosen by Den Hartog [8] to illustrate Rayleigh's method. Even though it seems to be a single term approximation, strictly there are two terms attached with the two unknown coefficients, $y_0$ and $h$, and hence it corresponds to the Rayleigh–Ritz method. If $\omega^2$ is optimized with respect to $h$ using the Rayleigh's method formulation, there are two resulting natural frequencies, 0 and $22{\cdot}676(EI/ml^4)^{1/2}$. If the chosen single term is $y = h$, this will give zero natural frequency in the Rayleigh's method. If two terms as in equation (21) are used in the Rayleigh–Ritz method, then the two resulting frequencies are upper bounds for the first two natural frequencies. Badrakhan reported that assuming $h = 0{\cdot}7y_0$ gave a frequency of $22{\cdot}21(EI/ml^4)^{1/2}$ which is lower than the exact value of the first bending natural frequency. However, by preassigning a value to $h$, the resulting expression

$$y = y_0 (\sin \pi x_0 - 0{\cdot}7) \tag{22}$$

becomes a single term approximation for the deflection shape. Even though this expression is assumed in the hope that it will represent the first bending mode of the free–free beam, the Rayleigh analysis tries to represent a rigid body mode with frequency of zero with this expression, rather unsuccessfully. An intuitive single term approximation for a higher mode providing a lower value for the natural frequency than that provided by the Rayleigh–Ritz analysis, was also reported by Leissa [9]. An explanation for this type of result was provided by Bhat [10], wherein he expanded the single term approximation in terms of a generalized Fourier series using the exact natural mode functions and showed that the computed frequency is influenced by the contribution of different natural modes in the assumed deflection function. Applying that reasoning, it is clear that when the contribution of the rigid body mode in the expression in equation (22) is maximized, the contribution of the sine term also increases automatically. Hence the resulting frequency of $22{\cdot}21(EI/ml^4)^{1/2}$ is a very poor approximation for the natural frequency of the rigid body mode, even though it appears close to that of the first bending mode. When the two term shape of equation (21) is used, $\partial\omega^2/\partial h = 0$ provides a minimum for the Rayleigh quotient at $\omega = 0$, but a maximum at $\omega = 22{\cdot}676(EI/ml^4)^{1/2}$. The reason for this is obvious because at $h = 2y_0/\pi$ obtained by the Rayleigh–Ritz analysis, one has $\partial^2\omega^2/\partial h^2 < 0$ at the frequency of $22{\cdot}676(EI/ml^4)^{1/2}$. This illustrates clearly that the computed natural frequency need not be a minimum with respect to the arbitrary coefficients. The condition of $\partial\omega^2/\partial \mathrm{Q}_j = 0$ is only a condition of stationarity and not the condition for minimum natural frequency with respect to the arbitrary coefficients $\mathrm{Q}_j$.

## 3. VARIATION OF NATURAL FREQUENCIES WITH NATURAL MODES

The general variation in $w(x, t)$ is given in equation (6). Out of this, the variation in the deflection function $X(x)$ resulted in the expression for the natural frequency given in equation (12). The function $X(x)$ itself was obtained by solving equation (11) in terms of the natural modes $\phi_i(x)$ and the corresponding natural frequencies $\omega_i$. When an exact solution for $X(x)$ is not possible, an approximate solution is considered as a linear combination of assumed deflection functions each of which satisfy at least the geometrical boundary conditions. The frequency expression in equation (12) is made stationary with respect to the arbitrary coefficients $Q_j$, one at a time. However, we know that each natural mode is a specific combination of coefficients $Q_j$. Hence, variation in $X(x)$, i.e. $\delta X$, can be accomplished by varying the form of the deflection function itself, by cruising along the frequency domain, in contrast to varying only the coefficients $Q_j$ in the linear combination of constant functions $f_j(x)$. In this way the deflection function passes through different modes successively. This can be accomplished as follows.

Let $f_i(x)$ in equation (14) be the exact natural modes $\phi_i(x)$, $i = 1, 2, 3, \ldots$ with the corresponding natural frequencies $p_i$ such that $p_1 < p_2 < p_3 <, \ldots, < p_n$. Then equation (14) becomes

$$X(x) = \sum_{i=1}^{\infty} \mathrm{Q}_i f_i(x). \tag{23}$$

The terms $\mathrm{Q}_i$ are in fact the modal co-ordinates. In order to represent $X(x)$ all along the frequency range, Hurty and Rubinstein [11] used the Lagrangian interpolation polynomials [12] and consequently, in the place of equation (23), they used

$$X(x, p) = \sum_{i=1}^{n} L_i(p)\phi_i(x), \tag{24}$$

where $p$ is frequency and $L_i(p)$ are the Lagrangian coefficient polynomials

$$L_i(p) = \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{(p - p_k)}{(p_i - p_k)} \tag{25}$$

with $p_i$ as the $i$th natural frequency. Lagrange coefficient polynomials have the property given by

$$L_i(p) = \begin{cases} 1, & p = p_i, \\ 0, & p \neq p_i. \end{cases} \tag{26}$$

Consequently, at $p = p_r$ one obtains

$$X(x) = L_r(p)\phi_r(x). \tag{27}$$

Substituting from equation (25) into equation (24) and using equation (12) one gets $\omega^2$ as a function of a single parameter $p$ as

$$\omega^2(p) = \sum_{i=1}^{n} K_i L_i^2(p) \Bigg/ \sum_{i=1}^{n} M_i L_i^2(p), \tag{28}$$

where

$$K_i = EI \int_0^L [\phi''(x)]^2 \, \mathrm{d}x, \qquad M_i = m \int_0^L \phi_i^2(x) \, \mathrm{d}x. \tag{29}$$

Note that if $p = p_r$, then

$$\omega^2(p_r) = K_r / M_r = \omega_r^2. \tag{30}$$

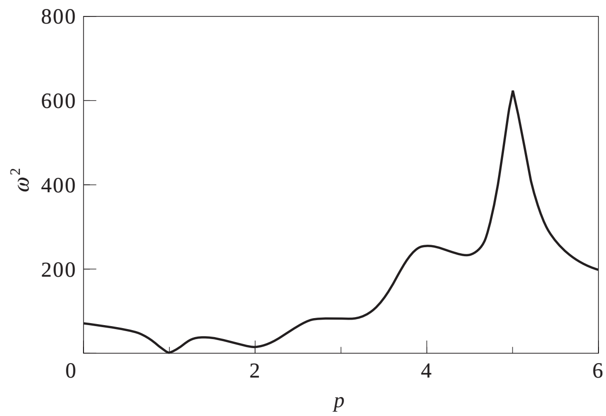


Figure 2. Frequency variation with deflection shape interpolated between natural modes using Lagrangian polynomials, with exact natural modes for a simply-supported beam.

Upon differentiation of $\omega^2$ in equation (28) with respect to $p$, it is found that the derivative vanishes at $p = p_r$, showing that the squared natural frequency $\omega^2$ is stationary at the natural modes.

In order to illustrate the nature of $\omega^2$ at the natural modes, the example of a simply-supported beam is used. The natural modes are

$$\phi_i(x) = \sqrt{2}\sin i\pi x \tag{31}$$

and $p_i = i^2\pi^2\sqrt{(EI/mL^4)}$. Further, $K_i = EIi^4\pi^4/mL^3$, $M_i = mL$, and the frequency is

$$\omega^2(p) = \omega_1^2 \sum_{i=1}^{n} i^4 L_i^2(p) \Big/ \sum_{i=1}^{n} L_i^2(p), \tag{32}$$

where $\omega_1^2 = \pi^4 EI/ml^4$, the square of the fundamental natural frequency. For numerical computations $p_i$ may be assigned integral values for convenience, since this maneuver only assigns a unique scale factor to the $p$-axis. Plots of $(\omega^2(p)/\omega_1^2)$ against $p$ are given in Figure 2 for $i = 1$–6. It can be seen that $\omega^2$ is a minimum, in general, at the natural modes. However, it becomes a maximum at the last one or two natural modes, where it is truncated. This is because equation (24) cannot represent the deflection satisfactorily in the neighbourhood of the last few modes where the series is truncated.

Lagrangian polynomials provide one way of smoothly changing the deflection shape from one natural mode to another and in between. However, it is not unique. Even though $L_r(p_r) = 1$ and $L_r(p_s) = 0$ for $s \neq r$, the variation of these coefficient polynomials in between the natural frequencies is not satisfactory as can be seen from Figure 3. A device is proposed here in which the response of a beam to a point force (located in such a way as to excite all the modes under consideration) is used to represent the smooth change of the deflection shape in between and at the natural modes. The steady state response will be identical to the natural modes at the corresponding natural frequencies and will vary smoothly and realistically in between the natural frequencies. This is accomplished as follows. The differential equation of motion for the beam is given by

$$EIw^{iv}(x, t) + m\ddot{w}(x, t) = f(x, t), \tag{33}$$

where $f(x, t)$ is the distributed load on the beam. Taking Fourier transforms on both sides one has

$$EIX^{iv}(x, p) - mp^2X(x, p) = F(x, p). \tag{34}$$

From the homogeneous form of equation (34) given by

$$EIX^{iv}(x, p) - mp^2X(x, p) = 0 \tag{35}$$

one can solve for the natural frequencies $p_i$ and the corresponding natural modes $\phi_i(x)$. Using them in equation (35) one gets

$$EI\phi_i^{iv}(x, p) = mp^2\phi_i(x, p). \tag{36}$$

The deflection of the beam for the non-homogeneous case is assumed in the form given in equation (23) with $X(x)$ expressed as $X(x, p)$, from which one gets

$$Q_i(p) = \left[\int_0^L X(x, p)\phi_i(x)\,\mathrm{d}x\right] \Big/ \left[\int_0^L \phi_i^2(x)\,\mathrm{d}x\right]. \tag{37}$$

Substituting this in equation (34) and considering a point force at $x = x_0$, one gets

$$Q_i(p) = EIF\phi_i(x_0)/mL^4(p_i^2 - p^2), \tag{38}$$

where $F$ is the amplitude of the point force. Proceeding as in the previous case, one obtains for a simply-supported beam

$$\omega^2(p) = \omega_1^2 \sum_{i=1}^{n} i^4 \frac{\sin^2 i\pi x_0}{(p_i^2 - p^2)} \Big/ \sum_{i=1}^{n} \frac{\sin^2 i\pi x_0}{(p_i^2 - p^2)}. \tag{39}$$

At any natural frequency, say $p = p_k$.

$$\omega^2(p_k) = \omega_1^2 \left[k^4 \frac{\sin^2 k\pi x_0}{(p_k^2 - p^2)}\right] \Big/ \left[\frac{\sin^2 k\pi x_0}{(p_k^2 - p^2)}\right] = \omega_1^2 k^4. \tag{40}$$

A very small damping term is used for the purpose of numerical evaluation of equation (39). Accordingly, the denominator is taken as $(p_i^2 - p^2 + \varepsilon)$, where $\varepsilon$ is arbitrarily taken as $10^{-10}$. Again $(\omega^2/\omega_1^2)$ is plotted in Figure 4 for $i = 1$–6. It is seen that $\omega^2$ is minimum

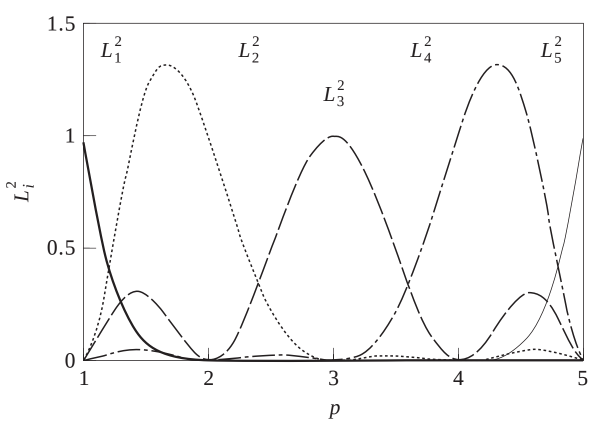


Figure 3. Lagrange coefficient polynomials.

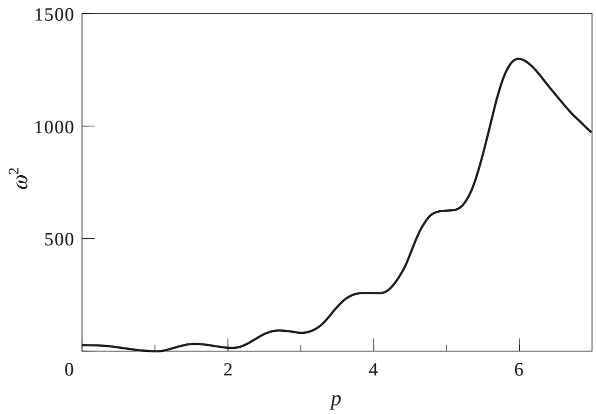


Figure 4. Frequency variation with deflection shape interpolated between natural modes using modal response functions, with exact natural modes for a simply-supported beam.

at the natural frequencies except for the last few modes in the truncated sequence. The modal response expressions, $Q_i(p)$, are plotted against $p$ in Figure 5 that can be compared with the Lagrangian coefficient polynomials shown in Figure 3. It is seen that the modal response $Q_i(p)$ is more realistic for the representation of deflection shape in between the natural modes.

If the natural frequencies and the natural modes computed using the Rayleigh–Ritz method are used in equation (23), the expression for $\omega^2(p)$ in equation (28) is still valid except that the terms $K_i$ and $M_i$ are the generalized stiffness and generalized mass for the $i$th mode. They can be easily obtained as the $i$th diagonalized elements of the generalized stiffness and generalized mass matrices $[\boldsymbol{\phi}]^T[\mathbf{K}][\boldsymbol{\phi}]$ and $[\boldsymbol{\phi}]^T[\mathbf{M}][\boldsymbol{\phi}]$. This is applied on the example of the cantilever beam using the deflection function described in equation (19) and using the stiffness and mass matrices given in equation (20). The result is shown in Figure 6. It is seen that the first computed natural frequency is minimum at the first computed natural mode, however, the higher natural frequencies become maximum at the corresponding natural modes. Further, an example of a simply-supported square plate is considered to study the variation of $\omega^2$ with the natural modes considering 36 exact natural modes of the type $\phi_{mn}(x) = \sin(m\pi x/L)\sin(n\pi y/L)$ with their corresponding natural frequencies, and also the natural modes and natural frequencies obtained by using

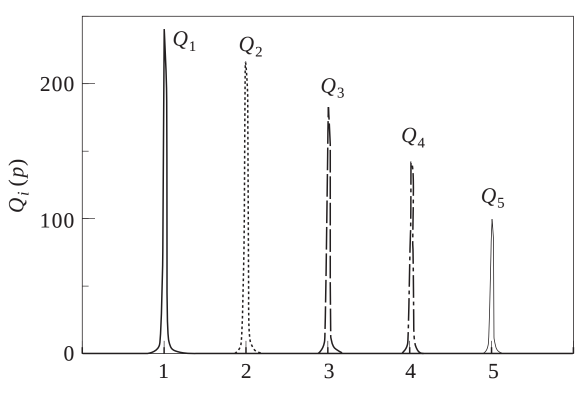


Figure 5. Modal response functions.

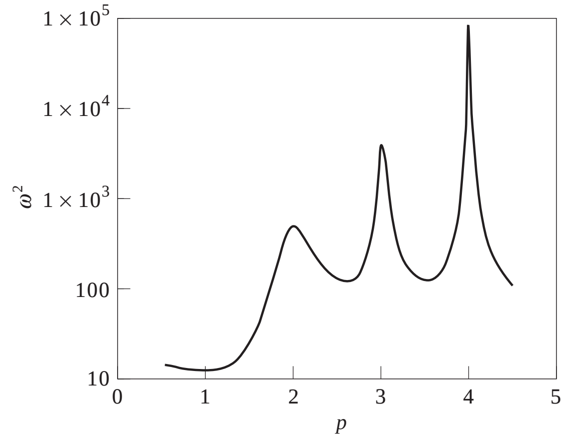


Figure 6. Frequency variation with deflection shape interpolated between natural modes using Lagrangian polynomials, with natural modes computed by the Rayleigh–Ritz method for a cantilever beam.

boundary characteristic orthogonal polynomials in the Rayleigh–Ritz method [13]. The results are shown in Figures 7 and 8. The natural frequencies are minimum with respect to the natural modes up to the 15th mode in both cases. However, after that they are maximum at the natural modes. This is quite understandable because the series representation of the deflection in equation (24) is valid only when an infinite number of terms are considered. When the series is truncated, the representation of the deflection shape will be poor for higher modes. From Figures 7 and 8, it can also be seen that the natural frequencies computed by the Rayleigh–Ritz method are very high compared to their exact values. This discrepancy can be reduced by choosing the assumed deflection functions as close to the exact mode shapes as possible. One way of achieving this for plates is to use the plate characteristic functions for the assumed deflection shapes [14, 15].

Further, it is noted that the method of optimizing an exponent to obtain the natural frequency as proposed by Lord Rayleigh [16] originally and popularized by Schmidt [17], Bert [18] and Laura [19] involves varying the form of the space function $X(x)$ itself to arrive

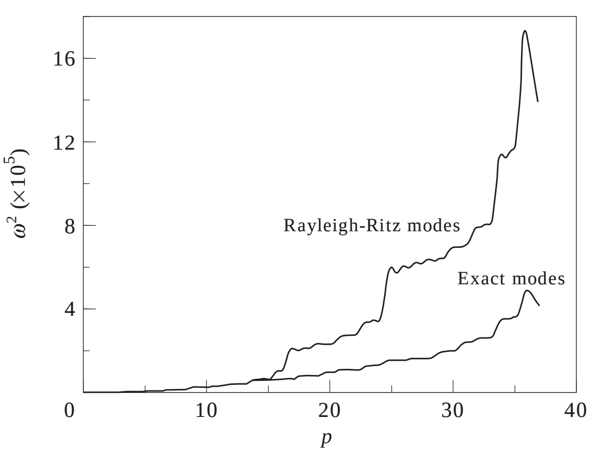


Figure 7. Frequency variation with deflection shape interpolated between natural modes using Lagrangian polynomials, with exact natural modes as well as those computed using the Rayleigh–Ritz method, for a simply-supported plate.

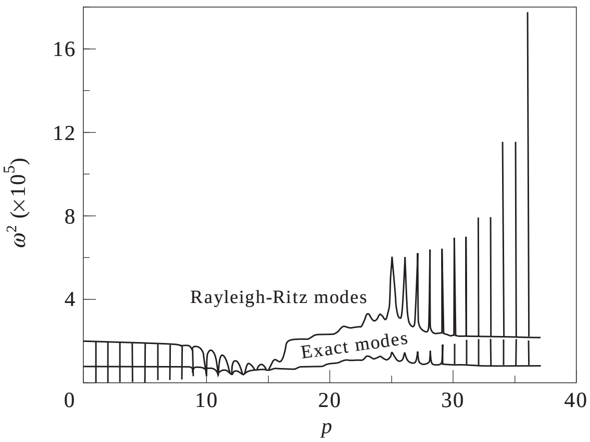


Figure 8. Frequency variation with deflection shape interpolated between natural modes using modal response functions, with exact natural modes as well as those computed using the Rayleigh–Ritz method, for a simply-supported plate.

at a minimum for the natural frequency at the natural mode. Laura and Cortinez [20] extended the method of optimizing an exponent in an ingenious way to optimize the space functions $X(x)$ corresponding to the higher natural modes also, for the first time, and achieved accurate higher eigenvalues. The above findings lead to a possibility of checking the accuracy of the computed natural frequencies obtained by energy techniques such as the Rayleigh–Ritz method, the Galerkin method, and the finite element methods, by examining whether the computed natural frequencies are a minimum at the computed natural modes.

## 4. CONCLUSIONS

A variational formulation of the Rayleigh–Ritz method is presented. The stationarity of the natural frequencies at the natural mode is examined. It is found that the natural frequencies need not be minimum with respect to the arbitrary coefficients in the linear combination of the assumed deflection functions. However, the natural frequencies are a minimum at the natural modes. When the number of natural modes used to represent the deflection function is limited, the natural frequencies are maximum with respect to higher modes, which is due to the reduction in accuracy in representing the deflection function. The accuracy of the computed natural frequencies by energy techniques such as the Rayleigh–Ritz, Galerkin, and finite element methods can be checked by examining whether the computed natural frequencies are minimum at the corresponding computed natural modes.

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