Journal of Sound and Vibration (1997) **203**(3), 373–387

THE EFFECT OF NON-LINEAR INERTIA ON THE STEADY STATE RESPONSE OF A BEAM SYSTEM SUBJECTED TO COMBINED EXCITATIONS

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(*Received* 9 *January* 1996, *and in final form* 9 *August* 1996)

In this paper a weakly non-linear beam system subjected simultaneously to parametric and harmonic excitations is studied. There are many types of resonant phenomena under these loading conditions. The weakly non-linear differential equation is derived by use of the averaging method and the method of multiple scales. Even for the first approximate solutions, it is found that different results arise due to the existence of a non-linear inertia factor of the governing equation. The transient amplitudes obtained by these two methods are compared with those obtained by the Runge–Kutta method. Steady state responses are also shown for the various cases of resonances.

1. INTRODUCTION

A system which consists of a beam with simply supported ends subjected to a perodically axial force can be reduced to the Mathieu equation from the partial differential equation by using mode separation. Bolotin [1] studied various non-linear effects on the steady state responses for such a system by the harmonic balance method. A comprehensive review of the response of single- and multi-degree-of-freedom systems subjected to the parametric excitations has been given by Evan-Iwanowski [2], Ibrahim and Barr [3, 4] and Nayfeh and Mook [5].

A harmonic forced system with cubic non-linearities may have resonances at primary, one-third subharmonic and superharmonic resonances of order 3. Ness [6] classified various types of resonant phenomena for a weakly non-linear, single-degree-of-freedom system subjected to one time-varying force and one parameteric excitation. Troger and Hsu [7] studied the response of a non-linear system under combined parametric and forcing excitations which are of the same frequency. Sato *et al*. [8] applied the harmonic balance method to solve the parametric response of a simply supported, horizontal beam, carrying a concentrated mass under the influence of gravity. They investigated the effects of the weight and the location of the concentrated mass on the steady state, free and parametric responses of the beam, but neglecting the non-linear terms. Lau *et al*. [9] applied the variable parameter incrementation method to determine the parametric instability boundary of linear and non-linear elastic columns. Yagasaki *et al*. [10] applied the averaging method to establish the existence of an invariant tori and to analyze the stability properties of a weakly non-linear single-degree-of-freedom system subjected to combined parametric and external excitations. In these papers mentioned above, the non-linear inertia term was not considered.

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In certain dynamic problem one must consider not only the non-linear elasticity but also the non-linear inertia force. The existence of non-linear elasticity leads to an increase of the frequency with amplitude; non-linear inertia causes a decrease of the natural frequency. Evensen and Evan-Iwanowski [11] performed analytical and experimental investigations of the effect of longitudinal inertia upon the elastic column. Atluri [12] used the perturbation method of multiple time scales to investigate the non-linear vibration of a hinged beam, including the non-linear inertia effect. Calculated results showed that the predominant non-linearity due to the non-linear longitudinal inertia is of the softening type. Sunakawa and Higuchi [13] showed the parametrically unstable phenomenon of a simply supported thin column can be evaluated using a relatively simple non-linear equation. Ashworth and Barr [14] used the generation solution, which includes slowly varying mean terms and linear forced response components, to analyze the resonances of structures with quadratic inertia non-linearity under both direct and parametric excitations. They showed that the linear parametric terms and inertial non-linearities can have the same ranking of importance when the structure is at external *and/or* internal resonances.

In this paper, we try to investigate a beam system as shown in Figure 1, subjected to one axially and one laterally harmonic excitations with different frequencies. These two excitations have the same order in magnitude. The subsystem, including the spring, the extra mass and the guideway, is added on the top of the beam, to affect the dynamic properties of the main beam system. The subsystem components are the main sources of the non-linearities occurring in the ordinary differential equation. In this paper, two methods, the averaging method and the method of multiple scales, are employed to study

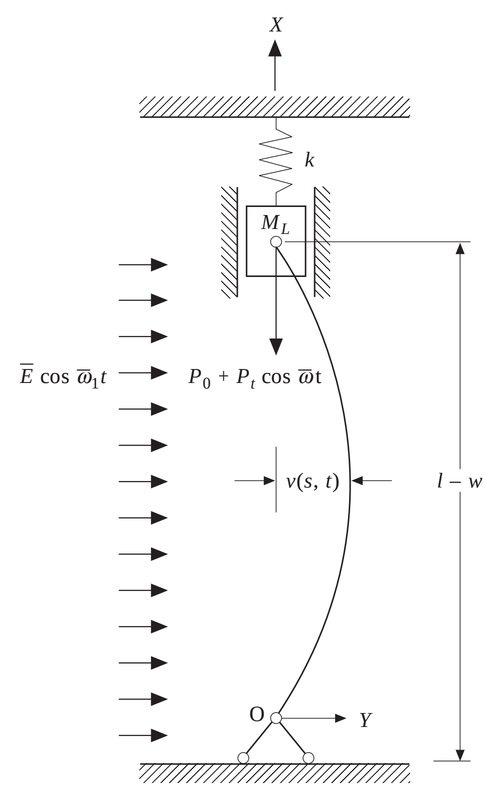

Figure 1. An analytical model of the oscillation beam.

the difference, which may come out between these two approaches. Our attention is focused on the non-linear inertia factor, which causes different results between the averaging method and the method of multiple scales, even if only the first approximation is retained. The two analytical methods are used to obtain the results and compared with the results obtained by the direct numerical integration which we used as the exact solution for comparison.

2. ANALYTIC WORK

The analytic model, as shown in Figure 1, is a simply supported beam subjected simultaneously to an axial load $P_0 + P_t \cos \bar{\omega} t$ and a laterally distributed force $\bar{E} \cos \bar{\omega}_1 t$. By using the Euler beam theory and considering the viscous damping during vibration, the equation of motion of the beam will be written as

$$EI \frac{\partial^4 v(s, t)}{\partial s^4} + q(t) \frac{\partial^2 v(s, t)}{\partial s^2} + m \frac{\partial^2 v(s, t)}{\partial t^2} + C \frac{\partial v(s, t)}{\partial t} = \bar{E} \cos \bar{\omega}_1 t, \tag{1}$$

where $v(s, t)$ is the transverse deflection, EI is the bending stiffness, $q(t)$ is the total longitudinal force, m is the mass per unit length of the beam and l is its total length. The influence of large curvature [1] is equivalent to that of a longitudinal elasticity coupling with a spring stiffness $\pi^2 EI/2l^3$ and an inertial mass coupling with a mass $(\frac{1}{3} - 3/8\pi^2)ml$ added at the upper end of the beam. During vibration, (a) the spring force, $-kW$, which arises from the reaction of the spring; (b) the resistance force, $-C_L W$, which occurs due to the guidance of the moving support and (c) the inertia force, $-M_L \ddot{W}$, are added to the system. Thus, the total longitudinal force is

$$q(t) = P_0 + P_t \cos \bar{\omega} t - KW - C_L \dot{W} - M\ddot{W}, \tag{2}$$

where

$$M = M_L + \left(\frac{1}{3} - \frac{3}{8\pi^2}\right) ml, \qquad K = k + \pi^2 \frac{EI}{2l^3},$$

and

$$W = l - \int_0^l \sqrt{1 - (\partial v/\partial s)^2}\, \mathrm{d}s$$

is the relationship between the longitudinal displacement W and the lateral deflection v.

Since experimental evidence of Sunakana and Higuchi [13] and Somerset and Evan-Iwanowski [15] showed that the first spatial mode is dominant, the first mode solution for (1) could be sought in the form

$$v(s, t) = f(t) \sin (\pi s/l), \tag{3}$$

which satisfies the simply supported end conditions

$$v(0, t) = v(l, t) = \frac{\partial^2 v(0, t)}{\partial s^2} = \frac{\partial^2 v(l, t)}{\partial s^2} = 0. \tag{4}$$

Substituting equation (3) into equation (1) and using the orthogonal conditions, we obtain

$$\ddot{f} + \Omega^2 f + 2\bar{\zeta}\dot{f} - 2\Omega^2 \bar{\mu} \cos \bar{\omega} t f + \bar{\gamma} f^3 + 2\bar{\varepsilon}_L f^2 \dot{f} + 2\bar{\alpha}(f^2 \ddot{f} + f\dot{f}^2) = \bar{e} \cos \bar{\omega}_1 t, \tag{5}$$

where

$$\bar{e} = \frac{2}{ml}\int_0^l \bar{E}\sin\frac{\pi s}{l}\,\mathrm{d}s,$$

Ω is the mode frequency, $\bar{\zeta}$ is the damping factor, and $\bar{\mu}$ is the parametric load parameter. Coefficients of non-linear terms, including the factor of non-linear elasticity $\bar{\gamma}$, the factor of non-linear damping $\bar{\varepsilon}_L$, and the factor of non-linear inertia $\bar{\alpha}$, are non-negative and are small constants (see Appendix 3). It should be noticed that equation (5) reduces to the forced Mathieu equation when the parametric excitation $\bar{\mu} \neq 0$ and $\bar{\gamma} = \bar{\varepsilon}_L = \bar{\alpha} = 0$, to the forced Duffing equation when the non-linear stiffness $\bar{\gamma} \neq 0$ and $\bar{\mu} = \bar{\varepsilon}_L = \bar{\alpha} = 0$, and to the forced van der Pol equation when the non-linear damping $\bar{\varepsilon}_L \neq 0$ and $\bar{\mu} = \bar{\gamma} = \bar{\alpha} = 0$. The Mathieu, Duffing and van der Pol equations are well documented in the analyses of steady state responses. In the present paper, we will focus on the study of the effect of the non-linear inertia.

Here we are interested only in a weakly non-linear system with a small dimensionless parameter ε, $0 < \varepsilon \ll 1$. The excitational amplitude $\bar{E}$ is assumed to be a constant, not a function of s, and $\bar{e}/\Omega^2$ is so small that it is of order $O(\varepsilon)$. Non-dimensionalizing equation (5) by introducing the time transformation $\tau = \Omega t$, one obtains

$$f'' + f = \varepsilon Q(f, f', f'', \tau) + O(\varepsilon^2), \tag{6}$$

where

$$Q(f, f', f'', \tau) = -2\zeta f' + 2\mu\cos\omega\tau f - \gamma f^3 - 2\varepsilon_L f^2 f' - 2\alpha(f^2 f'' + f f'^2) + e\cos\omega_1\tau,$$

$$\bar{\zeta}/\Omega = \varepsilon\zeta, \qquad \bar{\mu} = \varepsilon\bar{\mu}, \qquad \bar{\gamma}/\Omega^2 = \varepsilon\gamma, \qquad \bar{\varepsilon}_L/\Omega = \varepsilon\varepsilon_L, \quad \bar{\alpha} = \varepsilon\alpha, \qquad \bar{e}/\Omega^2 = \varepsilon\, e,$$

and $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}\tau$. The dimensionless parameters ζ, μ, γ, ε_L and α and the excitational frequencies $\omega = \bar{\omega}/\Omega$ and $\omega_1 = \bar{\omega}_1/\Omega$ all have the same order $O(1)$. Except for the external excitation term $\varepsilon e\cos\omega_1\tau$, equation (6) is similar to equation (8) of reference [16], which was approached by the averaging method.

It should be noticed that the term $2\varepsilon\alpha f^2 f''$ due to non-linear inertia in $\varepsilon Q(f, f', f'', \tau)$ could be considered as a feedback of f''. We put f'' and $2\varepsilon\alpha f^2 f''$ together to be $(1 + 2\varepsilon\alpha f^2)f''$. Since coefficient $1 + 2\varepsilon\alpha f^2$ is always not zero, by dividing every term of equation (6) by it and neglecting the terms of order $O(\varepsilon^2)$ and higher, equation (6) becomes

$$f'' + f = \varepsilon M(f, f', \tau) + O(\varepsilon^2), \tag{7}$$

where

$$M(f, f', \tau) = -2\zeta f' + 2\mu\cos\omega\tau f - (\gamma - 2\alpha)f^3 - 2\varepsilon_L f^2 f' - 2\alpha f f'^2 + e\cos\omega_1\tau.$$

The $\varepsilon M(f, f', \tau)$ in equation (7) is relatively small and can be regarded as a perturbation of the linear homogeneous differential equation. In the next section, the averaging method will be applied to equation (7), not to equation (6). The term $\varepsilon M(f, f', \tau)$ does not contain the inertia term f'', but f'' was included in the perturbation terms (8) of reference [16]. This is the main difference between the governing equation being studied in the present paper and that of reference [16]. It is seen that, from the procedure of equations (6) and (7), the non-linear inertia term $2\varepsilon\alpha f^2 f''$ in $\varepsilon Q(f, f', f'', \tau)$ is replaced by the equivalent non-linear stiffness $-2\varepsilon\alpha f^3$ in $\varepsilon M(f, f', \tau)$. This means that equation (6) is equivalent to equation (7) only if $f'' = -f$; that is, the solution $f(\tau)$ is a harmonic function with unit angular frequency.

3. AVERAGING METHOD

The term $\varepsilon M(f, f', \tau)$ includes two different frequencies of harmonic and parametric excitations. It is convenient to assume that the solution of equation (7) has frequency θ, which is near the natural frequency 1, and has certain relationships with ω and ω_1. The relationships between θ and ω and/or ω_1 cause many resonant phenomena, which will be discussed in the next section. To determine an approximate solution of equation (7) for ε small but different from zero, we start with the Krylov–Bogoliubov technique [17] to transform the dependent variable from $f(\tau)$ to $a(\tau)$ and $\phi(\tau)$, where

$$f(\tau) = a(\tau)\cos\Theta + O(\varepsilon),$$

$$f'(\tau) = -\theta a(\tau)\sin\Theta, \qquad a'(\tau)\cos\Theta - a(\tau)\phi'\sin\Theta = 0,$$

$$f''(\tau) = -\theta a'(t)\sin\Theta - \theta(\theta + \phi'(\tau))a(\tau)\cos\Theta$$

$$\Theta = \theta\tau + \phi(\tau). \tag{8a–e}$$

From the above equation, it is seen that $f'' \neq -f$ and that different results will be expected from equations (6) and (7). Now, we substitute equation (8) into (7), and obtain the relationships in the form

$$a'(\tau) = \frac{-1}{\theta}[(\theta^2 - 1) + \varepsilon M(f, f', \tau)]\sin\Theta, \tag{9a}$$

$$a\phi'(\tau) = \frac{-1}{\theta}[(\theta^2 - 1) + \varepsilon M(f, f', \tau)]\cos\Theta. \tag{9b}$$

If $\theta^2 - 1 = O(\varepsilon)$, then equation (9) is in the correct form for averaging. We set $\theta^2 = 1 + \varepsilon\sigma$, where $\sigma = O(1)$ is the "detuning" parameter which quantitatively describes the nearness of θ to 1. Equation (9) can be rewritten as

$$a'(\tau) = -\frac{\varepsilon}{\theta}\sin\Theta N(f, f', \tau) \equiv \varepsilon F(a, \phi, \tau), \tag{10a}$$

$$a\phi'(\tau) = -\frac{\varepsilon}{\theta}\cos\Theta N(f, f', \tau) \equiv \varepsilon G(a, \phi, \tau), \tag{10b}$$

where

$$N(f, f', \tau) = -2\zeta f' + (\sigma + 2\mu\cos\omega\tau)f - (\gamma - 2\alpha)f^3 - 2\varepsilon_L f^2 f' - 2\alpha f f'^2 + e\cos\omega_1\tau.$$

Thus the original second order differential equation (7) for $f(\tau)$ has been replaced by two first order differential equations (10a) and (10b) for the amplitude $a(\tau)$ and the phase $\phi(\tau)$, respectively. Since ε is small, the terms on the right sides of equation (10) are so small that both $a(\tau)$ and $\phi(\tau)$ vary slowly with time, and the averaging method can be applied to the standard form (10). Let the averaged amplitude A and the averaged phase angle Φ satisfy the averaged equations

$$A' = \varepsilon \lim_{T\to\infty} T^{-1}\int_0^T F(A, \Phi, \tau)\,d\tau, \tag{11a}$$

$$A\Phi' = \varepsilon \lim_{T\to\infty} T^{-1}\int_0^T G(A, \Phi, \tau)\,d\tau, \tag{11b}$$

During integration, A and Φ are taken to be constants. The explicit form of equation (11) after some calculations is as follows:

$$A' = -\frac{\varepsilon}{2\theta}\left\{2\zeta A\theta + \frac{\varepsilon_L}{2}A^3\theta + \mu A\delta_{2\theta,\omega}\sin 2\Phi + e\delta_{\theta,\omega_1}\sin\Phi\right\}, \tag{12a}$$

$$A\Phi' = -\frac{\varepsilon}{2\theta}\left\{\sigma A + \left[\frac{\alpha}{2}(3-\theta^2) - \frac{3\gamma}{4}\right]A^3 + \mu A\delta_{2\theta,\omega}\cos 2\Phi + e\delta_{\theta,\omega_1}\cos\Phi\right\}, \tag{12b}$$

where $\delta_{i,j}$ is the Kronecker delta. Equation (12) is the averaged equation of (7), and its constant solutions correspond to the periodic solutions of equation (7) with period $2\pi/\theta$.

4. CLASSIFICATION OF AVERAGED SYSTEMS

In this section, we are interested in the resonant oscillations of equation (5) and thus will concentrate only on the constant solution of (12). Special note should be taken of those terms in equation (12) when certain definite relationships existed between θ, ω and ω_1. The occurrence of any of these terms in equation (12) indicates the possibility of existence of a resonant oscillation in system (5). Therefore, the oscillation behavior of the system can be categorized as follows: (a) $\omega \neq \theta \neq \omega_1$, non-resonant oscillation; (b) $\theta = \omega_1$, harmonic resonance; (c) $\theta = \frac{1}{2}\omega$, parametric resonance; (d) $\theta = \omega_1 = \frac{1}{2}\omega$, both harmonic and parametric resonances.

It has been shown that the basic system (5) admits four cases of resonant oscillations. Moreover, these cases have been classified according to the relationships existing between the linear natural frequency of the system and the frequencies of the parametric and harmonic excitations imposed upon it. The vibration analysis of a system in a given resonant condition classified above is the determination of the response of the system; that is, the relationship existing between the relevant frequencies and the amplitude of the system oscillations.

5. STEADY STATE RESPONSES OF AVERAGED SYSTEMS

Let us now study the averaged equations (12a, b) and investigate their steady state responses associated with the four different cases mentioned previously. Four relationships exist between the natural frequency of the system and the frequencies of the parametric and harmonic excitations. Each frequency relationship that appeared in equation (12) will result in a set of first order differential amplitude and phase equations. The steady state amplitudes and phases of such a set of equations will be obtained by letting $A' = \Phi' = 0$ and will be denoted by a subscript 0.

5.1. NON-RESONANT SYSTEM

From equation (12), and using the relationship $\omega \neq \theta \neq \omega_1$, the governing averaged equations in this case become

$$A' = -\frac{\varepsilon}{2\theta}\left\{2\zeta A\theta + \frac{\varepsilon_L}{2}A^3\theta\right\}, \qquad A\Phi' = -\frac{\varepsilon}{2\theta}\left\{\sigma A + \left[\frac{\alpha}{2}(3-\theta^2) - \frac{3\gamma}{4}\right]A^3\right\}. \tag{13a, b}$$

The terms in e and μ do not appear in equation (13) and here, without loss of generality, θ can be taken to be equal to 1, and then $\sigma = 0$. The amplitudes and phases of the steady state solutions do not vary with time, so if we let $A' = 0$ and $\Phi' = 0$ in equation (13), the constant solutions can be readily obtained as follows:

$$A_0 = 0, \qquad \Phi_0 = \text{arbitrary value}.$$

5.2. HARMONIC SYSTEM

In this case, substituting $\theta = \omega_1$ into equation (8), we obtain the first approximation as $f(\tau) = a(\tau)\cos[\omega_1\tau + \phi(\tau)] + O(\varepsilon)$; and substituting into equation (12), the equations of the averaged amplitude A and the averaged phase Φ are given, respectively, by

$$A' = -\frac{\varepsilon}{2\omega_1}\left\{2\zeta A\omega_1 + \frac{\varepsilon_L}{2}A^3\omega_1 + e\sin\Phi\right\} \tag{14a}$$

$$A\Phi' = -\frac{\varepsilon}{2\omega_1}\left\{\sigma A + \left[\frac{\alpha}{2}(3-\omega_1^2) - \frac{3\gamma}{4}\right]A^3 + e\cos\Phi\right].\tag{14b}$$

Setting $A' = \Phi' = 0$ in equations (14a, b), we obtain

$$a_6A_0^6 + a_4A_0^4 + a_2A_0^2 + a_0 = 0, \qquad \Phi_0 = \tan^{-1}\frac{X}{\mu + Y}, \tag{15a, b}$$

where the amplitude equation (15a) is cubic in A_0^2, and for the coefficients a_i, $i = 0, 2, \ldots, 6$, and X and Y, see Appendix 1. If the damping terms vanish, $\zeta = \varepsilon_L = 0$, and equations (15a, b) then become

$$\left[\frac{\alpha}{2}(3-\omega_1^2) - \frac{3\gamma}{4}\right]A_0^3 + \sigma A_0 \pm e = 0, \quad \Phi = n\pi, \qquad n = 0, 1, 2, \ldots.$$

5.3. PARAMETRIC SYSTEM

In this case, substituting $\theta = \frac{1}{2}\omega$ into equation (8), we obtain the first approximation as $f(\tau) = a(\tau)\cos[\frac{1}{2}\omega\tau + \phi(\tau)] + O(\varepsilon)$; and substituting into equation (12), the equations of the averaged amplitude A and the averaged phase Φ are given, respectively, by

$$A' = -\frac{\varepsilon}{\omega}\left\{\zeta A\omega + \frac{\varepsilon_L}{4}A^3\omega + \mu A\sin 2\Phi\right\}, \tag{16a}$$

$$A\Phi' = -\frac{\varepsilon}{\omega}\left\{\sigma A + \left[\frac{\alpha}{2}(3-\tfrac{1}{4}\omega^2) - \frac{3\gamma}{4}\right]A^3 + \mu A\cos 2\Phi\right\}. \tag{16b}$$

Setting $A' = \Phi' = 0$ in equations (16a, b), we obtain

$$b_4A_0^4 + b_2A_0^2 + b_0 = 0, \quad \Phi_0 = \tfrac{1}{2}\tan^{-1}\frac{X}{\mu + Y}, \tag{17a, b}$$

where the amplitude is quadratic in A_0^2, and for the coefficients b_i, $i = 0, 2, 4$, see Appendix 1. The constant solutions are readily found as

$$A_0 = \pm\sqrt{\frac{-b_2 \pm \sqrt{b_2^2 - 4b_4b_0}}{2b_4}}.$$

If the damping terms vanish, $\zeta = \varepsilon_L = 0$, and these responses become

$$A_0 = \pm\sqrt{(-\sigma \pm \mu)\Big/\left[\frac{\alpha}{2}(3 - \tfrac{1}{4}\omega^2) - \frac{3\gamma}{4}\right]}, \qquad \Phi_0 = 0.$$

5.4. SYSTEM BOTH HARMONIC AND PARAMETRIC

In this case, $\theta = \omega_1 = \frac{1}{2}\omega$. The governing averaged equations are

$$A' = -\frac{\varepsilon}{2\theta}\left\{2\zeta A\theta + \frac{\varepsilon_L}{2}A^3\theta + \mu A \sin 2\Phi + e \sin \Phi\right\}, \tag{18a}$$

$$A\Phi' = -\frac{\varepsilon}{2\theta}\left\{\sigma A + \left[\frac{\alpha}{2}(3 - \theta^2) - \frac{3\gamma}{4}\right]A^3 + \mu A \cos 2\Phi + e \cos \Phi\right\}. \tag{18b}$$

As might be expected, the analysis of the system (18a, b) is more complicated than those of the systems (14) and (16). Setting $A' = \Phi' = 0$ in equation (18a, b), the response equations for this system are found as follows:

$$c_{10}A_0^{10} + c_8A_0^8 + c_6A_0^6 + c_4A_0^4 + c_2A_0^2 + c_0 = 0, \qquad \Phi_0 = \tan^{-1}(X/Y), \tag{19a, b}$$

where the amplitude equation is pentamerous in A_0^2, and for the coefficients c_i, $i = 0, 2, \ldots, 10$, see Appendix 1. If the damping terms vanish, $\zeta = \varepsilon_L = 0$, and the response equation (19a) becomes

$$\{[y_1A_0^3 + (\sigma + \mu)A_0]^2 - e^2\}(y_1A_0^2 + \sigma - \mu)^2 = 0.$$

6. METHOD OF MULTIPLE SCALES

In this section, we will determine a uniform first order approximate solution of the original system (6) by the method of multiple scales [5]. Consider the driving frequency θ near natural frequency 1. The proximity of θ to unity can be expressed as

$$\theta^2 = 1 + \varepsilon\sigma, \tag{20}$$

where σ is a detuning parameter. The natural frequency of the linear oscillator in equation (6) can be written in terms of θ by using equation (20). Sequentially, we obtain the equation in the following form:

$$f'' + \theta^2 f = \varepsilon[\sigma f - 2\zeta f' + 2\mu\cos\omega\tau f - \gamma f^3 - 2\varepsilon_L f^2 f' - 2\alpha(f^2 f'' + ff'^2) + e\cos\omega_1\tau]. \tag{21}$$

We let

$$f(\tau; \varepsilon) = f_0(T_0, T_1) + \varepsilon f_1(T_0, T_1) + \ldots, \tag{22}$$

where $T_0 = \tau$ is a fast scale associated with changes occurring at the frequencies 1, ω and ω_1, and $T_1 = \varepsilon\tau$ is a slow scale associated with modulations in the amplitude and phase caused by the non-linearity, damping and resonances. In terms of the multiple time scales T_n, the time derivatives becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = D_0 + \varepsilon D_1 + \ldots, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} = D_0^2 + \varepsilon 2D_0D_1 + \ldots, \tag{23a, b}$$

where $D_n = \partial/\partial T_n$. Substituting equations (22) and (23) into equation (21) and equating coefficients of like powers of ε, one obtains

$$D_0^2 f_0 + \theta^2 f_0 = 0, \tag{24}$$

$$D_0^2 f_1 + \theta^2 f_1 = \sigma f_0 - 2D_1 D_0 f_0 - 2\zeta D_0 f_0 + 2\mu \cos \omega T_0 f_0 - \gamma f_0^3 - 2\varepsilon_L f_0^2 D_0 f_0 - 2\alpha[f_0^2 D_0^2 f_0 + f_0(D_0 f_0)^2] + e \cos \omega_1 T_0. \tag{25}$$

The solution of equation (24) can be expressed in the complex form

$$f_0(T_0, T_1) = a(T_1)\,\mathrm{e}^{\mathrm{i}\theta T_0} + \bar{a}(T_1)\,\mathrm{e}^{-\mathrm{i}\theta T_0}, \tag{26}$$

where $a(T_1)$ is an undetermined function at this point, which will be determined by eliminating the secular terms from f_1. Based on the solution of (26), we have

$$-2\alpha f_0^2 D_0^2 f_0 = 2\alpha\theta^2 f_0^3.$$

Meanwhile, equation (25) can be rewritten as

$$D_0^2 f_1 + \theta^2 f_1 = \sigma f_0 - 2D_1 D_0 f_0 - 2\zeta D_0 f_0 + 2\mu \cos \omega T_0 f_0 - (\gamma - 2\alpha\theta^2) f_0^3 - 2\varepsilon_L f_0^2 D_0 f_0 - 2\alpha f_0 (D_0 f_0)^2 + e \cos \omega_1 T_0. \tag{27}$$

Substituting equation (26) into equation (27) and expressing $2\mu \cos \omega T_0$ and $e \cos \omega_1 T_0$ in complex form, we have

$$\begin{aligned} D_0^2 f_1 + \theta^2 f_1 = {} & [-\mathrm{i}2\theta D_1 a + \sigma a - \mathrm{i}2\zeta a\theta + \mu(a\delta_{\omega,0} + \bar{a}\delta_{\omega,2\theta}) \\ & + 3(2\alpha\theta^2 - \gamma)a^2\bar{a} - \mathrm{i}2\varepsilon_L a^2\bar{a}\theta - 2\alpha a^2\bar{a}\theta^2 + \tfrac{1}{2}e\delta_{\omega_1,\theta}]\,\mathrm{e}^{\mathrm{i}\theta T_0} \\ & + [\mu(a\delta_{w,2\theta} + \bar{a}\delta_{\omega,4\theta}) + (2\alpha\theta^2 - \gamma)a^3 \\ & - \mathrm{i}2\varepsilon_L a^3\theta + 2\alpha a^3\theta^2 + \tfrac{1}{2}e\delta_{\omega_1,3\theta}]\,\mathrm{e}^{\mathrm{i}3\theta T_0} + c.c., \end{aligned} \tag{28}$$

where $c.c.$ denotes the complex conjugate of the preceding term and θ is near 1. Depending on the function $a(T_1)$, the particular solution of equation (28) may contain secular terms. The condition for the elimination of these secular terms (that is, the solvability condition) is

$$-\mathrm{i}2\theta D_1 a + \sigma a - \mathrm{i}2\zeta a\theta + \mu\bar{a} + 3(2\alpha\theta^2 - \gamma)a^2\bar{a} - \mathrm{i}2\varepsilon_L a^2\bar{a}\theta - 2\alpha a^2\bar{a}\theta^2 + e/2 = 0. \tag{29}$$

To obtain the first order approximation, a is considered as a function of T_1 only, and, expressing $a(T_1)$ in the polar form,

$$a(T_1) = A(T_1)\,\mathrm{e}^{\mathrm{i}\Phi(T_1)}, \tag{30}$$

where $A(T_1)$ and $\Phi(T_1)$ are, respectively, the amplitude and phase with the fundamental frequency. Substituting equation (30) into equation (29) and separating the real and the imaginary parts, we have

$$A' = -\frac{\varepsilon}{2\theta}\left\{2\zeta A\theta + \frac{\varepsilon_L}{2}A^3\theta + \mu A\delta_{2\theta,\omega}\sin 2\Phi + e\delta_{\theta,\omega_1}\sin\Phi\right\}, \tag{31a}$$

$$A\Phi' = -\frac{\varepsilon}{2\theta}\left\{\sigma A + \left(\alpha\theta^2 - \frac{3\gamma}{4}\right)A^3 + \mu A\delta_{2\theta,\omega}\cos 2\Phi + e\delta_{\theta,\omega_1}\cos\Phi\right\}, \tag{31b}$$

where $(\,\cdot\,)' = \mathrm{d}(\,\cdot\,)/\mathrm{d}T_1$. The system (31) could also be reduced to the various systems such as those in section 5.

The first order solution then becomes

$$f(\tau) = f_0(T_0, T_1) = A(T_1)\cos(\theta\tau + \Phi(T_1)), \tag{32}$$

where $A(T_1)$ and $\Phi(T_1)$ are determined from equations (31a, b). Thus, the uniform first order approximate solution (32) is obtained for the original system (6) by the method of multiple scales. Since $D_0^2 f = -\theta^2 f$ could be obtained from equations (24) and (32), and equation (6) is equivalent to equation (7), we will have the same resultant solution as equation (31) if the method of multiple scales is directly applied to equation (7).

7. NUMERICAL RESULTS AND DISCUSSION

The system (7) has been shown to be reduced to two different sets of first order differential equations (12a, b) and (31a, b) by the averaging method and the method of multiple scales, respectively. The coefficients of the non-linear damping factor ε_L and the non-linear elasticity factor γ are the same in equations (12a, b) and (31a, b). However, the coefficients of the non-linear inertia factor α are different in the phase differential equations (12b) and (31b). Hence, it is expected that the result discrepancies will happen in the phase and then in the amplitude between the averaging method and method of multiple scales.

In order to obtain the explicit results for steady state responses of the various resonant systems, we consider an Al alloy beam which is the same as that of reference [13] (see Appendix 2). It was seen from references [10, 16] that the manner in which the resonant curve is bent depends on the sign of the parameter

$$p = \left(\frac{3\gamma}{4\alpha\theta^2}\right)_{\theta=1} = \frac{3\bar{\gamma}}{4\bar{\alpha}\Omega^2}. \tag{33}$$

From the above equation, the system is hard or soft, depending on the three parameters, $\bar{\gamma}$, $\bar{\alpha}$ and Ω. By adjusting these three parameters, the beam system can be of the soft spring type when $p < 1$, or the hard spring type when $p > 1$.

If there are no mass (M_L) and spring (k) components in the subsystem to affect the main system, equation (33) can be rewritten as

$$p = 0{\cdot}1287/(1 - P_0/P_E). \tag{34}$$

In addition, for the case with static loading $P_0 = 0$, the system is the soft spring type, $p = 0{\cdot}1287$. As the static loading is increased to greater than $0{\cdot}8713P_E$, the system becomes of the hard spring type.

In Figures 2 and 3 are shown the transient amplitudes of the beam system with both harmonic and parametric resonance where the detuning parameter $\varepsilon\sigma = -0{\cdot}02$ and $-0{\cdot}2$, respectively. These curves are obtained by the use of the fourth order Runge–Kutta method in direct integration of equations (6), (12a, b) and (31a, b), which are the equations of the original system, the averaging method and the method of multiple scales, respectively.

The difference between the non-linear inertia terms in equations (12b) and (31b) is $\frac{3}{2}\varepsilon^2\alpha\sigma \simeq 0{\cdot}0025$ in Figures 2(a), 2(b) and 2(c). From Figure 2(a), it can be seen that the beating amplitudes build up and then diminish in the regular pattern in the transient amplitudes. The differences between the transient amplitudes are shown in Figures 2(b) and 2(c). These curves are obtained by subtracting the amplitudes of the original system (6) from those of equations (8a) and (32), obtained by the averaging method and the method of multiple scales, respectively. Comparing Figure 2(b) with Figure 2(c), it is observed that the amplitude differences are almost the same, but the amplitudes are not in the same phase.

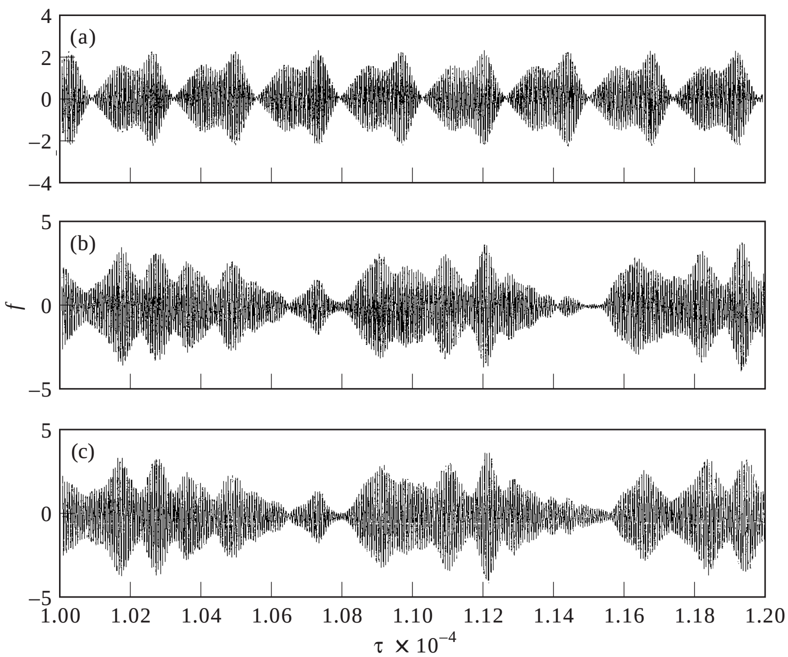

Figure 2. Transient amplitudes. (a) Numerical integration of the original system (6); (b) amplitude differences between the original system (6) and equation (8a), obtained by the averaging method; (c) amplitude differences between the original system (6) and equation (32), obtained by the method of multiple scales. $\varepsilon\zeta = \varepsilon\varepsilon_L = 0$, $\varepsilon\gamma = 0{\cdot}05483$, $\varepsilon\alpha = 0{\cdot}08218$, $\omega = 1{\cdot}7888$, $\omega_1 = 0{\cdot}8944$, $\varepsilon\sigma = -0{\cdot}02$, $\varepsilon\mu = \varepsilon e = 0{\cdot}05$, $f(0) = 0{\cdot}01$ and $f'(0) = 0$.

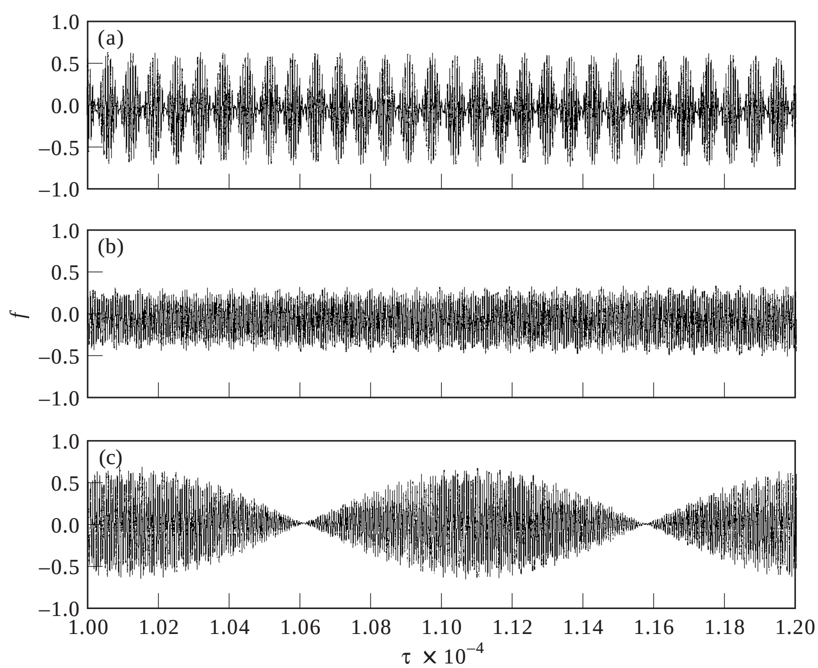

Figure 3. Transient amplitudes. (a) Numerical integration of the original system (6); (b) amplitude differences between the original system (6) and equation (8a), obtained by the averaging method; (c) amplitude differences between the original system (6) and equation (32), obtained by the method of multiple scales. $\varepsilon\sigma = -0{\cdot}2$: the other parameters are as in Figure 2.

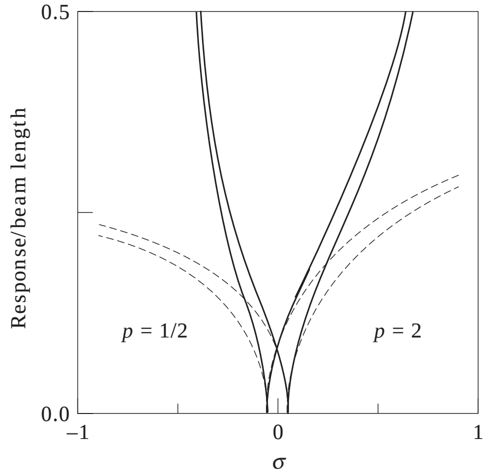

Figure 4. Steady state responses per beam length of the harmonic system ($\varepsilon e = 0{\cdot}05$) with respect to the detuning parameter $\varepsilon\sigma$. ——, the averaging method; – – –, the method of multiple scales. $p = 1/2$, soft spring type system; $p = 2$, hard spring type system.

Figures 3(a)–(c) are similar to Figures 2(a)–(c), except that $\varepsilon\sigma = -0{\cdot}2$. Comparing Figure 3(a) with Figure 2(a), it can be seen that as the detuning parameter increases, the transient amplitude decreases. Figures 3(b) and 3(c) reveal the differences in the transient amplitudes. Comparing Figure 3(b) with Figure 3(c), it can be seen that the amplitude differences are smaller when obtained by the averaging method than by the method of multiple scales.

The steady state amplitude A_0 could be obtained from the cubic algebraic equation (15) in A_0^2 for the harmonic system, from the quadratic equation (17) in A_0^2 for the parametric system, and from the pentamerous equation (19) in A_0^2 for both the harmonic and parametric system. The steady state amplitude A_0 in equations (15), (17) and (19), based on the physical considerations, are constrained to be positive real number and in the interval $[0, l/2]$. In Figures 4–6 are shown the frequency–response relationships of the

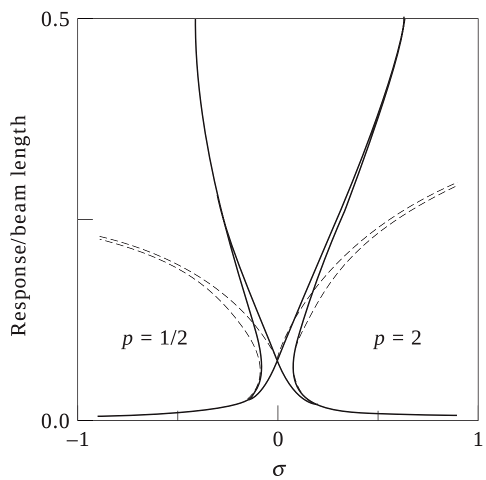

Figure 5. Steady state responses per beam length of the parametric system ($\varepsilon\mu = 0{\cdot}05$) with respect to the detuning parameter $\varepsilon\sigma$. Key and f values as in Figure 4.

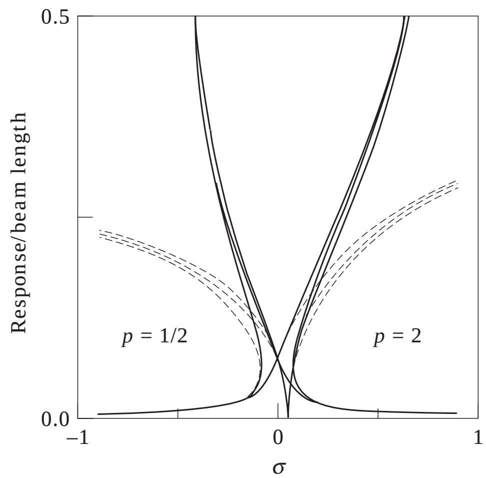

Figure 6. Steady state responses per beam length of the system that is both harmonic and parametric ($\varepsilon e = \varepsilon\mu = 0{\cdot}05$) with respect to the detuning parameter $\varepsilon\sigma$. Key and f values as in Figure 4.

harmonic, parametric, and both harmonic and parametric resonance systems, respectively. The response curve is bent toward increasing frequency for the hard spring type system, and bent toward decreasing frequency for the soft spring type system.

In order to equate equations (12b) and (31b), it is found that the sufficient condition is $\theta^2 = 1$. This means that the detuning parameter must have the value $\sigma = 0$ in the proximity of the driving frequency equation (20). Hence, in Figures 4–6 the responses have the same value at $\sigma = 0$ for both the averaging method and the method of multiple scales. The greater value of the absolute detuning parameter $|\sigma|$ will yield a greater difference between the two methods. Since the system is hard or soft depending on the value p calculated at $\theta = 1$ in equation (33), the system is independent of the method being used.

8. CONCLUSIONS

The dynamic behavior of a simply supported beam under both harmonic and parametric excitations has been investigated by the averaging method and the method of multiple scales. From the results and comparisons, we conclude the following.

1. Based upon the first order solution assumptions, there exist different transient amplitudes and steady state responses between the averaging method and the method of multiple scales, if the non-linear inertial force of the system is considered. Otherwise, the two methods will lead to the same results.
2. The transient amplitudes obtained by the averaging method and the method of multiple scales are different in both the amplitude and the phase angle.
3. At the same value of the detuning parameter, the steady state responses obtained by the averaging method will be greater than those obtained by the method of multiple scales. However, the steady state responses will be equal at the zero detuning parameter value.
4. Whether the system is hard or soft is independent of the method being used.

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APPENDIX 1: COEFFICIENTS OF AMPLITUDE AND PHASE EQUATIONS

$$X = x_1 A_0^2 + x_2, \qquad Y = y_1 A_0^2 + y_2,$$

$$x_1 = \tfrac{1}{2}\varepsilon_L \theta, \qquad x_2 = 2\zeta\theta, \qquad y_1 = \alpha/2(3-\theta^2) - 3\gamma/4, \qquad y_2 = \sigma - \mu,$$

$$z_1 = x_1^2 + y_1^2, \qquad z_2 = 2(x_1 x_2 + y_1 y_2), \qquad z_3 = x_2^2 + y_2^2,$$

$$a_6 = z_1, \qquad a_4 = z_2 + 2\mu y_1, \qquad a_2 = z_3 + 2\mu y_2 + \mu^2, \qquad a_0 = -e^2,$$

$$b_4 = z_1, \qquad b_2 = z_2 + 2\mu y_1, \qquad b_0 = z_3 + 2\mu y_2,$$

$$c_{10} = z_1^2, \qquad c_8 = 2z_1(z_2 + 2\mu y_1),$$

$$c_6 = z_1(z_3 + 2\mu y_2 + \mu^2) + z_2(z_2 + 2\mu y_1) + z_1 z_3 - z_1\mu^2 + 2\mu y_1 z_2 + 2\mu y_2 z_1 + 4\mu^2 y_1^2,$$

$$c_4 = z_2(z_3 + 2\mu y_2 + \mu^2) + z_3(z_2 + 2\mu y_1) - z_2\mu^2 + 2\mu y_1 z_3 + 2\mu y_2 z_2 - e^2 z_1 + 8\mu^2 y_1 y_2,$$

$$c_2 = z_3(z_3 + 2\mu y_2 + \mu^2) - z_3\mu^2 + 2\mu y_2 z_3 - e^2 z_2 + 4\mu^2 y_2^2, \qquad c_0 = -e^2 z_3.$$

APPENDIX 2: THE THIN BEAM

Dimensions = 15·0 × 1·0 × 0·1 cm.
Material: 5052 (Al alloy).
Young's modulus = 7E08 g/cm^2.
Density = 2·7 g/cm^3.
Static buckling load, $P_E = 2559$ g.
$\omega_0/2\pi = 101{\cdot}6$ Hz.

APPENDIX 3: NOMENCLATURE

$\bar{\zeta} = C/2m$, damping factor
$\Omega = \omega_0\sqrt{1 - P_0/P_E}$, mode frequency
$\omega_0 = (\pi^2/l^2)\sqrt{(EI/m)}$, natural frequency
$P_E = \pi^2 EI/l^2$, Euler buckling load
$\bar{\mu} = P_t/[2(P_E - P_0)]$, dimensionless load parameter
$\bar{\gamma} = \pi^4 K/4ml^3$, non-linear elasticity factor
$\bar{\varepsilon}_L = \pi^4 C_L/4ml^3$, non-linear damping factor
$\bar{\alpha} = \pi^4 M/4ml^3$, non-linear inertia factor
$\tau = \Omega t$, non-dimensional time