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ON THE ENERGY DECAY OF TWO COUPLED STRINGS THROUGH A JOINT DAMPER

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The stabilization of two coupled strings with fixed boundaries attached to a linear damper at their joint of connection is studied. The transformation used in a previous approach to convert the second order hyperbolic equations into a first order system results in a loss of equivalence. It is shown that when the ratio of the wave speeds of the two strings is an irrational number, the system is asymptotically but not exponentially, stable. The method is applied to the other cases with different joint and boundary conditions.

1. INTRODUCTION

Many engineering structures, such as power transmission lines, and aerial and suspension cables, are commonly modelled as a chain of coupled strings. In the construction of these systems, active and passive damping devices can often be installed at the joints of connection to suppress the deleterious vibration. The model of coupled vibrating strings is described by a system of partial differential equations, with associated joint and boundary conditions. The study of vibration damping of such distributed parameter systems is of both practical and theoretical interest.

The mechanism of energy dissipation was first studied for the simple model consisting of two coupled strings with equal length and wave speed and a linear damper at their joint of connection [1]. It was shown that both uniform and non-uniform rates of decay of the energy of vibration can occur depending on the internal and boundary conditions. In some cases in which the system is symmetric with respect to the damper, the energy of vibration does not decay asymptotically; while in some other cases with special damping constants, all the energy of vibration is dissipated in finite time. In the general case in which the wave speeds of the two strings are different, an abstract approach using the frequency domain method was developed to predict the stability of the coupled system [2]. The stability of the system was readily inferred when the ratio of the wave speeds of the two strings, d, is a rational number. The main difficulty was to handle the case in which d is irrational.

A flaw in the proof of Theorem 4.2 in [2] is identified first in this paper for the case in which the two coupled strings with fixed boundaries are attached to a linear damper at their joint of connection. It is found that the transformation used therein did not lead to an equivalent, first order system, because the prescribed internal and boundary conditions were replaced by their time differential forms. As a result, the transformed system admits an extraneous zero eigenvalue that is not present in the original system. It is shown that the energy of vibration of the system decays asymptotically when the ratio of the wave speeds of the two strings is an irrational number. Although all of the eigenvalues remain strictly in the left half-plane in this case, it is shown that an infinite number of them

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approach the imaginary axis. Hence the energy of vibration does not decay uniformly exponentially. The method is applicable to the other cases considered in the previous analyses, as illustrated in section 5.

2. EQUATIONS OF MOTION

Consider two coupled equal-length strings described, respectively, by the wave equations

$$m_1 y_{tt}(x, t) - T_1 y_{xx}(x, t) = 0, \qquad x \in (0, 1), \quad t > 0,$$
$$m_2 y_{tt}(x, t) - T_2 y_{xx}(x, t) = 0, \qquad x \in (1, 2), \quad t > 0, \tag{1}$$

where m_i and T_i $(i = 1, 2)$ are the mass density and tension in each string. The corresponding wave speeds are $c_i = \sqrt{T_i/m_i}$ $(i = 1, 2)$. Without loss of generality, the length of each string is normalized to unity. The boundary conditions at the two ends $x = 0, 2$ are either free or fixed, resulting in three possibilities:

$$y(0, t) = y(2, t) = 0, \qquad y(0, t) = y_x(2, t) = 0, \qquad y_x(0, t) = y_x(2, t) = 0. \quad \text{(2a–c)}$$

The internal conditions at the joint $x = 1$ are described by either

$$T_2 y_x(1^+, t) = T_1 y_x(1^-, t), \qquad T_1 y_{1x}(1^-, t) = k[y_{2t}(1^+, t) - y_{1t}(1^-, t)] \tag{3a}$$

or

$$y(1^-, t) = y(1^+, t), \qquad T_2 y_x(1^+, t) - T_1 y_x(1^-, t) = k y_t(1^+, t), \tag{3b}$$

where $k > 0$ is the damping constant. At the joint $x = 1$, either the two strings are connected end to end through a dashpot in equation (3a), or both ends of the strings are attached to one end of a dashpot in equation (3b). The initial conditions are given by

$$y(x, 0) = y_0(x), \qquad y_t(x, 0) = y_1(x). \tag{4}$$

For each combination of internal and boundary conditions, the energy of vibration

$$E(t) = \tfrac{1}{2}\int_0^1 [m_1 y_t^2(x, t) + T_1 y_x^2(x, t)]\,\mathrm{d}x + \tfrac{1}{2}\int_1^2 [m_2 y_t^2(x, t) + T_2 y_x^2(x, t)]\,\mathrm{d}x \tag{5}$$

satisfies $\mathrm{d}E(t)/\mathrm{d}t \leqslant 0$ [1]. The system is dissipative and all eigenvalues have non-positive real parts.

The conditions that ensure asymptotic and exponential rates of decay of the energy of vibration were examined in [2], as summarized in Table 1. For cases I and VI, when $d = c_1/c_2 = (4p + 1)/(4q \pm 1)$, where $p, q \in \mathbf{Z}$, a branch of eigenvalues lies on the imaginary

TABLE 1

Stability for different combinations of internal and boundary conditions

Case	Boundary conditions	Internal conditions	Asymptotic stability	Exponential stability
I	(2a)	(3a)	$d \neq (4p + 1)/(4q \pm 1)$	$d \neq (4p + 1)/(4q \pm 1)$, $d \in \mathbf{Q}$
II	(2b)	(3a)	$d \neq (4p + 2)/(4q \pm 1)$	$d \neq (4p + 2)/(4q \pm 1)$, $d \in \mathbf{Q}$
III	(2c)	(3a)	No	No
IV	(2a)	(3b)	$d \neq p/q$	No
V	(2b)	(3b)	$d \neq (4p + 1)/(4q \pm 2)$	$d \neq (4p + 1)/(4q \pm 2)$, $d \in \mathbf{Q}$
VI	(2c)	(3b)	$d \neq (4p + 1)/(4q \pm 1)$	$d \neq (4p + 1)/(4q \pm 1)$, $d \in \mathbf{Q}$

axis; the systems are not asymptotically stable. For other rational d, all of the eigenvalues are distributed along a finite number of lines in the left half-plane parallel to the imaginary axis; the systems are asymptotically as well as exponentially stable. For irrational d, the systems are asymptotically, but not exponentially, stable. Similar conclusions hold for cases II and V. The system in case III is not asymptotically stable because zero is an eigenvalue of the system. The same prediction for case IV, however, as presented in Theorem 4.2 in [2], is not always true. In what follows it is shown that the system is not asymptotically stable when d is rational. For irrational d, the system is asymptotically, but not exponentially, stable.

3. ASYMPTOTIC STABILITY

The flaw in the proof of the Theorem 4.2 in [2] is indicated first. Through the transformation

$$\begin{aligned} w_1(x,t) &= \sqrt{m_1}[-c_1 y_x(x,t) + y_t(x,t)]/2,\\ w_2(x,t) &= \sqrt{m_2}[-c_2 y_x(2-x,t) + y_t(2-x,t)]/2,\\ w_3(x,t) &= \sqrt{m_1}[c_1 y_x(x,t) + y_t(x,t)]/2,\\ w_4(x,t) &= \sqrt{m_2}[c_2 y_x(2-x,t) + y_t(2-x,t)]/2, \end{aligned} \tag{6}$$

the system in case IV reduces in [2] to an evolution equation in the state space $(L^2(0,1))^4$:

$$\partial \mathbf{w}(x,t)/\partial t = \mathbf{A}\mathbf{w}(x,t), \qquad \mathbf{w}(x,0) = \mathbf{w}_0(x), \tag{7}$$

where

$$\mathbf{w}(x,t) = [w_1(x,t), w_2(x,t), w_3(x,t), w_4\,(x,t)]^{\mathrm{T}}, \tag{8}$$

$$\begin{aligned} \mathbf{A}\boldsymbol{\Phi}(x) &= \operatorname{diag}(-c_1, c_2, c_1, -c_2)\frac{\partial}{\partial x}\boldsymbol{\Phi}(x),\\ \forall\boldsymbol{\Phi}(x) &= [\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)]^{\mathrm{T}} \in D(\mathbf{A}), \end{aligned} \tag{9}$$

$$\begin{aligned} D(\mathbf{A}) = \Big\{ &\boldsymbol{\Phi}(x) \in (L^2(0,1))^4 \mid \phi_1(0) + \phi_3(0) = \phi_2(0) + \phi_4(0) = 0,\\ &\sqrt{m_2}[\phi_1(1) + \phi_3(1)] = \sqrt{m_1}[\phi_2(1) + \phi_4(1)],\\ &\sqrt{T_1}[\phi_3(1) - \phi_1(1)] - \sqrt{T_2}[\phi_4(1) - \phi_2(1)] = -\frac{k}{\sqrt{m_1}}[\phi_1(1) + \phi_3(1)] \Big\}, \end{aligned} \tag{10}$$

$$\begin{aligned} \mathbf{w}_0(x) = \tfrac{1}{2}(&\sqrt{m_1}[-c_1 y_0'(x) + y_1(x)], \sqrt{m_2}[-c_2 y_0'(2-x) + y_1(2-x)],\\ &\sqrt{m_1}[c_1 y_0'(x) + y_1(x)], \sqrt{m_2}[c_2 y_0'(2-x) + y_1(2-x)])^{\mathrm{T}}. \end{aligned} \tag{11}$$

The well-posedness of the transformed equations (7)–(11) is established in [2] through the dissipativity of the operator $\mathbf{A}$ within the framework of the theory of semigroups. The principal observation here is that, instead of prescribing the boundary conditions (2a) and $(3b)_1$, equation (10) imposes the corresponding velocity relations

$$y_t(0,t) = y_t(2,t) = 0, \qquad y_t(1^-,t) = y_t(1^+,t). \tag{12, 13}$$

While zero is not an eigenvalue of the original system in case IV, it becomes an eigenvalue of the transformed system (7). Because all the solutions to the original system in case IV constitute a subset of those of the transformed one, the system in case IV is also well-posed. The energy functional in equation (5) is equivalent to the L^2-norm of $\mathbf{w}(\cdot, t)$ [1]:

$$E(t) = \|\mathbf{w}(\cdot\,, t)\|^2 \equiv \sum_{i=1}^{4} \int_0^1 |w_i(x, t)|^2 \, \mathrm{d}x. \tag{14}$$

Assuming a separable solution $y(x, t) = \phi(x)\,\mathrm{e}^{\lambda t}$, where λ is the eigenvalue and $\phi(x)$ is the eigenfunction, and substituting it into equations (1), (2a) and (3b), yields

$$\begin{aligned}
&\lambda^2\phi(x) - c_1^2\phi''(x) = 0, \qquad x\in(0, 1),\\
&\lambda^2\phi(x) - c_2^2\phi''(x) = 0, \qquad x\in(1, 2),\\
&\phi(0) = \phi(2) = 0,\\
&\phi(1^-) = \phi(1^+),\\
&T_1\phi'(1^-) - T_2\phi'(1^+) = -k\lambda\phi(1).
\end{aligned} \tag{15}$$

The eigenvalue problem (15) leads to the characteristic equation

$$(a + b + k)\,\mathrm{e}^{(1+d)\mu} + (b - a - k)\,\mathrm{e}^{\mu} + (a - b - k)\,\mathrm{e}^{d\mu} - a - b + k = 0, \tag{16}$$

where

$$a = T_1/c_1, \qquad b = T_2/c_2, \qquad d = c_1/c_2, \qquad \mu = 2\lambda/c_1. \tag{17}$$

Equation (16) is identical to the characteristic equation for the operator $\mathbf{A}$ as derived in [2]. $\lambda \neq 0$ is an eigenvalue of equation (15) if and only if it is also an eigenvalue of $\mathbf{A}$. Although $\lambda = 0$ indeed satisfies equation (16), a direct calculation shows that it is not an eigenvalue of equation (15), because the system does not permit any rigid body motion. However, it can be easily shown that $\lambda = 0$ is indeed an eigenvalue of $\mathbf{A}$ with the associated eigenvector

$$\mathbf{\Phi}_0 = (\sqrt{T_2}, \sqrt{T_1}, -\sqrt{T_2}, -\sqrt{T_1}). \tag{18}$$

The fact that the transformed system (7) possesses an extraneous zero eigenvalue does not preclude the asymptotic stability of the original system in case IV. The following theorem establishes the necessary and sufficient condition for all of the eigenvalues in case IV to remain strictly in the left half-plane.

Theorem 1. All of the eigenvalues of equation (15) satisfy $\mathrm{Re}\,\lambda < 0$ if and only if d is irrational.

Proof. Because $\mathrm{Re}\,\lambda \leqslant 0$ by equation (5), $\mathrm{Re}\,\lambda < 0$ if and only if there are no eigenvalues on the imaginary axis. Multiplying equations $(15)_1$ and $(15)_2$ by $\overline{\phi}(x)$, where the overbar denotes complex conjugation, integrating the resulting equations in [0, 1] and [1, 2] respectively, adding the two expressions and using the boundary conditions in equation (15), yields

$$k\lambda|\phi(1)|^2 + \int_0^1 [T_1|\phi'(x)|^2 + m_1\lambda^2|\phi(x)|^2]\,\mathrm{d}x + \int_1^2 [T_2|\phi'(x)|^2 + m_2\lambda^2|\phi(x)|^2]\,\mathrm{d}x = 0. \tag{19}$$

If $\lambda = \mathrm{i}\omega$, where ω is real, separating the imaginary part from equation (19) gives

$\phi(1)=0$. Then $T_1\phi'(1^-)=T_2\phi'(1^+)$ by $(15)_5$. Hence, by equation (15), the eigenfunctions are

$$\phi(x)=\begin{cases}A_1\sin\omega x/c_1, & x\in(0,1),\\ A_2\sin\omega(x-2)/c_2, & x\in(1,2).\end{cases} \tag{20}$$

Because $\phi(1)=0$, we have, by equation (20),

$$A_1\sin\omega/c_1=0,\qquad A_2\sin\omega/c_2=0. \tag{21}$$

Neither A_1 nor A_2 vanish. If $A_1=0$, then $\phi(x)=0$ for all $x\in[0,1]$. Since $\phi(x)$ cannot be identically zero in $x\in[1,2]$, $A_2\neq 0$. By equation (21), $\sin\omega/c_2=0$; hence $\cos\omega/c_2\neq 0$. However, $T_2\phi'(1^+)=(\omega/c_2)T_2A_2\cos(\omega/c_2)=T_1\phi'(1^-)=0$ yields a contradiction; $A_2=0$. Hence $A_1\neq 0$. Similarly $A_2\neq 0$. Then equation (21) implies that $\omega/c_1=q\pi$ and $\omega/c_2=p\pi$ for some integers p and q. Therefore $d=c_1/c_2=p/q$ is rational.

Conversely, if $d=c_1/c_2=p/q$, where p and q are co-prime positive integers, let $\omega=c_1q=c_2p$. It can be easily shown that

$$\lambda_n=\mathrm{i}n\omega=\mathrm{i}c_1qn\pi,\qquad n=\pm1,\pm2,\ldots \tag{22}$$

are a branch of imaginary eigenvalues of equation (15). □

For rational $d=p/q$ with p and q co-prime, all of the eigenvalues of equation (15) can be obtained exactly. Let $z=\mathrm{e}^{2\lambda/qc_1}$; equation (16) reduces to the following polynomial equation of degree $p+q$:

$$(a+b+k)z^{p+q}+(b-a-k)z^q+(a-b-k)z^p-a-b+k=0. \tag{23}$$

Note that $z=1$ is a root of equation (23). The corresponding branch of eigenvalues is purely imaginary, as given by equation (22). The eigenvalues corresponding to the root z_k ($z_k\neq 1$) of equation (23) are

$$\lambda_n=qc_1[\ln|z_k|+\mathrm{i}(\arg z_k+2\pi n)]/2,\qquad n=0,\pm1,\pm2,\ldots. \tag{24}$$

Each branch of eigenvalues in equation (24) lies on a straight line parallel to the imaginary axis and hence represents a constant rate of damping.

For the sake of mathematical rigor, in what follows we show that, for irrational d, the system in case IV is indeed asymptotically stable. Define $\mathbf{H}$ in $(L^2(0,1))^4$ orthogonal to $\mathbf{\Phi}_0$:

$$\mathbf{H}=\{\mathbf{\Phi}\in(L^2(0,1))^4\,|\,\langle\mathbf{\Phi},\mathbf{\Phi}_0\rangle=0\}, \tag{25}$$

where $\mathbf{H}$ is a Hilbert space with the same inner product as that in $(L^2(0,1))^4$. Let

$$\mathbf{A_H}=\mathbf{A}|_{\mathbf{H}} \tag{26}$$

be the restriction of the operator $\mathbf{A}$ in $\mathbf{H}$. Then $\mathbf{A}$ and $\mathbf{A_H}$ have the same non-zero spectrum. The main distinction between them is that $\lambda=0$ is not an eigenvalue of $\mathbf{A_H}$. It is crucial to impose the boundary condition (2a) on the transformed system. The velocity relation (12) as specified in equation (10) follows from equation (2a). By a direct integration using equations (6) and (2a), we have

$$\langle\mathbf{w}(\cdot,t),\mathbf{\Phi}_0\rangle=\sqrt{T_2}\int_0^1[w_1(x,t)-w_3(x,t)]\,\mathrm{d}x+\sqrt{T_1}\int_0^1[w_2(x,t)-w_4(x,t)]\,\mathrm{d}x=0. \tag{27}$$

Hence $\mathbf{w}(x, t) \in \mathbf{H}$. Instead of equation (7), we consider the new transformed system

$$\partial \mathbf{w}(x, t)/\partial t = \mathbf{A_H} \mathbf{w}(x, t), \qquad \mathbf{w}(x, 0) = \mathbf{w}_0(x) \tag{28}$$

in the state space $\mathbf{H}$. The extraneous zero eigenvalue in equation (7) is eliminated in equation (28). By Theorem 1 and Theorem 4.1 in [2], the system (28) is asymptotically stable when d is irrational; i.e., for any initial conditions $y_0(x)$ and $y_1(x)$, the energy $E(t) \to 0$ as $t \to \infty$.

4. NON-EXPONENTIAL RATE OF DECAY OF ENERGY

To show that the system in case IV is not exponentially stable for irrational d, the abstract approach in [2] by use of its Lemma 4.1 does not readily apply. A direct approach is developed here, which shows that there exists a branch of eigenvalues arbitrarily close to the imaginary axis. To this end, we introduce the following lemma on approximating an irrational number by rational fractions [3].

Lemma 1. For any irrational d, there exists an infinite number of rational fractions p_n/q_n with $(p_n, q_n) = 1$ and $n \in \mathbf{N}$, which satisfy $|d - p_n/q_n| < 1/|q_n^2|$.

Through the use of the well-known Rouché's theorem on the number of zeros inside a closed contour [4], we prove the following theorem.

Theorem 2. For irrational $d > 0$, there exists an infinite number of eigenvalues λ_n of equation (16), which satisfy $\operatorname{Re} \lambda_n \to 0$ as $n \to \infty$.

Proof. By Lemma 1, we write $d = [d] + p_n/q_n + r_n/q_n^2$, where $|r_n| \leqslant 1$. Let

$$f(\mu) = (a + b + k)\, e^{(1 + [d] + p_n/q_n)\mu} + (b - a - k)\, e^{\mu} + (a - b - k)\, e^{([d] + p_n/q_n)\mu} - a - b + k, \tag{29}$$

$$g(\mu) = [e^{r_n \mu/q_n^2} - 1][(a + b + k)\, e^{(1 + [d] + p_n/q_n)\mu} + (a - b - k)\, e^{([d] + p_n/q_n)\mu}]. \tag{30}$$

Then equation (16) becomes $f(\mu) + g(\mu) = 0$. Because $f(\mu) = 0$ has solutions $\mu_n = 2\pi l q_n \mathrm{i}$, where l is any positive integer, we define O_n around μ_n by $\mu = \mu_n + \alpha\, e^{\mathrm{i}\theta}/|\mu_n|$, where $0 \leqslant \theta \leqslant 2\pi$. For any μ on O_n, using the Taylor expansion we have

$$\begin{aligned} f(\mu) &= (a + b + k)\, e^{(1 + [d] + p_n/q_n)\alpha\, e^{\mathrm{i}\theta}/|\mu_n|} + (b - a - k)\, e^{\alpha e^{\mathrm{i}\theta}/|\mu_n|} \\ &\quad + (a - b - k)\, e^{([d] + p_n/q_n)\alpha e^{\mathrm{i}\theta}/|\mu_n|} - a - b + k \\ &= (a + b + k)(1 + [d] + p_n/q_n)\alpha\, e^{\mathrm{i}\theta}/|\mu_n| + (b - a - k)\alpha\, e^{\mathrm{i}\theta}/|\mu_n| \\ &\quad + (a - b - k)([d] + p_n/q_n)\alpha\, e^{\mathrm{i}\theta}/|\mu_n| + O(|\mu_n|^{-2}) \\ &= 2\alpha\, e^{\mathrm{i}\theta}[b + a([d] + p_n/q_n)]/|\mu_n| + O(|\mu_n|^{-2}), \end{aligned} \tag{31}$$

$$\begin{aligned} g(\mu) &= [e^{r_n(\mu_n + \alpha\, e^{\mathrm{i}\theta}/|\mu_n|)/q_n^2} - 1][(a + b + k)\, e^{(1 + [d] + p_n/q_n)\alpha\, e^{\mathrm{i}\theta}/|\mu_n|} \\ &\quad + (a - b - k)\, e^{([d] + p_n/q_n)\alpha\, e^{\mathrm{i}\theta}/|\mu_n|}] \\ &= [r_n \mu_n/q_n^2 + O(|\mu_n|^{-2})][(a + b + k)\, e^{(1 + [d] + p_n/q_n)\alpha\, e^{\mathrm{i}\theta}/|\mu_n|} \\ &\quad + (a - b - k)\, e^{([d] + p_n/q_n)\alpha\, e^{\mathrm{i}\theta}/|\mu_n|}] \\ &= -8a\pi^2 r_n/\mu_n + O(|\mu_n|^{-2}). \end{aligned} \tag{32}$$

Choose $\alpha > 4\pi^2 a/(b + ad)$; then there exists $N > 0$ such that when $n \geqslant N$, $|f(\mu)| > |g(\mu)|$ for all μ on O_n. By the theorem of Rouché, $f(\mu) + g(\mu)$ and $g(\mu)$ have the same number of zeros inside O_n for $n \geqslant N$. Hence there exists $\hat{\mu}_n$ in O_n such that when $n \geqslant N$, $|\hat{\mu}_n - \mu_n| < \alpha/|\mu_n|$. By equation (17) there exist eigenvalues λ_n of equation (15) with

$|\lambda_n - c_1 q_n l\pi \mathrm{i}| < \alpha c_1/4\pi |q_n|$. Because there is an infinite number of distinct q_n, $q_n \to \infty$ as $n \to \infty$. Therefore Re $\lambda_n \to 0$ as $n \to \infty$. □

Note that $c_1 q_n l\pi\mathrm{i}$ ($l = 1, 2, \ldots$) correspond to the infinite number of eigenvalues on the imaginary axis, as predicted by equation (22), when d is approximated by its rational fractions p_n/q_n. As $n \to \infty$, $p_n/q_n \to d$ and $c_1 q_n l\pi\mathrm{i}$ ($l = 1, 2, \ldots$) approach those eigenvalues corresponding to the irrational d.

For any positive constant α, by Theorem 2, there always exists an infinite number of eigenvalues of high modes with damping rates smaller than $\alpha/2$. Under the initial conditions that only one of these modes is excited, the energy of vibration $E(t)$ does not satisfy $E(t) \leqslant M\,\mathrm{e}^{-\alpha t} E(0)$ for all $t > 0$, where M is any positive constant. Hence, we conclude the following.

Corollary 1. The system in case IV is not exponentially stable.

5. OTHER COMBINATIONS OF INTERNAL AND BOUNDARY CONDITIONS

The method developed in section 4 is applicable to the other cases in Table 1 for irrational d. To demonstrate this, we consider the case V with non-symmetrical boundary conditions at two ends. The associated eigenvalue problem leads to the characteristic equation

$$(a + b + k)\,\mathrm{e}^{(1+d)\mu} + (a - b + k)\,\mathrm{e}^{\mu} + (a - b - k)\,\mathrm{e}^{d\mu} + a + b - k = 0, \tag{33}$$

where a, b and μ are those defined in equation (17). We prove first the following lemma similar to Lemma 1.

Lemma 2. For any irrational $d > 0$, there exists an infinite number of rational fractions p_n/q_n ($n \in \mathbf{N}$), where p_n is odd, q_n is even and $(p_n, q_n) = 1$, which satisfy $|d - p_n/q_n| < 12/q_n^2$.

Proof. For any irrational $d > 0$, $2d$ is also irrational. By Lemma 1, there exists an infinite number of irreducible pairs of $p_n > 0$ and $q_n > 0$ such that $|2d - p_n/q_n| < 1/q_n^2$.

If p_n is odd, then $|d - p_n/2q_n| < 2/(2q_n)^2$. If p_n is even, since $(p_n, q_n) = 1$, there exist integers f_n and g_n such that $p_n f_n - q_n g_n = 1$ (see, e.g., reference [3, p. 21]). If $|f_n| > q_n$, we write $|f_n| = u_n q_n + v_n$ for some integers u_n and v_n with $0 < v_n < q_n$. Hence $|p_n v_n - q_n w_n| = 1$ for some integers w_n. As p_n is even, w_n must be odd. Noting that $q_n + v_n \leqslant 2q_n$ implies $1/q_n \leqslant 2/(q_n + v_n)$, we have

$$\left|2d - \frac{p_n + w_n}{q_n + v_n}\right| \leqslant \left|2d - \frac{p_n}{q_n}\right| + \left|\frac{p_n + w_n}{q_n + v_n} - \frac{p_n}{q_n}\right| \leqslant \frac{1}{q_n^2} + \left|\frac{p_n v_n - q_n w_n}{q_n(q_n + v_n)}\right|$$

$$\leqslant \frac{1}{q_n^2} + \frac{1}{q_n(q_n + v_n)} \leqslant \frac{6}{(q_n + v_n)^2}. \tag{34}$$

Hence

$$\left|d - \frac{p_n + w_n}{2(q_n + v_n)}\right| \leqslant \frac{12}{[2(q_n + v_n)]^2}.$$

Because $|(p_n + w_n)v_n - (q_n + v_n)w_n| = 1$, $(p_n + w_n, q_n + v_n) = 1$. Also, $p_n + w_n$ is odd; hence $p_n + w_n$ and $2(q_n + v_n)$ are irreducible. Because q_n are infinite in number, so are $2(q_n + v_n)$. Hence, by equation (34), $p_n + w_n$ are also infinite in number. □

With Lemma 2, and using the theorem of Rouché, we are in a position to prove the following theorem.

Theorem 3. For irrational d, there exists an infinite number of eigenvalue λ_n of equation (33), which satisfy Re $\lambda_n \to 0$ as $n \to \infty$.

Proof. Write $d = [d] + p_n/q_n + r_n/q_n^2$, where $|r_n| \leqslant 12$. Let

$$f(\mu) = (a+b+k)\,\mathrm{e}^{(1+[d]+p_n/q_n)\mu} + (a-b+k)\,\mathrm{e}^{\mu} + (a-b-k)\,\mathrm{e}^{([d]+p_n/q_n)\mu} + a + b - k, \tag{35}$$

$$g(\mu) = [\mathrm{e}^{r_n\mu/q_n^2} - 1][(a+b+k)\,\mathrm{e}^{(1+[d]+p_n/q_n)\mu} + (a-b-k)\,\mathrm{e}^{([d]+p_n/q_n)\mu}]. \tag{36}$$

Then equation (33) becomes $f(\mu) + g(\mu) = 0$. Because q_n is even and p_n is odd, $f(\mu) = 0$ has solutions $\mu_n = q_n\pi\mathrm{i}$. Define O_n around μ_n by $\mu = \mu_n + \alpha\,\mathrm{e}^{\mathrm{i}\theta}/|\mu_n|$, where $0 \leqslant \theta \leqslant 2\pi$. For any μ on O_n, using the Taylor expansion we have

$$\begin{aligned} f(\mu) &= -(a+b+k)\,\mathrm{e}^{(1+[d]+p_n/q_n)\alpha\,\mathrm{e}^{\mathrm{i}\theta}/|\mu_n|} + (a-b+k)\,\mathrm{e}^{\alpha\mathrm{e}^{\mathrm{i}\theta}/|\mu_n|} \\ &\quad - (a-b-k)\,\mathrm{e}^{([d]+p_n/q_n)\alpha\mathrm{e}^{\mathrm{i}\theta}/|\mu_n|} + a + b - k \\ &= -(a+b+k)(1+[d]+p_n/q_n)\alpha\,\mathrm{e}^{\mathrm{i}\theta}/|\mu_n| + (a-b+k)\alpha\,\mathrm{e}^{\mathrm{i}\theta}/|\mu_n| \\ &\quad - (a-b-k)([d]+p_n/q_n)\alpha\,\mathrm{e}^{\mathrm{i}\theta}/|\mu_n| + O(|\mu_n|^{-2}) \\ &= -2\alpha\,\mathrm{e}^{\mathrm{i}\theta}[b + a([d]+p_n/q_n)]/|\mu_n| + O(|\mu_n|^{-2}), \end{aligned} \tag{37}$$

$$\begin{aligned} g(\mu) &= -[\mathrm{e}^{r_n(\mu_n+\alpha\mathrm{e}^{\mathrm{i}\theta}/|\mu_n|)/q_n^2} - 1][(a+b+k)\,\mathrm{e}^{(1+[d]+p_n/q_n)\alpha\mathrm{e}^{\mathrm{i}\theta}/|\mu_n|} \\ &\quad + (a-b-k)\,\mathrm{e}^{([d]+p_n/q_n)\alpha\mathrm{e}^{\mathrm{i}\theta}/|\mu_n|}] \\ &= -[r_n\mu_n/q_n^2 + O(|\mu_n|^{-2})][(a+b+k)(1+O(|\mu_n|^{-1})) + (a-b-k)(1+O(|\mu_n|^{-1}))] \\ &= 2\pi^2 a r_n/|\mu_n| + O(|\mu_n|^{-2}). \end{aligned} \tag{38}$$

Choose $\alpha > 12\pi^2 a/(b+ad)$; there exists $N > 0$ such that when $n \geqslant N$, $|f(\mu)| > |g(\mu)|$ for all μ on O_n. By Rouché's theorem, $f(\mu) + g(\mu)$ has one zero $\hat{\mu}_n$ inside O_n for $n \geqslant N$, with $|\hat{\mu}_n - \mu_n| < \alpha/|\mu_n|$. By equation (17), there exists an infinite number of eigenvalues λ_n of equation (33) such that $|\lambda_n - c_1 q_n\pi\mathrm{i}/2| \leqslant c_1\alpha/2\pi|q_n|$. Because $q_n \to \infty$ as $n \to \infty$, Re $\lambda_n \to 0$ as $n \to \infty$. □

Following the same reasoning as that in section 4, we deduce Theorem 4.4 in [2] from Theorem 3, as follows.

Corollary 2. The system in case V is not exponentially stable for irrational d.

6. CONCLUDING REMARKS

In case IV, $\lambda = 0$ is not an eigenvalue, although it does satisfy the characteristic equation. The system is asymptotically stable only when d is irrational. To show the system is not exponentially stable, a direct approach by applying Rouché's theorem to the characteristic equation is presented. In instances such as those considered here, a practical question can often arise as to how the behavior of the system for rational and irrational d can be so different, when these numbers can be made arbitrarily close. The explanation lies in the fact that as the rational approximation to an irrational number improves, both the numerator and denominator become arbitrarily large. Although the behavior of the system associated with rational and irrational d can be made arbitrarily close for any finite number of low modes, that corresponding to the infinite number of higher modes remains distinct, leading to differing global behavior of the distributed models.

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