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FREE VIBRATION ANALYSIS OF SANDWICH ARCHES WITH ELASTIC OR VISCOELASTIC CORE AND VARIOUS KINDS OF AXIS-SHAPE AND BOUNDARY CONDITIONS

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In this paper, a method of analysis for the free vibration of a three-layer sandwich arch with an elastic or viscoelastic core, and with various kinds of axis-shape and boundary conditions is presented. The characteristic equation of the free vibration is derived by applying Green functions. The Green functions are obtained in a discrete form for various kinds of sandwich arches with non-uniform cross-section and radius of curvature. They enable the setting up of the frequency equation in eigenvalue form.

1. INTRODUCTION

The theory of the free vibration of a three-layer sandwich arch has been considered by only a few investigators. Ahmed [1, 2] has analysed the flexural vibration characteristics of a curved sandwich beam with an elastic core by the finite element displacement method.

It is the purpose of this paper to present an analytical method of the free vibration of a three-layer sandwich arch with an elastic or viscoelastic core, and with an arbitrary axis-shape and various boundary conditions. The characteristic equation of the free vibration is derived by applying Green functions which are the two pairs of the normal and tangential displacements of a sandwich arch under the individual action of a tangential concentrated load and a normal concentrated load. The Green functions are obtained as the discrete type solutions of the differential equations governing the behaviour of a sandwich arch in this paper. The discrete type solutions give solutions at each discrete point uniformly distributed on a sandwich arch axis, and they can be obtained for a sandwich arch with non-uniform cross-section and radius of curvature as well as a sandwich arch with uniform cross-section and radius of curvature. They enable the setting up of the frequency equation in eigenvalue form. It is shown that by applying the characteristic equation; the behaviour of the free vibration of a sandwich arch can be analysed efficiently without any calculation using a trial and error method, and that the numerical solution has a uniform convergency and a good accuracy. Moreover, the effect of the elastic or viscoelastic core shear modulus and the depth of the core to the natural frequency and the loss factor of a sandwich arch are evaluated. The analysis is carried out according to the assumptions. (1) The face plates are elastic. (2) The core is elastic or linearly viscoelastic, with a shear modulus of G or $G = G_0(1 + i\nu)$. (3) There is no slipping between the face plates and the core at their interfaces. (4) Shear strains in the face plates are negligible, and longitudinal direct stresses in the core are negligible. (5) The shear strain

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is constant across the depth of the core. (6) Transverse direct strains in both core and face plates are negligible.

2. FUNDAMENTAL EQUATION OF SANDWICH ARCH

The equilibrium equations of a three-layer sandwich arch having a normal load $p(s)$ and a tangential load $q(s)$ whose element is shown in Figure 1(a) are given by the following equations:

$$\mathrm{d}Q/\mathrm{d}s + N/R + p(s) = 0, \qquad \mathrm{d}N/\mathrm{d}s - Q/R + q(s) = 0, \qquad \mathrm{d}M/\mathrm{d}s = Q, \quad \text{(1a–c)}$$

where Q, N and M are the shear force, normal force and bending moment of a sandwich arch, R the radius of curvature of a sandwich arch axis, and s is the axial co-ordinate of a sandwich arch whose origin is set at the left end.

Next, the relations between the shear force, axial force and bending moment of a sandwich arch and those of the lower and upper face plates are obtained from Figure 1(a) as follows:

$$Q = Q_1 + Q_2 + bh\tau, \qquad N = N_1 + N_2, \qquad M = M_1 + M_2 + N_1 d_1 + N_2 d_2, \qquad \text{(2a–c)}$$

where Q_1, Q_2, N_1, N_2 and M_1, M_2 are the shar forces, axial forces and bending moments of the lower and upper face plates, τ is the shear stress of the core, b is the width of a sandwich arch, h is the depth of the core, d_1 and d_2 are the distances between the centroidal axis of a sandwich arch and the centroidal axes of the lower and upper face plates. The distances d_1 and d_2 are as follows:

$$d_1 + d_2 = H, \qquad d_2 = (E_1 A_1/E_2 A_2)\mathrm{d}_1, \qquad \text{(3a, b)}$$

where $H = h + t_1/2 + t_2/2$, and t_1 and t_2 are the thicknesses of the lower and upper face plates, $E_1 A_1$ and $E_2 A_2$ are the axial rigidities of the lower and upper face plates.

The relations between the angular, tangential, normal displacements of a sandwich arch and those of the lower and upper face plates are given by the following equations:

$$\theta_1 = \theta_2 = \theta, \qquad w_1 + w_2 = 2w, \qquad u_1 = u_2 = u, \qquad \text{(4a–c)}$$

where θ_1, θ_2 and θ are the angular displacements at the centroidal axes of the lower, upper face plates and for the sandwich arch, w_1, u_1, w_2, u_2 and w, u are the tangential, normal displacements at the centroids of the lower, upper face plates and for the sandwich arch.

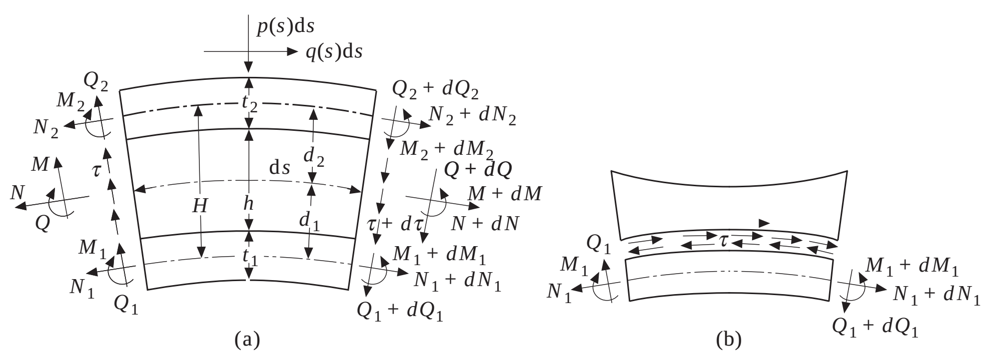

Figure 1. Element of sandwich arch.

The relations between the angular displacement and the bending moments of the lower and upper face plates are given by the following equations:

$$M_1 = -E_1 I_1 \, \mathrm{d}\theta/\mathrm{d}s_1, \qquad M_2 = -E_2 I_2 \, \mathrm{d}\theta/\mathrm{d}s_2, \tag{5a, b}$$

where $E_1 I_1$, $E_2 I_2$ and s_1, s_2 are the flexural rigidities and the axial co-ordinates of the lower and upper face plates, $\mathrm{d}s_1 = (R_1/R)\,\mathrm{d}s$, $\mathrm{d}s_2 = (R_2/R)\,\mathrm{d}s$, and R_1, R_2 are the radii of curvature of lower, upper face plate axes.

From the equations (2b, c), (5a, b) the following equation is obtained:

$$(E_1 I_1 (R/R_1) + E_2 I_2 (R/R_2))\,\mathrm{d}\theta/\mathrm{d}s = -\mathrm{d}_2 N - M + H N_1. \tag{6}$$

The relations between the tangential and normal displacements and the axial forces of the lower and upper face plates are given by the following equations:

$$N_1 = E_1 A_1 (\mathrm{d}w_1/\mathrm{d}s_1 - u/R_1), \qquad N_2 = E_2 A_2 (\mathrm{d}w_2/\mathrm{d}s_2 - u/R_2), \tag{7a, b}$$

From the equations (2b), (4b), (7a, b) the following equations are obtained:

$$E_1 A_1 \frac{R}{R_1}\left(\frac{\mathrm{d}w_1}{\mathrm{d}s} - \frac{u}{R}\right) = N_1, \qquad E_2 A_2 \frac{R}{R_2}\left(2\frac{\mathrm{d}w}{\mathrm{d}s} - \frac{\mathrm{d}w_1}{\mathrm{d}s} - \frac{u}{R}\right) = N - N_1. \tag{8a, b}$$

Next, the relations between the displacements and the shear forces of the lower and upper face plates are given by the following equations:

$$Q_1 = \frac{G_1 A_1}{\kappa_1}\left(\frac{\mathrm{d}u}{\mathrm{d}s_1} + \frac{w_1}{R_1} - \theta\right), \qquad Q_2 = \frac{G_2 A_2}{\kappa_2}\left(\frac{\mathrm{d}u}{\mathrm{d}s_2} + \frac{w_2}{R_2} - \theta\right), \tag{9a, b}$$

where G_1, A_1, κ_1 and G_2, A_2, κ_2 are the elastic shear moduli, the cross-sectional areas, the shear factors of the lower and upper face plates.

The shear stress τ of the core of a three-layer sandwich arch shown in Figure 1(a) is given by the following equation [1] under the assumption that the shear strain is constant across the depth of the core

$$\tau = \frac{G}{h}\left[H\frac{\mathrm{d}u}{\mathrm{d}s} + \left(1 + \frac{h}{2R}\right)w_1 - \left(1 - \frac{h}{2R}\right)w_2\right], \tag{10}$$

where G is the core shear modulus.

From the equations (2a), (4b), (9a, b) and (10), the following equation is obtained:

$$\begin{aligned}\left(\frac{G_1 A_1}{\kappa_1}\frac{R}{R_1} + \frac{G_2 A_2}{\kappa_2}\frac{R}{R_2} + GHb\right)\frac{\mathrm{d}u}{\mathrm{d}s} &= Q + \left(\frac{G_1 A_1}{\kappa_1} + \frac{G_2 A_2}{\kappa_2}\right)\theta - 2\left[\frac{G_2 A_2}{\kappa_2}\frac{1}{R_2} - Gb\left(1 - \frac{h}{2R}\right)\right]w \\ &\quad - \left(\frac{G_1 A_1}{\kappa_1}\frac{1}{R_1} - \frac{G_2 A_2}{\kappa_2}\frac{1}{R_2} + 2Gb\right)w_1.\end{aligned} \tag{11}$$

From the tangential equilibrium equation of the lower face plate shown in Figure 1(b), $\mathrm{d}N_1/\mathrm{d}s = Q_1/R_1 + b\tau$, the following equation is obtained:

$$\mathrm{d}N_1/\mathrm{d}s = Q_1/R + (R_1/R)b\tau. \tag{12}$$

From the equations (4b), (10) and (12) the following equation is obtained:

$$\left(\frac{G_1A_1}{\kappa_1}\frac{1}{R_1} + Gb\frac{H}{h}\frac{R_1}{R}\right)\frac{\mathrm{d}u}{\mathrm{d}s} - \frac{\mathrm{d}N_1}{\mathrm{d}s} = \frac{G_1A_1}{\kappa_1}\frac{1}{R}\theta + 2G\frac{b}{h}\frac{R_1}{R}\left(1 - \frac{h}{2R}\right)w$$

$$-\left(\frac{G_1A_1}{\kappa_1}\frac{1}{R}\frac{1}{R_1} + 2G\frac{b}{h}\frac{R_1}{R}\right)w_1. \tag{13}$$

3. DISCRETE TYPE GREEN FUNCTION OF SANDWICH ARCH

By introducing the following non-dimensional expressions:

$$X_{d1} = -\frac{Ql^2}{E_0I_0}, \quad X_{d2} = -\frac{Nl^2}{E_0I_0}, \quad X_{d3} = -\frac{Ml}{E_0I_0}, \quad X_{d4} = \theta, \quad X_{d5} = \frac{w}{l}, \quad X_{d6} = \frac{u}{l}$$

$$X_{d7} = \frac{N_1h}{GHbl}, \quad X_{d8} = \frac{w_1}{H}, \quad \eta = \frac{s}{l}, \quad d = 1 \text{ or } 2: \quad l \text{ is length of arch axis,}$$

the non-dimensional Green functions are defined by the following two pairs of tangential and normal displacements of a sandwich arch with a concentrated load P_1 or P_2 at a position $s = z$, $(z/l = \xi)$ on the arch modelled in this paper

$$X_{15}(\eta, \xi) = \frac{P_1l^2}{E_0I_0}W_1(\eta, \xi), \qquad X_{16}(\eta, \xi) = \frac{P_1l^2}{E_0I_0}U_1(\eta, \xi) \tag{14a, b}$$

$$X_{25}(\eta, \xi) = \frac{P_2l^2}{E_0I_0}W_2(\eta, \xi), \qquad X_{26}(\eta, \xi) = \frac{P_2l^2}{E_0I_0}U_2(\eta, \xi). \tag{14c, d}$$

The first pair $X_{15}(\eta, \xi)$ and $X_{16}(\eta, \xi)$ are obtained as the two displacements with the other quantities $X_{11}(\eta, \xi), \ldots, X_{18}(\eta, \xi)$ for a normal concentrated load P_1 acting at a position $z/l = \xi$ on a sandwich arch axis; $p(\eta) = P_1\delta(\eta - \xi)/l$ and the second pair $X_{25}(\eta, \xi)$ and $X_{26}(\eta, \xi)$ are obtained as the two displacements with the other quantities $X_{21}(\eta, \xi), \ldots, X_{28}(\eta, \xi)$ for a tangential concentrated load P_2 acting at a position $z/l = \xi$ on a sandwich arch axis; $q(\eta) = P_2\delta(\eta - \xi)/l$, and they satisfy the following simultaneous differential equation obtained by arranging the equations (1a–c), (6), (8a, b), (11) and (13)

$$\frac{\mathrm{d}X_{dt}}{\mathrm{d}\eta} = \sum_{e=1}^{8} G_{te}X_{de} + \frac{P_dl^2}{E_0I_0}\delta_{dt}\delta(\eta - \xi), \qquad d = 1, 2, \quad t = 1 \sim 8, \tag{15}$$

where $\delta(\eta - \xi)$ is Dirac's delta function, δ_{d1}, δ_{d2} are Kronecker's deltas, E_0I_0 is standard flexural rigidity, G_{te} is given in Appendix A.

By integrating the equation (15) over the interval $[0, \eta]$, the following integral equation is obtained:

$$X_{dt}(\eta) = X_{dt}(0) + \int_0^{\eta}\sum_{e=1}^{8} G_{te}(\zeta)X_{de}(\zeta)\,\mathrm{d}\zeta + \frac{P_dl^2}{E_0I_0}\delta_{dt}\mathrm{u}(\eta - \xi), \tag{16}$$

where $\mathrm{u}(\eta - \xi)$ is a unit step function.

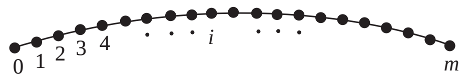

Figure 2. Discrete points on arch axis.

Next, the bounded interval $0 \leqslant \eta \leqslant 1$ is divided into m equal-length parts, and each divisional point is distinguished by a number from 0–m as shown in Figure 2.

By applying the numerical integration method using equally spaced argument values, the equation (16) is discretely expressed as follows:

$$X_{dti} = X_{dt0} + \sum_{j=0}^{i} \sum_{e=1}^{8} \beta_{ij} G_{tej} X_{dej} + \frac{P_d l^2}{E_0 I_0} \delta_{dt} \mathrm{u}(i - x), \tag{17}$$

where $\mathrm{u}(i - x) = 0$ $(i < x)$, $0{\cdot}5$ $(i = x)$ or 1 $(i > x)$, x being the position of the concentrated load P_d, β_{ij} is the weight coefficient of numerical integration, $i = 0$–m and X_{dti} is the value of the function $X_{dt}(\eta)$ at a discrete point i on the sandwich arch axis shown in Figure 2.

The discrete type solution [8] of the simultaneous differential equation (15) can be obtained by the following form:

$$X_{dti} = \sum_{k=1}^{8} a_{dtki} X_{dk0} + a_{dt9i} \frac{P_d l^2}{E_0 I_0}, \qquad d = 1, 2, \quad t = 1\text{–}8, \quad i = 0\text{–}m. \tag{18}$$

By substituting the equations $X_{dt0}, X_{dt1}, X_{dt2}, \ldots, X_{dti}$ given by equation (17) for equation (18) in numbered order, the simultaneous equations to evaluate the elements a_{dtki} and a_{dt9i} in the discrete type solution (18) are obtained finally as follows:

$$a_{dtki} = \delta_{kt} + \sum_{j=0}^{i} \sum_{e=1}^{8} \beta_{ij} G_{tej} a_{dekj} + \delta_{dt} \delta_{k9} \mathrm{u}(i - x), \qquad k = 1\text{–}9, \tag{19}$$

The integral constants $X_{d10}, X_{d20}, \ldots, X_{d80}$ being involved in the discrete type solution (18) are to be evaluated by the boundary conditions of a sandwich arch.

The boundary conditions of a hinged end, fixed end and free end of a sandwich arch can be expressed simply as follows:

$M = w = u = M_1 = M_2 = 0 \qquad (X_{d3} = X_{d5} = X_{d6} = G_{42} X_{d2} + G_{47} X_{d7} = 0)$: simply supported end,

$\theta = w = u = w_1 = 0 \qquad (X_{d4} = X_{d5} = X_{d6} = X_{d8} = 0)$: fixed end,

$Q = N = M = N_1 = 0 \qquad (X_{d1} = X_{d2} = X_{d3} = X_{d7} = 0)$: free end.

By using these boundary conditions and the discrete type solution (18), the two pairs of discrete type non-dimensional Green functions X_{d5i} and X_{d6i} defined by the equations (14a–d) are obtained for a sandwich arch with various kinds of boundary condition as follows:

$$X_{d5i} = \frac{P_d l^2}{E_0 I_0} W_{dix}, \qquad X_{d6i} = \frac{P_d l^2}{E_0 I_0} U_{dix}, \qquad d = 1, 2, \tag{20a, b}$$

where W_{dix} and U_{dix} are the values of the functions $W_d(\eta, \xi)$ and $U_d(\eta, \xi)$ defined by the equations (14a–d) at a discrete point i on a sandwich arch with a normal concentrated load P_1 or a tangential concentrated load P_2 at a discrete point x, and they become as follows:

$$W_{dix} = t_1 a_{d5oi} + t_2(a_{d5pi} + \alpha a_{d57i}) + t_3 a_{d5qi} + t_4 a_{d5ri} + a_{d59i} \tag{21a}$$

$$U_{dix} = t_1 a_{d6oi} + t_2(a_{d6pi} + \alpha a_{d67i}) + t_3 a_{d6qi} + t_4 a_{d6ri} + a_{d69i}, \tag{21b}$$

where

$o = 1, \quad p = 2, \quad q = 4, \quad r = 8, \quad \alpha = -G_{420}/G_{470}$: 2-hinge arch
$o = 1, \quad p = 2, \quad q = 3, \quad r = 7, \quad \alpha = 0$: fixed arch
$o = 1, \quad p = 2, \quad q = 4, \quad r = 8, \quad \alpha = -G_{420}/G_{470}$: hinged-fixed arch
$o = 4, \quad p = 5, \quad q = 6, \quad r = 8, \quad \alpha = 0$: free-fixed curved beam

t_1, t_2, t_3, t_4 are listed in Appendix B.

4. CHARACTERISTIC EQUATION OF THE FREE VIBRATION OF A SANDWICH ARCH

From equations (1a–c), (6), (8a, b), (11) and (13), the differential equations of the normal functions $\bar{Q}$, $\bar{N}$, $\bar{M}$, $\bar{\theta}$, $\bar{W}$, $\bar{u}$, $\bar{N}_1$ and $\bar{w}_1$ of the harmonic free vibration of a sandwich arch are obtained as follows:

$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}s} + \frac{\bar{N}}{R} + \rho\omega^2\bar{u} = 0, \qquad \frac{\mathrm{d}\bar{N}}{\mathrm{d}s} - \frac{\bar{Q}}{R} + \rho\omega^2\bar{w} = 0, \qquad \frac{\mathrm{d}\bar{M}}{\mathrm{d}s} = \bar{Q},$$

$$\left(E_1 I_1 \frac{R}{R_1} + E_2 I_2 \frac{R}{R_2}\right)\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}s} = -\bar{M} - d_2\bar{N} + H\bar{N}_1,$$

$$E_1 A_1 \frac{R}{R_1}\left(\frac{\mathrm{d}\bar{w}_1}{\mathrm{d}s} - \frac{\bar{u}}{R}\right) = \bar{N}_1,$$

$$E_2 A_2 \frac{R}{R_2}\left(2\frac{\mathrm{d}\bar{w}}{\mathrm{d}s} - \frac{\mathrm{d}\bar{w}_1}{\mathrm{d}s} - \frac{\bar{u}}{R}\right) = \bar{N} - \bar{N}_1, \qquad \left(\frac{G_1 A_1}{\kappa_1}\frac{R}{R_1} + \frac{G_2 A_2}{\kappa_2}\frac{R}{R_2} + GHb\right)\frac{\mathrm{d}\bar{u}}{\mathrm{d}s}$$

$$= \bar{Q} + \left(\frac{G_1 A_1}{\kappa_1} + \frac{G_2 A_2}{\kappa_2}\right)\bar{\theta} - 2\left[\frac{G_2 A_2}{\kappa_2}\frac{1}{R_2} - Gb\left(1 - \frac{h}{2R}\right)\right]\bar{w}$$

$$- \left(\frac{G_1 A_1}{\kappa_1}\frac{1}{R_1} - \frac{G_2 A_2}{\kappa_2}\frac{1}{R_2} + 2Gb\right)\bar{w}_1,$$

$$\left(\frac{G_1 A_1}{\kappa_1}\frac{1}{R_1} + Gb\frac{H}{h}\frac{R_1}{R}\right)\frac{\mathrm{d}\bar{u}}{\mathrm{d}s} - \frac{\mathrm{d}\bar{N}_1}{\mathrm{d}s} = \frac{G_1 A_1}{\kappa_1}\frac{1}{R}\bar{\theta} + 2G\frac{b}{h}\frac{R_1}{R}\left(1 - \frac{h}{2R}\right)\bar{w}$$

$$- \left(\frac{G_1 A_1}{\kappa_1}\frac{1}{R}\frac{1}{R_1} + 2G\frac{b}{h}\frac{R_1}{R}\right)\bar{w}_1, \tag{22a–h}$$

where ρ is the mass per unit length of the sandwich arch.

By using the following non-dimensional normal functions Y_1–Y_8,

$$Y_1 = -\bar{Q}l^2/E_0I_0, \qquad Y_2 = -\bar{N}l^2/E_0I_0, \qquad Y_3 = -\bar{M}l/E_0I_0, \qquad Y_4 = \bar{\theta},$$
$$Y_5 = \bar{w}/l, \qquad Y_6 = \bar{u}/l, \qquad Y_7 = \bar{N}_1 h/GHbl, \qquad Y_8 = \bar{w}_1/H,$$

the following simultaneous differential equation is derived from the equations (22a–h):

$$\mathrm{d}Y_t/\mathrm{d}\eta = \sum_{e=1}^{8} F_{te} Y_e, \qquad t = 1\text{–}8, \tag{23}$$

where $F_{16} = \lambda^4$, $F_{25} = \lambda^4$, other $F_{te} = G_{te}$ are listed in Appendix A.

$$\lambda^4 = \frac{\rho\omega^2 l^4}{E_0 I_0}, \qquad \omega^2 = \begin{cases} \omega_0^2: & \text{elastic core} \\ \omega_0^2(1 + \mathrm{i}\mu): & \text{viscoelastic core} \end{cases}$$

ω_0, μ are the circular frequency and loss factor of the sandwich arch.

The discrete type solution of the simultaneous differential equation (23) is obtained by the same method at the third section as follows:

$$Y_{ti} = \sum_{k=1}^{8} a_{tki} Y_{k0}, \qquad t = 1\text{–}8, \quad i = 0\text{–}m, \tag{24}$$

where

$$a_{tki} = \delta_{kt} + \sum_{j=0}^{i} \sum_{e=1}^{8} \beta_{ij} F_{tej} a_{ekj}, \qquad k = 1\text{–}8.$$

By using the frequency equation derived from the discrete type solutions (24) and the boundary conditions, the values of the natural frequency parameter λ of the free vibration of a sandwich arch are evaluated basically, but it needs a calculation using a trial and error method. Therefore, to avoid this, a method setting up the frequency equation in eigenvalue form is proposed as follows.

By applying the Green functions defined by the equations (14a–d),

$$X_{d5}(\eta, \xi) = \frac{P_d l^2}{E_0 I_0} W_d(\eta, \xi), \qquad X_{d6}(\eta, \xi) = \frac{P_d l^2}{E_0 I_0} U_d(\eta, \xi), \qquad d = 1, 2$$

the following simultaneous integral equations concerning the non-dimensional normal functions $Y_5(\xi)$ and $Y_6(\xi)$ of the harmonic free vibration of a sandwich arch are obtained according to Betti's law as shown in Appendix C

$$Y_6(\xi) = \lambda^4 \int_0^1 [U_1(\eta, \xi) Y_6(\eta) + W_1(\eta, \xi) Y_5(\eta)]\, \mathrm{d}\eta, \tag{25a}$$

$$Y_5(\xi) = \lambda^4 \int_0^1 [U_2(\eta, \xi) Y_6(\eta) + W_2(\eta, \xi) Y_5(\eta)]\, \mathrm{d}\eta. \tag{25b}$$

By applying the numerical integration method using the $(m + 1)$ equally spaced argument values, the equations (25a, b) are discretely expressed as

$$Y_{6x} = \lambda^4 \sum_{j=0}^{m} \beta_{mj}(U_{1jx} Y_{6j} + W_{1jx} Y_{5j}), \tag{26a}$$

$$Y_{5x} = \lambda^4 \sum_{j=0}^{m} \beta_{mj}(U_{2jx} Y_{6j} + W_{2jx} Y_{5j}) \qquad x = 0\text{–}m. \tag{26b}$$

From the equations (26a, b), the homogeneous linear equations in $2(m+1)$ unknowns, Y_{50}–Y_{5m}, Y_{60}–Y_{6m}, are obtained as

$$\sum_{j=0}^{m} [(\beta_{mj} U_{1jx} - \kappa\delta_{xx}) Y_{6j} + \beta_{mj} W_{1jx} Y_{5j}] = 0, \tag{27a}$$

$$\sum_{j=0}^{m} [\beta_{mj} U_{2jx} Y_{6j} + (\beta_{mj} W_{2jx} - \kappa\delta_{xx}) Y_{5j}] = 0, \qquad x = 0\text{–}m, \tag{27b}$$

where $\kappa = 1/\lambda^4$.

The characteristic equation of the free vibration of a sandwich arch is obtained from equations (27a, b) as

$$\left|\begin{array}{cccc:cccc}
\beta_{m0} U_{100} - \kappa & \beta_{m1} U_{110} & \cdots & \beta_{mm} U_{1m0} & \beta_{m0} W_{100} & \beta_{m1} W_{110} & \cdots & \beta_{mm} W_{1m0} \\
\beta_{m0} U_{101} & \beta_{m1} U_{111} - \kappa & \cdots & \beta_{mm} U_{1m1} & \beta_{m0} W_{101} & \beta_{m1} W_{111} & \cdots & \beta_{mm} W_{1m1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\beta_{m0} U_{10m} & \beta_{m1} U_{11m} & \cdots & \beta_{mm} U_{1mm} - \kappa & \beta_{m0} W_{10m} & \beta_{m1} W_{11m} & \cdots & \beta_{mm} W_{1mm} \\
\hdashline
\beta_{m0} U_{200} & \beta_{m1} U_{210} & \cdots & \beta_{mm} U_{2m0} & \beta_{m0} W_{200} - \kappa & \beta_{m1} W_{210} & \cdots & \beta_{mm} W_{2m0} \\
\beta_{m0} U_{201} & \beta_{m1} U_{211} & \cdots & \beta_{mm} U_{2m1} & \beta_{m0} W_{201} & \beta_{m1} W_{211} - \kappa & \cdots & \beta_{mm} W_{2m1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\beta_{m0} U_{20m} & \beta_{m1} U_{21m} & \cdots & \beta_{mm} U_{2mm} & \beta_{m0} W_{20m} & \beta_{m1} W_{21m} & \cdots & \beta_{mm} W_{2mm} - \kappa
\end{array}\right| = 0. \tag{28}$$

By applying the characteristic equation (28), the values of the natural frequency parameter λ and the loss factor μ of a sandwich arch with a viscoelastic core can be evaluated efficiently without a calculation using a trial and error method.

5. NUMERICAL RESULTS

In the numerical analysis, the following equation is used as the standard moment of inertia of the cross-sectional area I_0 of a three-layer sandwich arch

$$I_0 = 2(bt^3/12 + h^2bt/4).$$

This equation is the moment of inertia of area of the idealized I-section which consists of two flanges corresponding to both face plates of a sandwich arch and a web of negligible area and a height of the core depth of a sandwich arch.

5.1. CONVERGENCY AND ACCURACY OF NUMERICAL SOLUTIONS

Numerical solutions of the frequency parameter λ for some sandwich circular or parabolic arches are given in Tables 1–5 with the other theoretical solutions for the flat curved beams similar to the straight beams from references [1] and [2]. The dimensions and material properties of these sandwich arches are: length of arch $l = 0{\cdot}7112$ m, core

TABLE 1

Convergency of frequency parameter λ and comparison of natural frequency f(Hz) to the other theoretical value of 2-hinge sandwich circular arch with rise ratio $f/L = 0{\cdot}02084$ ($l = 0{\cdot}7112$ m, $h = 12{\cdot}7$ mm, $t = 0{\cdot}4572$ mm, $G/E = 0{\cdot}0012$)

	Present study						Ahmed
	m					f	f
Mode	32	40	48	56	64	64	Ref. [1]
1	4·338	4·338	4·338	4·337	4·337	182·7	199·5
2	6·029	6·023	6·019	6·017	6·016	351·4	394
3	8·687	8·668	8·657	8·651	8·647	726·1	746
4	11·022	10·982	10·960	10·947	10·939	1162	1175
5	13·114	13·044	13·006	12·984	12·969	1633	1639
6	14·996	14·886	14·828	14·793	14·770	2118	—
7	16·724	16·565	16·480	16·430	16·397	2611	—
8	18·318	18·108	17·993	17·924	17·880	3104	—

thickness $h = 12{\cdot}7$ mm, face thickness $t_1 = t_2 = t = 0{\cdot}4572$ mm face elastic modulus $E_1 = E_2 = E_0 = E = 6{\cdot}89 \times 10^{10}$ N/m^2, core shear modulus $G = 0{\cdot}0012$E core density $\rho_c = 32{\cdot}8$ kg/m^3, face density $\rho_f = 2680$ kg/m^3.

The numerical solution has been obtained by applying the trapezoidal rule, and has a uniform convergency as shown in Tables 1–5. In Table 1 the discrepancies between the authors' values and those calculated by Ahmed [1] are of the order of 10% for the two lower frequencies, but for the others the discrepancies are small. In Table 2 the authors' values have a good agreement with those calculated by Ahmed [1, 2]. In Table 3 the italic values calculated by Ahmed [2] are the frequencies for a free-fixed straight sandwich beam, and the authors' values for the flat free-fixed curved beam similar to the straight beam are not in good agreement with the values for the free-fixed curved sandwich beam by Ahmed [1] but with the italic values by Ahmed [2].

TABLE 2

Convergency of frequency parameter λ and comparison of natural frequency f(Hz) to the other theoretical value of fixed sandwich circular arch with rise ratio $f/L = 0{\cdot}02084$ ($l = 0{\cdot}7112$ m, $h = 12{\cdot}7$ mm, $t = 0{\cdot}4572$ mm, $G/E = 0{\cdot}0012$)

	Present study						Ahmed	
	m					f	f	
Mode	32	40	48	56	64	64	Ref. [1]	Ref. 2
1	5·023	5·021	5·020	5·019	5·019	244·6	264·2	240
2	7·093	7·083	7·078	7·074	7·072	485·6	522	474
3	9·459	9·435	9·423	9·415	9·410	859·8	889	843
4	11·553	11·509	11·486	11·471	11·462	1276	1312	1253
5	13·479	13·406	13·367	13·343	13·328	1725	1767	1697
6	15·250	15·138	15·079	15·043	15·020	2190	—	—
7	16·909	16·748	16·662	16·611	16·578	2668	—	—
8	18·411	18·242	18·128	18·059	18·014	3151	—	—

TABLE 3

Convergency of frequency parameter λ and comparison of natural frequency f(Hz) to the other theoretical values of free-fixed sandwich circular curved and straight beams ($l = 0{\cdot}7112$ m, $h = 12{\cdot}7$ mm, $t = 0{\cdot}4572$ mm, $G/E = 0{\cdot}0012$, $f/L = 0{\cdot}02084$, $f/L = 0$)

	Present study						Ahmed	
	m					f	f	
Mode	32	40	48	56	64	64	Ref. [1]	Ref. [2]
1	1·866	1·865	1·865	1·865	1·865	33·8	179	*33·97*
2	4·528	4·525	4·523	4·523	4·522	198·5	266	*200·5*
3	7·289	7·278	7·272	7·269	7·266	513	546	*517*
4	9·735	9·708	9·694	9·685	9·679	910	934	*918*
5	11·922	11·871	11·844	11·827	11·816	1356	1379	*1368*
6	13·065	13·063	13·063	13·062	13·062	1657	—	—
7	13·904	13·820	13·775	13·748	13·731	1831	—	*1844*
8	15·707	15·580	15·511	15·471	15·444	2316	—	*2331*

5.2. FREE VIBRATION OF SANDWICH ARCH WITH ELASTIC CORE

5.2.1. *Frequency curve and free vibrational mode of a sandwich circular arch*

The frequency curves of 2-hinge and fixed sandwich circular arches with $l = 0{\cdot}7112$ m $t_1 = t_2 = t = 0{\cdot}4572$ mm, $h = 12{\cdot}7$ mm, $E_1 = E_2 = E_0 = E$ and $G/E = 0{\cdot}0012$ are shown in Figures 3 and 4. The dotted lines are the frequency curves of the arch with the corresponding idealized I-section, which consists of two flanges corresponding to both face plates of the sandwich arch and a web of negligible area and a height of the core depth of the sandwich arch. For some 2-hinge sandwich arches in Figure 3, the free vibrational u-mode is shown in Figure 5.

It has been shown that the difference of the frequency curves between sandwich arch and I-sectional arch becomes large at higher degrees of free vibration, and that the transition of the free vibrational modes between the extensional modes and the flexural modes arises on the sandwich arch as well as the usual arch.

TABLE 4

Convergency of frequency parameter λ of 2-hinge sandwich circular arch with high rise ratio $f/L = 0{\cdot}15$ ($l = 0{\cdot}7112$ m, $h = 12{\cdot}7$ mm, $t = 0{\cdot}4572$ mm, $G/E = 0{\cdot}0012$)

	m				
Mode	32	40	48	56	64
1	5·788	5·782	5·778	5·776	5·775
2	8·220	8·203	8·194	8·188	8·185
3	10·913	10·873	10·851	10·838	10·829
4	10·994	10·986	10·982	10·979	10·978
5	13·174	13·107	13·070	13·049	13·035
6	14·926	14·818	14·760	14·725	14·702
7	16·715	16·556	16·471	16·421	16·388
8	18·218	18·018	17·909	17·844	17·802

TABLE 5

Convergency of frequency parameter λ of 2-hinge sandwich parabolic arch with high rise ratio $f/L = 0{\cdot}15$ ($l = 0{\cdot}7112$ m, $h = 12{\cdot}7$ mm, $t = 0{\cdot}4572$ mm, $G/E = 0{\cdot}0012$)

	m				
Mode	32	40	48	56	64
1	5·806	5·800	5·796	5·794	5·793
2	8·391	8·374	8·364	8·358	8·354
3	10·791	10·787	10·785	10·784	10·783
4	10·941	10·900	10·878	10·865	10·857
5	13·119	13·050	13·013	12·991	12·976
6	14·947	14·838	14·780	14·745	14·722
7	16·709	16·549	16·464	16·414	16·381
8	18·255	18·051	17·939	17·873	17·829

5.2.2. *Effect of core shear modulus*

The values of the lowest eight natural frequency parameter λ of a 2-hinge sandwich circular arch with $l = 0{\cdot}7112$ m, $t_1 = t_2 = t = 0{\cdot}4572$ mm, $h = 12{\cdot}7$ mm, $E_1 = E_2 = E_0 = E$, $f/L = 0{\cdot}15$ have been evaluated for a wide range of the core shear modulus to face elastic modulus ratio G/E. The results are summarized in Table 6. In Table 6 the values of the left end column have been calculated by using the method given by reference [4], and they give the values of the frequency parameter of the arch with corresponding idealized I-section.

It has been shown that as the ratio G/E increases the natural frequency parameter λ of the sandwich arch approaches that of the corresponding idealized I-sectional arch, and that the natural frequency parameter λ of the sandwich arch becomes small compared with that of the corresponding idealized I-sectional arch below the value $G/E = 0{\cdot}0001$.

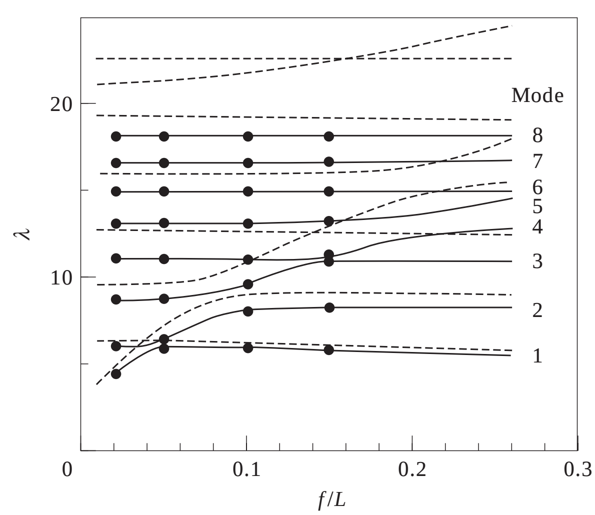

Figure 3. Frequency curve of 2-hinge sandwich circular arch: ——, $G/E = 0{\cdot}0012$; – – –, arch with I-section; ●, points of illustration of mode.

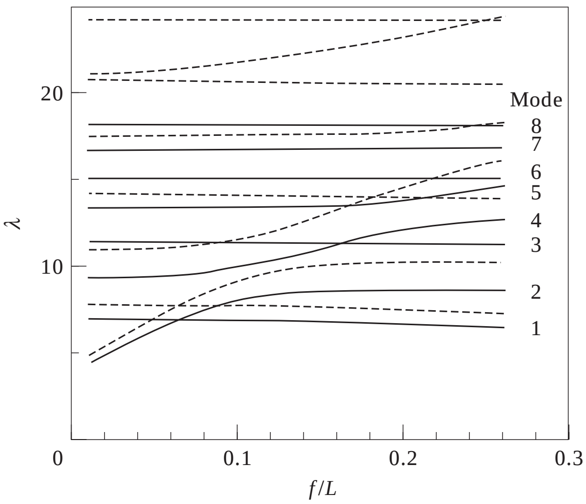

Figure 4. Frequency curve of fixed sandwich circular arch: ——, $G/E = 0{\cdot}0012$; ---, arch with I-section.

5.2.3. *Effect of core thickness*

The values of the lowest eight natural frequency parameter λ of a 2-hinge sandwich circular arch with $l = 0{\cdot}7112$ m, $t_1 = t_2 = t = 0{\cdot}4572$ mm, $E_1 = E_2 = E_0 = E$, $G/E = 0{\cdot}0012$, $f/L = 0{\cdot}15$ have been evaluated for a wide range of core thickness to face thickness ratio h/t. The results are summarized in Table 7. It has been shown that the natural frequency parameter λ of the sandwich arch becomes relatively small below the value $h/t = 10$.

5.3. FREE VIBRATION OF SANDWICH ARCH WITH VISCOELASTIC CORE

The frequency curve of fixed sandwich circular arch with viscoelastic core of the shear modulus $G = G_0(1 + \mathrm{i}v)$, and with $l = 0{\cdot}7112$ m, $t_1 = t_2 = t = 0{\cdot}4572$ mm, $h = 12{\cdot}7$ mm, $E_1 = E_2 = E_0 = E$, $G_0/E = 0{\cdot}0012$, $v = 0{\cdot}4$ is shown in Figure 6 with the frequency curve of the sandwich arch with an elastic core. The natural frequency parameter λ_0 has the definitions $\lambda_0^4 = \rho\omega_0^2 l^4/E_0 I_0$, $\omega^2 = \omega_0^2(1 + \mathrm{i}\mu)$ and the numerical values of the natural frequency parameter λ_0 and the loss factor μ of the fixed circular sandwich arch for a case of rise ratio $f/L = 0{\cdot}15$ are shown in Table 8 for a wide range of loss factor v of the viscoelastic core material.

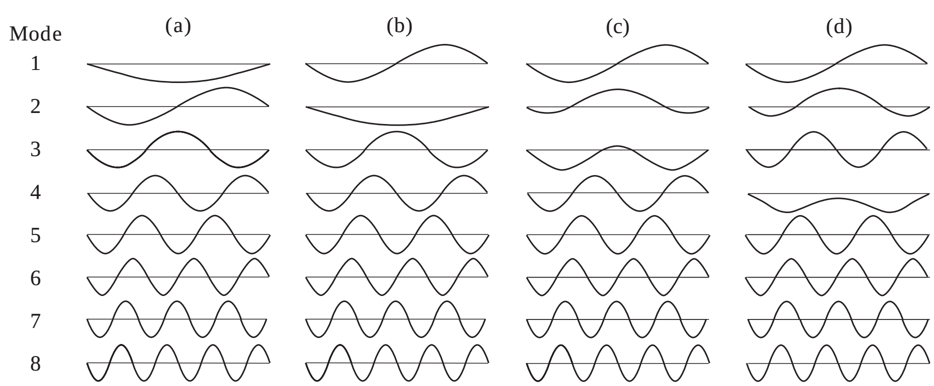

Figure 5. Vibrational u-mode of 2-hinge sandwich arch: $f/L =$ (a) $0{\cdot}02084$; (b) $0{\cdot}05$; (c) $0{\cdot}10$; (d) $0{\cdot}15$.

TABLE 6

Frequency parameter λ of 2-hinge sandwich circular arch with various G/E ratios ($l = 0{\cdot}7112$ m, $h = 12{\cdot}7$ mm, $t = 0{\cdot}4572$ mm, $f/L = 0{\cdot}15$)

		G/E				
Mode	I-sec. Ref. [4]	1/10	$1/10^2$	$1/10^3$	$1/10^4$	$1/10^5$
1	6·033	6·030	5·999	5·735	4·720	3·910
2	8·869	8·862	8·774	8·088	6·128	4·930
3	11·169	11·167	11·143	10·637	7·508	5·951
4	12·490	12·469	12·221	10·946	8·453	6·684
5	15·800	15·758	15·287	12·774	9·481	7·462
6	18·615	18·581	17·989	14·343	10·198	8·098
7	19·353	19·313	19·096	15·960	11·143	8·793
8	22·284	22·165	20·931	17·333	11·173	9·384

TABLE 7

Frequency parameter λ of 2-hinge sandwich circular arch with various h/t ratios ($l = 0{\cdot}7112$ m, $t = 0{\cdot}4572$ mm, $G/E = 0{\cdot}0012$, $f/L = 0{\cdot}15$)

	h/t					
Mode	50	40	30	20	10	5
1	7·462	6·779	5·978	4·994	3·674	2·742
2	9·857	9·297	8·432	7·204	5·410	4·079
3	11·634	11·280	11·033	9·577	7·284	5·546
4	13·399	12·394	11·166	10·702	8·788	6·805
5	15·792	14·709	13·400	11·841	10·414	8·131
6	17·660	16·542	15·127	13·248	10.965	9·270
7	19·076	18·365	16·860	14·881	11·977	10·503
8	19·591	18·960	18·280	16·281	13·131	11·011

TABLE 8

Frequency parameter λ_0 and loss factor μ of fixed circular sandwich arch with viscoelastic core ($l = 0{\cdot}7112$ m, $h = 12{\cdot}7$ mm, $t = 0{\cdot}4572$ mm, $G_0/E = 0{\cdot}0012$, $f/L = 0{\cdot}15$)

	v								
	0	0·1		0·2		0·4		0·8	
Mode	λ_0	λ_0	μ	λ_0	μ	λ_0	μ	λ_0	μ
1	6·879	6·883	0·017	6·896	0·033	6·943	0·061	7·082	0·096
2	8·604	8·609	0·017	8·626	0·034	8·688	0·064	8·875	0·100
3	11·120	11·123	0·006	11·130	0·013	11·155	0·024	11·224	0·040
4	11·371	11·380	0·027	11·407	0·053	11·510	0·101	11·839	0·169
5	13·484	13·494	0·028	13·524	0·056	13·637	0·109	14·023	0·188
6	15·000	15·013	0·033	15·053	0·065	15·201	0·125	15·689	0·216
7	16·667	16·682	0·036	16·726	0·071	16·892	0·136	17·440	0·237
8	18·016	18·041	0·034	18·111	0·069	18·343	0·137	18·990	0·248

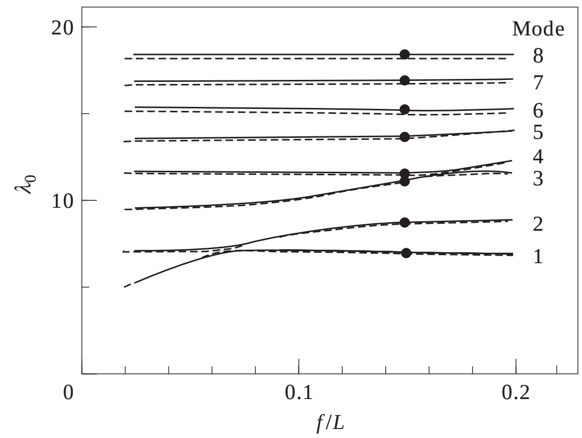

Figure 6. Frequency curve of fixed sandwich circular arch with viscoelastic core: ——, $G_0/E = 0{\cdot}0012$, $\nu = 0{\cdot}4$; ---, with elastic core; ●, points of illustration of loss factor.

It has been shown that the loss factor μ of a sandwich arch is small in the extensional vibration compared with the flexural vibration.

6. CONCLUSIONS

A method of analysis for the free vibration of a three-layer sandwich arch with an elastic or viscoelastic core, and with various kinds of axis shape and boundary conditions has been presented in this paper. The characteristic equation of the free vibration was derived by applying Green functions comprising two pairs of tangential and normal displacements of a sandwich arch under the individual action of a normal concentrated load and a tangential concentrated load. The Green functions were obtained as discrete type solutions of the differential equations governing the behaviour of a sandwich arch. The discrete type solutions gave the solutions at each discrete point uniformly distributed on a sandwich arch axis, and they can be obtained for a sandwich arch with non-uniform cross-section and radius of curvature as well as a sandwich arch with uniform cross-section and radius of curvature and enabled setting up of the frequency equation in eigenvalue form. By applying the characteristic equation, the behaviour of the free vibration of a sandwich arch with an elastic or viscoelastic core could be analysed efficiently without any calculation using a trial and error method. It was shown that the numerical solution had a uniform convergency and a good accuracy, and the effect of an elastic or viscoelastic core shear modulus and the depth of the core to the natural frequency and the loss factor of a sandwich arch were evaluated.

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APPENDIX A

$$G_{12} = -g_1, \quad G_{21} = g_1, \quad G_{31} = 1, \quad G_{42} = g_0/h_1, \quad G_{43} = 1/h_1, \quad G_{47} = f_7 g_7 g_9/h_1,$$

$$G_{86} = g_1/g_7, \quad G_{87} = f_7 g_6/(f_3 g_4), \quad G_{52} = -0{\cdot}5/(f_4 g_5), \quad G_{56} = g_1,$$

$$G_{57} = f_7 g_9 (1/(f_3 g_4) - 1/(f_4 g_5))/2, \quad G_{61} = -1/h_2, \quad G_{64} = (f_5 + f_6)/h_2,$$

$$G_{65} = -2[f_6 g_3 - f_7(1 - g_8)]/h_2, \quad G_{68} = -(f_5 g_2 - f_6 g_3 + 2f_7) g_7/h_2, \quad G_{71} = h_3 G_{61},$$

$$G_{74} = h_3 G_{64} - f_5 g_1/(f_7 g_9), \quad G_{75} = h_3 G_{65} - 2(1 - g_8)/(g_4 g_7),$$

$$G_{78} = h_3 G_{68} + f_5 g_1 g_2 g_7/(f_7 g_9) + 2/g_4, \quad \text{other}$$

$$G_{te} = 0; \quad h_1 = f_1 g_4 + f_2 g_5, \quad h_2 = f_5 g_4 + f_6 g_5 + f_7 g_7, \quad h_3 = f_5 g_2/(f_7 g_9) + 1/g_4,$$

$$f_1 = \frac{E_1 I_1}{E_0 I_0}, \qquad f_2 = \frac{E_2 I_2}{E_0 I_0}, \qquad f_3 = \frac{E_1 A_1 l^2}{E_0 I_0}, \qquad f_4 = \frac{E_2 A_2 l^2}{E_0 I_0}, \qquad f_5 = \frac{G_1 A_1 l^2}{\kappa_1 E_0 I_0},$$

$$f_6 = \frac{G_2 A_2 l^2}{\kappa_2 E_0 I_0}, \qquad f_7 = \frac{G b l^3}{E_0 I_0}, \quad g_1 = \frac{l}{R}, \qquad g_2 = \frac{l}{R_1}, \qquad g_3 = \frac{l}{R_2}, \qquad g_4 = \frac{R}{R_1},$$

$$g_5 = \frac{R}{R_2}, \qquad g_6 = \frac{l}{h}, \qquad g_7 = \frac{H}{l}, \qquad g_8 = \frac{h}{2R}, \qquad g_9 = \frac{H}{h}, \qquad g_0 = \frac{d_2}{l}.$$

APPENDIX B

B.1. 2-hinge sandwich arch

Boundary conditions of the left support;

$$X_{d30} = X_{d50} = X_{d60} = G_{420} X_{d20} + G_{470} X_{d70} = 0.$$

Boundary conditions of the right support;

$$X_{d3m} = 0: \qquad a_{d31m} X_{d10} + a_{d32m} X_{d20} + a_{d34m} X_{d40} + a_{d37m} X_{d70} + a_{d38m} X_{d80} + a_{d39m} X_{d90} = 0.$$

Hence,

$$a_{d31m} X_{d10} + (a_{d32m} + \alpha a_{d37m}) X_{d20} + a_{d34m} X_{d40} + a_{d38m} X_{d80} = -a_{d39m} X_{d90},$$

where, $\alpha = -G_{420}/G_{470}$;

$$X_{d5m} = 0: \qquad a_{d51m} X_{d10} + (a_{d52m} + \alpha a_{d57m}) X_{d20} + a_{d54m} X_{d40} + a_{d58m} X_{d80} = -a_{d59m} X_{d90};$$

$$X_{d6m} = 0: \qquad a_{d61m} X_{d10} + (a_{d62m} + \alpha a_{d67m}) X_{d20} + a_{d64m} X_{d40} + a_{d68m} X_{d80} = -a_{d69m} X_{d90};$$

$$G_{42m} X_{d2m} + G_{47m} X_{d7m} = 0:$$

$$(a_{d71m} + \beta a_{d21m})X_{d10} + [(a_{d72m} + \alpha a_{d77m}) + \beta(a_{d22m} + \alpha a_{d27m})]X_{d20}$$
$$+ (a_{d74m} + \beta a_{d24m})X_{d40} + (a_{d78m} + \beta a_{d28m})X_{d80} = -(a_{d79m} + \beta a_{d29m})X_{d90},$$

where, $\beta = G_{42m}/G_{47m}$.

Hence, the equations for t_1–t_4 are obtained as

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} =$$

$$\begin{bmatrix} a_{d31m} & a_{d32m} + \alpha a_{d37m} & a_{d34m} & a_{d38m} \\ a_{d51m} & a_{d52m} + \alpha a_{d57m} & a_{d54m} & a_{d58m} \\ a_{d61m} & a_{d62m} + \alpha a_{d67m} & a_{d64m} & a_{d68m} \\ a_{d71m} + \beta a_{21m} & a_{d72m} + \alpha a_{d77m} + \beta(a_{d22m} + \alpha a_{d27m}) & a_{d74m}\beta a_{d24m} & a_{d78m} + \beta a_{d28m} \end{bmatrix}^{-1} \cdot$$

$$\begin{bmatrix} -a_{d39m} \\ -a_{d59m} \\ -a_{d69m} \\ -a_{d79m} - \beta a_{d29m} \end{bmatrix}$$

B.2. Fixed sandwich arch

Boundary conditions of the left support;

$$X_{d40} = X_{d50} = X_{d60} = X_{d80} = 0.$$

Boundary conditions of the right support;

$$X_{d4m} = 0: \quad a_{d41m}X_{d10} + a_{d42m}X_{d20} + a_{d43m}X_{d30} + a_{d47m}X_{d70} = -a_{d49m}X_{d90},$$
$$X_{d5m} = 0: \quad a_{d51m}X_{d10} + a_{d52m}X_{d20} + a_{d53m}X_{d30} + a_{d57m}X_{d70} = -a_{d59m}X_{d90},$$
$$X_{d6m} = 0: \quad a_{d61m}X_{d10} + a_{d62m}X_{d20} + a_{d63m}X_{d30} + a_{d67m}X_{d70} = -a_{d69m}X_{d90},$$
$$X_{d8m} = 0: \quad a_{d81m}X_{d10} + a_{d82m}X_{d20} + a_{d83m}X_{d30} + a_{d87m}X_{d70} = -a_{d89m}X_{d90}.$$

Hence, the equations for t_1–t_4 are obtained as follows:

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} a_{d41m} & a_{d42m} & a_{d43m} & a_{d47m} \\ a_{d51m} & a_{d52m} & a_{d53m} & a_{d57m} \\ a_{d61m} & a_{d62m} & a_{d63m} & a_{d67m} \\ a_{d81m} & a_{d82m} & a_{d83m} & a_{d87m} \end{bmatrix}^{-1} \cdot \begin{bmatrix} -a_{d49m} \\ -a_{d59m} \\ -a_{d69m} \\ -a_{d89m} \end{bmatrix}.$$

B.3. Hinged-fixed sandwich arch

Boundary conditions of the left support;

$$X_{d30} = X_{d50} = X_{d60} = G_{420}X_{d20} + G_{470}X_{d70} = 0.$$

Boundary conditions of the right support;

$$X_{d4m}=0:\quad a_{d41m}X_{d10}+(a_{d42m}+\alpha a_{d47m})X_{d20}+a_{d44m}X_{d40}+a_{d48m}X_{d80}=-a_{d49m}X_{d90},$$

$$X_{d5m}=0:\quad a_{d51m}X_{d10}+(a_{d52m}+\alpha a_{d57m})X_{d20}+a_{d54m}X_{d40}+a_{d58m}X_{d80}=-a_{d59m}X_{d90},$$

$$X_{d6m}=0:\quad a_{d61m}X_{d10}+(a_{d62m}+\alpha a_{d67m})X_{d20}+a_{d64m}X_{d40}+a_{d68m}X_{d80}=-a_{d69m}X_{d90},$$

$$X_{d8m}=0:\quad a_{d81m}X_{d10}+(a_{d82m}+\alpha a_{d87m})X_{d20}+a_{d84m}X_{d40}+a_{d88m}X_{d80}=-a_{d89m}X_{d90},$$

where, $\alpha=-G_{420}/G_{470}$.

Hence, the equations for t_1–t_4 are obtained as follows:

$$\begin{bmatrix} t_1\\ t_2\\ t_3\\ t_4\end{bmatrix}=\begin{bmatrix} a_{d41m} & a_{d42m}+\alpha a_{d47m} & a_{d44m} & a_{d48m}\\ a_{d51m} & a_{d52m}+\alpha a_{d57m} & a_{d54m} & a_{d58m}\\ a_{d61m} & a_{d62m}+\alpha a_{d67m} & a_{d64m} & a_{d68m}\\ a_{d81m} & a_{d82m}+\alpha a_{d87m} & a_{d84m} & a_{d88m}\end{bmatrix}^{-1}\cdot\begin{bmatrix} -a_{d49m}\\ -a_{d59m}\\ -a_{d69m}\\ -a_{d89m}\end{bmatrix}.$$

B.4. FREE-FIXED SANDWICH ARCH

Boundary conditions of the left support;

$$X_{d10}=X_{d20}=X_{d30}=X_{d70}=0.$$

Boundary conditions of the right support;

$$X_{d4m}=0:\quad a_{d44m}X_{d40}+a_{d45m}X_{d50}+a_{d46m}X_{d60}+a_{d48m}X_{d80}=-a_{d49m}X_{d90},$$

$$X_{d5m}=0:\quad a_{d54m}X_{d40}+a_{d55m}X_{d50}+a_{d56m}X_{d60}+a_{d58m}X_{d80}=-a_{d59m}X_{d90},$$

$$X_{d6m}=0:\quad a_{d64m}X_{d40}+a_{d65m}X_{d50}+a_{d66m}X_{d60}+a_{d68m}X_{d80}=-a_{d69m}X_{d90},$$

$$X_{d8m}=0:\quad a_{d84m}X_{d40}+a_{d85m}X_{d50}+a_{d86m}X_{d60}+a_{d88m}X_{d80}=-a_{d89m}X_{d90}.$$

Hence, the equations for t_1–t_4 are obtained as

$$\begin{bmatrix} t_1\\ t_2\\ t_3\\ t_4\end{bmatrix}=\begin{bmatrix} a_{d44m} & a_{d45m} & a_{d46m} & a_{d48m}\\ a_{d54m} & a_{d55m} & a_{d56m} & a_{d58m}\\ a_{d64m} & a_{d65m} & a_{d66m} & a_{d68m}\\ a_{d84m} & a_{d85m} & a_{d86m} & a_{d88m}\end{bmatrix}^{-1}\cdot\begin{bmatrix} -a_{d49m}\\ -a_{d59m}\\ -a_{d69m}\\ -a_{d89m}\end{bmatrix}.$$

APPENDIX C

Betti's law relating the normal concentrated load system shown in Figure C1a and the inertia force system shown in Figure C1c introduces directly the integral equation

$$P_1u(z)=\int_0^l \rho\omega^2[u_1(s,z)u(s)+w_1(s,z)w(s)]\,\mathrm{d}s. \tag{C1}$$

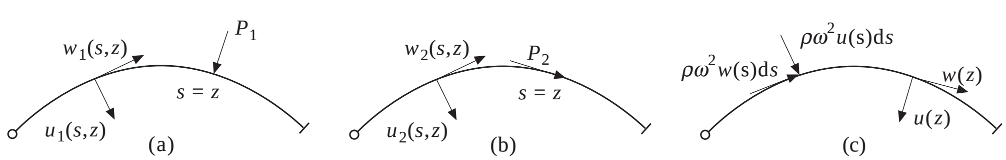

Figure C1. Three types of loading of sandwich arch.

From the tangential concentrated load system shown in Figure C1b and the inertia force system in Figure C1c, the following integral equation is obtained:

$$P_2 w(z) = \int_0^l \rho\omega^2 [u_2(s, z) u(s) + w_2(s, z) w(s)] \, \mathrm{d}s. \tag{C2}$$

By considering the following relations:

$$u(z) = lY_6(\xi), \qquad u(s) = lY_6(\eta), \qquad w(z) = lY_5(\xi), \qquad w(s) = lY_5(\eta), \qquad \mathrm{d}s = l\,\mathrm{d}\eta,$$

$$u_1(s, z) = \frac{P_1 l^3}{E_0 I_0} U_1(\eta, \xi), \qquad w_1(s, z) = \frac{P_1 l^3}{E_0 I_0} W_1(\eta, \xi), \qquad u_2(s, z) = \frac{P_2 l^3}{E_0 I_0} U_2(\eta, \xi),$$

$$w_2(s, z) = \frac{P_2 l^3}{E_0 I_0} W_2(\eta, \xi),$$

the following simultaneous integral equation is obtained from the equations (C1) and (C2):

$$Y_6(\xi) = \int_0^1 \frac{\rho\omega^2 l^4}{E_0 I_0} [U_1(\eta, \xi) Y_6(\eta) + W_1(\eta, \xi) Y_5(\eta)] \, \mathrm{d}\eta$$

$$Y_5(\xi) = \int_0^1 \frac{\rho\omega^2 l^4}{E_0 I_0} [U_2(\eta, \xi) Y_6(\eta) + W_2(\eta, \xi) Y_5(\eta)] \, \mathrm{d}\eta.$$