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IDENTIFYING NODES AND ANTI-NODES OF AN AXIALLY VIBRATING BAR WITH LUMPED MASS

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In a recent letter to the editor, Batan and Gürgöze [1] noted mistakes that were contained in the paper by Cutchins [2]. Specifically, they observed errors that are caused by numerical evaluations of the frequency equation of a fixed-free axially vibrating bar with a lumped mass attached at a point along the span of the bar (see Figure 1). The frequency equation and the corresponding mode shapes of the system of Figure 1 are given in reference [2]. Using another solver, Batan and Gürgöze recalculated the first three eigenfrequencies of the system of Figure 1 as a function of the location of the lumped mass. They noted that their frequency curves are similar in shape to those obtained in reference [2]. However, unlike the results given in reference [2], the minima of their frequency curves shift with varying values of the lumped mass. Comparing the two curves in Figure 2 of reference [1], it is also noted that the maxima of the two curves occur at the same lumped mass location. While Batan and Gürgöze used the minima of the frequency curves to highlight the difference between the two set of results, it is interesting to note that the location of these maxima and local minimum of the frequency curves also reveal certain physical characteristics of the system.

In a recent paper [3], the present authors found that by attaching a lumped virtual mass to a complicated structure and analyzing the free vibration of the resultant system as a function of the location (or co-ordinate) of the lumped virtual mass, one can readily determine the nodes (locations of zero displacement) and anti-nodes (locations of local maxima displacement) of the complicated structure. Specifically, when the frequency curves of the assembled system (complicated structure with lumped virtual mass) are plotted against the constraint location of the lumped virtual mass, the *maxima* of the frequency equations correspond to the *nodes* of the complicated structure. The locations of the local minima of the frequency curves are found to vary with the values of the lumped virtual mass, in agreement with the observation noted in reference [1]. However, as the lumped virtual mass approaches zero, the locations of the *local minima* of the frequency curves converge to those of the *anti-nodes* of the complicated structure.

The reason that the nodes of the complicated structure (or original system) correspond to the maxima of the frequency curves of the assembled system can be best explained as follows. The addition of a lumped mass to any system generally decreases the natural frequencies of the system since it leads to an increase in the overall mass of the structure. However, if the location of the lumped virtual mass coincides with a node of the system under consideration (the original system), it will not contribute to the total kinetic energy of the assembled system since the lumped virtual mass has zero velocity because it is located at a node. Thus, a plot of the natural frequencies of the assembled system as a function of the constrained location of the lumped virtual mass reveals the nodes of the original system as *maxima* on that plot. Interestingly, we can also extract the anti-nodes from the plot of ω_i versus x_v, where ω_i denotes the ith natural frequency of the assembled system and x_v represents the constraint location of the lumped virtual mass. The authors'

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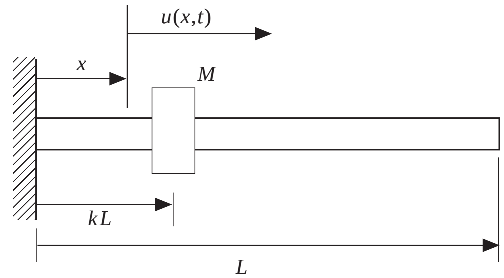

Figure 1. Uniform bar-mass assembly.

numerical experiments [3] show that the anti-nodes of the original system correspond to the *local maxima* on that plot, provided that the lumped virtual mass is sufficiently small. The proposed identification procedure thus allows us to locate the nodes and anti-nodes of a system without actually calculating the eigenvectors and examining the mode shapes.

Using the results of Figure 2 in reference [1], we can extract the nodes and anti-nodes of a uniform fixed-free bar under axial vibrations. For such a simple system, the nodes and anti-nodes can be obtained analytically. After some algebra (for brevity, the detailed derivations are omitted here), the ith normalized mode shape of the uniform fixed-free bar is given by

$$u_i(x) = \sqrt{2/\rho L}(\sin(2i-1)\pi/2)x/L, \qquad i = 1, 2, \ldots, \infty, \tag{1}$$

where ρ represents the mass per unit length of the bar and L its length. For the ith mode shape, the jth node and anti-node are given by, respectively,

$$(x_{\mathrm{N}}/L)_i^j = 2j/(2i-1), \qquad (x_{\mathrm{AN}}/L)_i^j = (2j-1)/(2i-1), \qquad j = 1, 2, \ldots, i-1, \tag{2, 3}$$

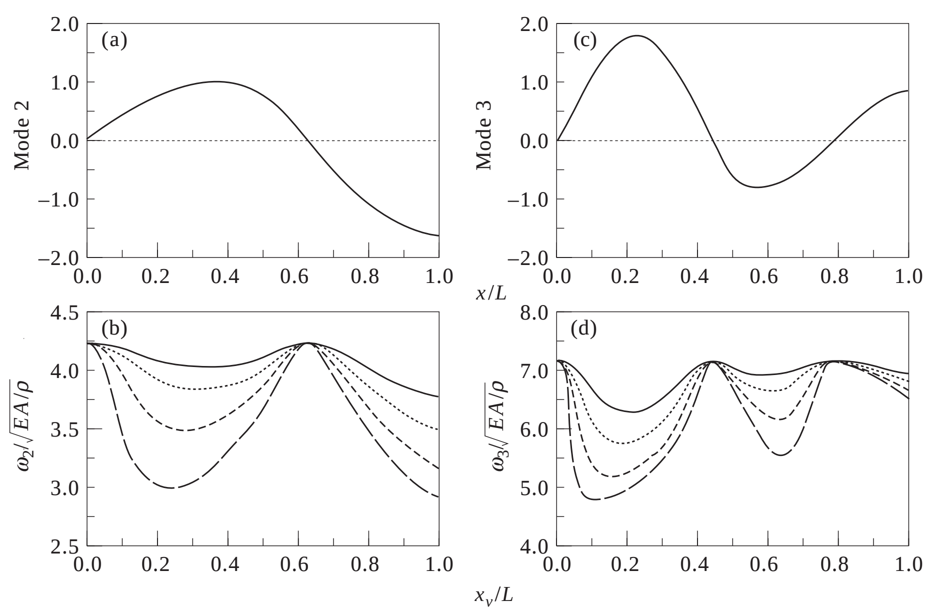

Figure 2. Second and third mode shapes for the system of Figure 1 and the corresponding frequency curves as a function of the constraint location. The lumped mass, $M = 0{\cdot}25\rho L$, is located at $0{\cdot}5L$. (a) Mode 2, (b) frequency curves for mode 2 (——, $m_v = 0{\cdot}1$; · · · ·, 0·2; ---, 0·4; – – –, 0·8), (c) mode 3, (d) frequency curves for mode 3 for different m_v (key as in (a)).

where the subscripts N and AN represent node and anti-node, respectively. Here one does not consider $x = L$ as an anti-node since it corresponds to the free end of the bar. Using the expressions above, one can easily verify that the maxima and local minima of the frequency curves of Figure 2 given in reference [1] correspond to the nodes and anti-nodes of the uniform fixed-free bar.

The present approach can be extended easily to locate the nodes and anti-nodes of the system of Figure 1. Consider the case where the lumped mass, M, has a value of $M = 0{\cdot}25\rho L$ and is located at $0{\cdot}5L$ (see Figure 3 of reference [1]). Figure 2 shows the second and third mode shapes of the system of Figure 1 and the natural frequency curves of the combined system as a function of the constraint location, x_v, of the lumped virtual mass, m_v. The sensitivity of the identification scheme to m_v is also examined by varying m_v. In Figure 2 one notes that the locations of the maxima of the natural frequency curves (of the combined system) correspond to the nodes of Figure 1 and they are independent of m_v. The locations of the local minima of the frequency curves vary with m_v. The magnitude of m_v affects the location of the first local minimum the most. Other local minima shift only slightly in comparison. As m_v becomes smaller, the local minima converge to the anti-nodes of the system in Figure 1. Also, as m_v increases, the frequency curve shifts downward because the mass of the combined system becomes larger. Finally, note that the mode shapes have a break at the point of mass attachment, as explained in reference [1]. Incidentally, in reference [3] the authors used the proposed scheme to locate the nodes and anti-nodes of a cantilevered Euler-Bernoulli beam with various attachments. For the systems analyzed in reference [3] the mode shapes do not have breaks like those of the modes of the present system because the boundary conditions for the beam system at the location of the attached lumped mass involve the third and fourth derivatives of the displacement.

Thus, in brief, by attaching a lumped virtual mass to a complicated structure and tracking the maxima and local minima of the frequency curves of the assembled system, one can locate the nodes and anti-nodes of the (original) complicated structure.

REFERENCES

1. H. Batan and M. Gürgöze 1996 *Journal of Sound and Vibration* **194**, 751–756. On the effect of an arbitrarily located mass on the longitudinal vibrations of a bar.
2. M. A. Cutchins 1980 *Journal of Sound and Vibration* **73**, 185–193. The effect of an arbitrarily located mass on the longitudinal vibrations of a bar.
3. P. D. Cha, C. L. Dym and W. Wong 1996 *Journal of Sound and Vibration*. Identifying nodes and anti-nodes of complex structures using virtual elements.