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VIBRATIONS OF COMPOSITE, DOUBLY CONNECTED SQUARE MEMBRANES

D. V. Bambill, R. H. Gutierrez, P. A. A. Laura and V. Jederlinic
Department of Engineering, Universidad Nacional del Sur and Institute of Applied Mechanics (CONICET), 8000-Bahia Blanca, Argentina

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1. INTRODUCTION

Composite, structural elements are commonly used in today's technology and, on the other hand, non-homogeneities may be generated by manufacturing procedures. The analysis of mechanical systems with varying material properties has been the subject of many recent investigations. In an excellent paper, Spence and Horgan [1] found upper and lower bounds for the natural frequencies of vibration of a circular membrane with stepped radial density and they showed that eigenvalue estimation techniques based on an integral equation approach are more effective than classical variational techniques. A conformal mapping approach was used in reference [2] in the case of composite membranes of regular polygonal shape whose inner circular core possesses a density ρ_1 while the remaining core is characterized by ρ_0.

In general, previous investigations deal with composites, simply connected membranes, an exception being made of the analysis reported in reference [3] where circular annular composite membranes have been considered. The present study deals with the doubly connected membrane shown in Figures 1(a) and (b). In the case of discontinuous variation of the density, in Figure 1(a) one has

$$\rho(x, y) = \left(\begin{array}{l} \rho_0 \text{ (constant) } (\bar{x}, \bar{y}) \varepsilon \bar{D}_0, \\ \rho_1 \text{ (constant) } (\bar{x}, \bar{y}) \varepsilon \bar{D}_1 - \bar{D}_0, \end{array} \right. \tag{1}$$

while, when dealing with continuous variation of the density it is assumed that it varies according to (see Figure 1(b))

$$\rho(x, y) = \rho_0[1 + \gamma/a(\sqrt{\bar{x}^2 + \bar{y}^2} - R_0)], \qquad \gamma > 0 \text{ (constant)}. \tag{2}$$

2. DETERMINATION OF THE FUNDAMENTAL FREQUENCY COEFFICIENT BY MEANS OF THE OPTIMIZED RAYLEIGH-RITZ METHOD

Assuming normal modes of vibration, the governing functional is given by the well known expression

$$J(W) = \iint_{\bar{D}_1} \left(W_{\bar{x}}^2 + W_{\bar{y}}^2 \right) \mathrm{d}\bar{x}\, \mathrm{d}\bar{y} - \frac{\omega^2}{S} \iint_{\bar{D}_1} \rho(\bar{x}, \bar{y}) W^2\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}, \tag{3}$$

where W is the amplitude of vibration. Introducing the dimensionless variables $x = \bar{x}/a$ and $y = \bar{y}/b$, expression (3) becomes

$$a^2 J(W) = \iint_{D_1} \left(W_x^2 + W_y^2 \right) \mathrm{d}x\, \mathrm{d}y - \lambda^2 \iint_{D_1} f(x, y) W^2\, \mathrm{d}x\, \mathrm{d}y, \tag{4}$$

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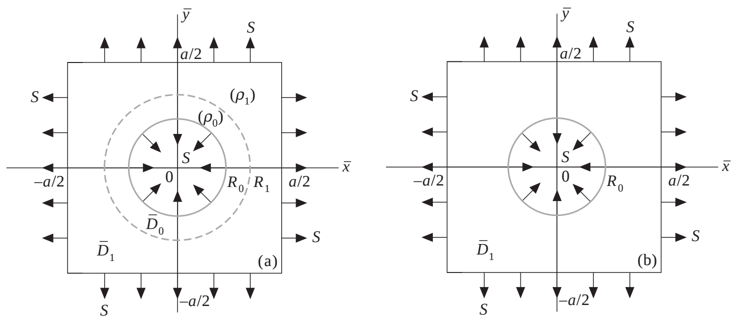

Figure 1(a). Doubly connected membrane: (a) case of discontinuously varying density; (b) case of continuously varying density; $\bar{D}_0 = \bar{D}_1$.

where

$$\lambda^2 = \rho_0 a^2 \omega^2 / S, \qquad f(x, y) = \begin{cases} 1, & (x, y) \varepsilon D_0, \\ \gamma, & (x, y) \varepsilon D_1 - D_0, \end{cases} \qquad \gamma = \rho_1/\rho_0$$

in the case of the configuration shown in Figure 1(a), and

$$f(x, y) = 1 + \gamma(\sqrt{x^2 + y^2} - r_0), \qquad r_0 = R_0/a,$$

when the density varies in a continuous fashion. The amplitude of the fundamental mode shape is approximated by means of the expression

$$W \cong W_\alpha = C_1 \varphi_1(x, y) + C_2 \varphi_2(x, y), \tag{5}$$

where

$$\varphi_1(x, y) = (1/2 - x^2)(1/2 - y^2)(\sqrt{x^2 + y^2} - r_0),$$
$$\varphi_2(x, y) = (1/2 - x^2)(1/2 - y^2)(\sqrt{x^2 + y^2} - r_0)^p,$$

and where p is Rayleigh's optimization parameter.

In accordance with the Rayleigh-Ritz method one requires the minimization condition

$$\frac{a^2}{2} \frac{\partial J}{\partial C_i} = \sum_{j=1}^{2} \left[\iint_{D_1} (\varphi_{jx} \varphi_{ix} + \varphi_{jy} \varphi_{iy}) \, \mathrm{d}x \, \mathrm{d}y - \lambda^2 \iint_{D_1} f \varphi_j \varphi_i \, \mathrm{d}x \, \mathrm{d}y \right] C_j = 0, \qquad (i = 1,2), \tag{6}$$

which from the non-triviality requirement yields a secular determinant in λ, the lower root being the fundamental frequency coefficient $\lambda_1 = \sqrt{\rho_0/S}\omega_1 a$. Since

$$\lambda_1 = \lambda_1(p), \tag{7}$$

by minimizing λ_1 with respect to p, one obtains an optimized value of the fundamental frequency coefficient.

3. ALTERNATIVE SOLUTION: USE OF AN APPROXIMATE CONFORMAL MAPPING APPROACH

As shown in reference [4] the domain shown in Figure 1(a) can be mapped with good engineering accuracy, onto an annular domain in the ξ-plane by means of the expression

$$z = a_p(x + \mathrm{i}y) = f(\xi) = a_p A_s \sum_{n=0}^{\infty} (-1)^n a_n \xi^{1+4n}, \tag{8}$$

where a_p is $a/2$ (apothem of the square), $A_s = 1{\cdot}0787$ [4] and $\xi = \mathrm{r}\,\mathrm{e}^{\mathrm{i}\theta}$.

The approximation is valid as long as R_0/a_p, $R_1/a_p \ll 1$, see Figure 1, but from the point of view of the determination of eigenvalues good accuracy is achieved for $R_1/a_p < 0{\cdot}5$ [4]. For $|\xi| = r \ll 1$ one obtains from equation (8):

$$|z| \simeq a_p A_s r. \tag{9}$$

Accordingly,

$$r_0 \simeq R_0/a_p A_s; \qquad r_1 \simeq R_1/a_p A_s. \tag{10}$$

Substituting equations (8), (9) and (10) in equation (3) one obtains after some manipulations

$$J[W(r,\theta)] = \int_{r_0}^{1}\int_0^{2\pi}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$- \frac{\omega^2\rho_0}{S}\left[a_p^2 A_s^2 \int_{r_0}^{r_1}\int_0^{2\pi} W^2 r\,\mathrm{d}r\,\mathrm{d}\theta + \frac{\rho_1}{\rho_0}\int_{r_1}^{1}\int_0^{2\pi} W^2 |f'|^2 r\,\mathrm{d}r\,\mathrm{d}\theta\right]. \tag{11}$$

TABLE 1

Fundamental eigenvalues λ_1 of the configuration shown in Figure 1(a)

		0·5		1		1·5		2	
$2R_1/a$	$2R_0/a$	(I)	(II)	(I)	(II)	(I)	(II)	(I)	(II)
0·20	0·05	7·88	7·87	5·62	5·65	4·60	4·64	3·99	4·03
	0·10	8·58	8·59	6·08	6·11	4·97	4·99	4·30	4·33
0·30	0·10	8·40	8·31	6·08	6·11	5·00	5·04	4·35	4·39
	0·20	9·66	9·86	6·87	7·02	5·62	5·74	4·87	4·97
0·40	0·10	7·94	7·79	6·08	6·11	5·10	5·16	4·47	4·54
	0·20	9·32	9·45	6·87	7·02	5·68	5·81	4·95	5·07
	0·40	10·99	11·33	7·82	8·09	6·40	6·62	5·55	5·74
0·50	0·10	7·40	–	6·08	–	5·25	–	4·67	–
	0·20	8·72	–	6·87	–	5·81	–	5·12	–
	0·30	10·53	–	7·82	–	6·49	–	5·66	–
	0·40	12·33	–	8·38	–	7·24	–	6·28	–
0·75	0·10	6·40	–	6·08	–	5·77	–	5·48	–
	0·20	7·36	–	6·87	–	6·40	–	5·99	–
	0·30	8·62	–	7·82	–	7·14	–	6·56	–
	0·40	10·04	–	8·83	–	7·88	–	7·14	–
	0·50	12·22	–	10·21	–	8·85	–	7·89	–

(I) Determined using the optimized Rayleigh-Ritz method in the physical plane.
(II) Determined by means of conformal mapping-optimized Rayleigh-Ritz method.

TABLE 2

Fundamental eigenvalues λ_1 of the configuration shown in Figure 1b. Determination using the optimized Rayleigh-Ritz method in the physical plane

$2R_0/a$	γ: 0·5	1	1·5	2
0·1	5·739	5·443	5·187	4·963
0·2	6·514	6·204	5·934	5·694
0·3	7·455	7·129	6·841	6·584
0·4	8·468	8·138	7·843	7·576
0·5	9·842	9·502	9·192	8·910

Following previous investigations [3, 4] it was found approximate $W(r, \theta)$ by means of the expression

$$W(r, \theta) \cong W_\alpha(r) = A_1(1 - r^p)(1 - r_0/r) + A_2(1 - r^{p+2})[1 - (r_0/r)^2] + A_3(1 - r^{p+4})[1 - (r_0/r)^4], \tag{12}$$

where p is, again, Rayleigh's optimization parameter. The determination of the optimized value of the fundamental eigenvalue, λ_1, follows the previously explained procedure.

3. NUMERICAL RESULTS

Table 1 depicts values of λ_1 for the case of discontinuously varying density. A comparison of eigenvalues obtained by the two approaches presented in this paper is shown for $2R_1/a = 0{\cdot}20$, 0·30 and 0·40 and several values of $2R_0/a$ and γ. The differences between values obtained by the different methodologies range from 0·1–4%. The conformal mapping approach yields lower values for $2R_1/a = 0{\cdot}30$ and 0·40 and $2R_0/a = 0{\cdot}10$ for $\gamma = 0{\cdot}5$. The agreement is, in general, quite acceptable. Table 2 shows values of λ_1 for the case of a continuously varying density (Figure 1(b)). The variation of the fundamental eigenvalue with respect to the intervening parameters appears to be quite reasonable.

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