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FORCED MOTION OF A STEPPED SEMI-INFINITE PLATE

A. P. Gupta and N. Sharma

Department of Mathematics, University of Roorkee, Roorkee 247 667, India

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Forced motion of a plate of infinite length whose thickness, density and elastic properties vary in steps along the finite breadth, is analysed by an eigenfunction method. The numerical results for transverse deflection computed for a clamped-clamped plate subjected to constant or half-sine pulse load are plotted in graphs.

1. INTRODUCTION

A large number of papers are available in the literature on the vibration of beams of constant and uniformly varying thickness. A few papers available on free vibration of beams of stepped thickness are given in the reference section [1–13]. The authors have not so far come across any paper on forced motion of beams of stepped thickness.

In the present paper, the forced motion of an isotropic plate of infinite length and finite breadth whose thickness, density and elastic property along the breadth vary in steps, is considered. The analysis is based on classical theory. The plate is assumed to be made up of n plate elements of infinite lengths and finite breadths, joined edge to edge and having, in general, different breadths, thicknesses, densities, Young's modulii and Poisson ratios. The arbitrary constants arising in the solution of equations of motion for free vibration are determined by the edge and continuity conditions. The forced motion is analysed by the eigenfunction method.

The forced motion of a plate clamped at both edges and subjected to constant or half-sine pulse load uniformly distributed over a portion of the plate is analysed as an example problem. The numerical results for transverse deflection computed for a plate made up of three plate elements by varying the breadths, thicknesses, densities and Young's modulii of the elements for the loads distributed uniformly over the whole plate are plotted in graphs. The variations in breadths, thicknesses and densities are taken in such a way that the total breadth, average thickness and average density of the plate remain constant.

2. EQUATION OF MOTION

An isotropic plate of infinite length and finite breadth a whose thickness, density and elastic property along the breadth vary in steps is considered. The plate is referred to Cartesian co-ordinates by taking the y-axis along the infinite length, the middle plane of the plate in the plane $z = 0$ and the two edges in the planes $x = 0$ and $x = a$. The plate is assumed to be made up of n plate elemets joined edge to edge with their middle planes lying in plane $z = 0$. The breadth, thickness, density, Young's modulus and Poisson ratio of the kth element ($k = 1, 2, \dots, n$) are taken as a_k, h_k, ρ_k, E_k and ν_k respectively and it

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lies from $x = x_{k-1}$ to $x = x_k$ where $x_k - x_{k-1} = a_k$, $x_0 = 0$ and $x_n = a$. Some of the thickness profiles of the plate along the breadth are shown in Figure 1.

The equations of motion of the plate elements according to classical theory are taken as

$$\frac{E_k h_k^3}{12(1-\nu_k^2)} w_{k,xxxx} + \rho_k h_k w_{k,tt} = p_k(x, t); \qquad x_{k-1} \leqslant x \leqslant x_k, \qquad k = 1, 2, \ldots, n, \quad (1)$$

where w_k and p_k are the transverse deflections and the loads per unit area respectively, and t is the time. A comma followed by a variable suffix denotes differentiation with respect to that variable.

Making the equations (1) non-dimensional, one gets

$$I_k W_{k,XXXX} + \gamma_k H_k W_{k,TT} = P_k(X, T); \qquad X_{k-1} \leqslant X \leqslant X_k, \qquad k = 1, 2, \ldots, n, \qquad (2)$$

where

$$X = x/a,\ X_k = x_k/a,\ H_k = h_k/a,\ \gamma_k = \rho_k/\rho_a,\ \varepsilon_k = E_k/E,\ P_k = p_k/E,$$
$$T = t\sqrt{(E/\rho_a a^2)},\ I_k = \varepsilon_k H_k^3/12(1-\nu_k^2),\ X_0 = 0,\ X_n = 1.$$

ρ_a is the average density of the plate and E is the Young's modulus of some standard material.

3. FREE VIBRATION ANALYSIS

3.1. SOLUTION

For free vibration, one takes

$$W_k(X, T) = W_{kj}(X)\, e^{i\Omega_j T} \qquad (3)$$

and substitutes in equation (2), after putting $P_k = 0$, to get

$$W_{kj,XXXX} - \omega_{kj}^4 W_{kj} = 0; \qquad \omega_{kj}^4 = \gamma_k H_k \Omega_j^2 / I_k, \qquad (4)$$

where Ω_j and W_{kj} are the circular frequency and mode shape function respectively in the jth normal mode of free vibration.

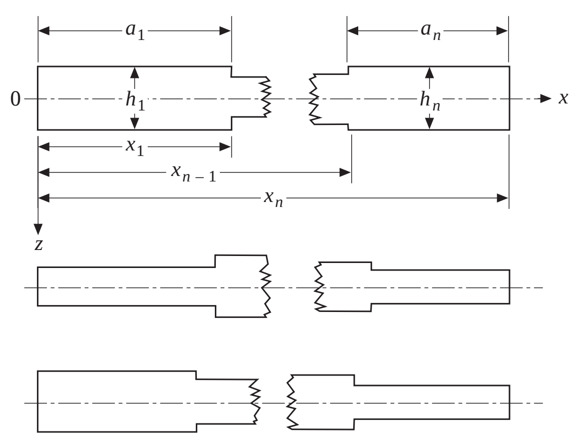

Figure 1. Thickness profiles of the plate.

For the sake of convenience the suffix j is suppressd in free vibration analysis and the solutions of equations (4) are taken as

$$W_k(X) = S_k(X)\mathrm{D}_k, \qquad \mathrm{D}_k = [d_{1k} \quad d_{2k} \quad d_{3k} \quad d_{4k}]',$$
$$S_k(X) = [\cosh \omega_k X \ \sinh \omega_k X \ \cos \omega_k X \ \sin \omega_k X], \tag{5}$$

where D_k are vectors of mode shape constants and prime denotes the transpose of a matrix.

The continuity conditions between the plate elements at $X = X_k$; $k = 1, 2, \ldots, n-1$ can be taken as

$$W_l(X_k) = W_k(X_k), \qquad W_{l,X}(X_k) = W_{k,X}(X_k),$$
$$I_l W_{l,XX}(X_k) = I_k W_{k,XX}(X_k), \qquad I_l W_{l,XXX}(X_k) = I_k W_{k,XXX}(X_k), \tag{6}$$

where $l = k + 1$.

From (5) and (6) one gets

$$\mathrm{D}_l = \mathrm{B}^{(l)}\mathrm{D}_k, \qquad \mathrm{B}^{(l)} = \mathrm{A}_l^{-1}(X_k)\mathrm{A}_k(X_k) \tag{7}$$

where the matrices $\mathrm{A}_k(X_k)$ and $\mathrm{A}_l(X_k)$ are given by

$$\mathrm{A}_k(X_k) = [S_k(X_k) \quad S_{k,X}(X_k) \quad I_k S_{k,XX}(X_k) \quad I_k S_{k,XXX}(X_\mathrm{K})]',$$
$$\mathrm{A}_l(X_k) = [S_l(X_k) \quad S_{l,X}(X_k) \quad I_l S_{l,XX}(X_k) \quad I_l S_{l,XXX}(X_k)]'. \tag{8}$$

From equation (7) one gets

$$\mathrm{D}_l = \mathrm{C}^{(l)}\mathrm{D}_1, \qquad \mathrm{C}^{(l)} = \mathrm{B}^{(l)}\mathrm{B}^{(l-1)} \ldots \mathrm{B}^{(2)} = [\mathrm{c}_{qr}^{(l)}]_{4\times 4}. \tag{9}$$

In this way the $4n$ constants arising in solutions (5) are reduced to 4. It should be noted that if the thicknesses, densities and elastic properties of the n plate elements are taken to be the same, the matrices $\mathrm{B}^{(l)}$ and $\mathrm{C}^{(l)}$ reduces to unit matrices and the whole problem reduces to that of a uniform plate.

3.2. EDGE CONDITIONS

The plate is taken to be clamped at both edges, for which the conditions are

$$W_1(0) = W_{1,X}(0) = W_n(1) = W_{n,X}(1) = 0. \tag{10}$$

3.3. FREQUENCY EQUATION

Using relations (9) in solutions (5) and then putting them in conditions (11), one gets

$$d_{11} + d_{31} = 0, \qquad d_{21} + d_{41} = 0, \qquad s_{11}d_{11} + s_{12}d_{21} + s_{13}d_{31} + s_{14}d_{41} = 0,$$
$$s_{21}d_{11} + s_{22}d_{21} + s_{23}d_{31} + s_{24}d_{41} = 0, \tag{11}$$

where

$$s_{1r} = S_3(1)[c_{qr}^{(n)}]_{4\times 1}, \qquad s_{2r} = S_{3,X}(1)[c_{qr}^{(n)}]_{4\times 1}, \qquad r = 1, 2, 3, 4. \tag{12}$$

For a non-trivial solution of equations (11) the determinant of the coefficient matrix must vanish, which gives rise to the following transcendental frequency equation

$$(s_{13} - s_{11})(s_{24} - s_{22}) - (s_{23} - s_{21})(s_{14} - s_{12}) = 0. \tag{13}$$

The denumerable infinity of roots of this equation for given dimensions, densities and elastic constants of the plate elements are frequencies Ω_j of various normal modes of free vibration of the plate.

3.4. ORTHONORMALITY CONDITION

The orthogonality condition for normal modes of free vibration of the plate be obtained. It is

$$\sum \gamma_k H_k \int_{X_{k-1}}^{X_k} W_{ki} W_{kj}\, \mathrm{d}X = 0, \qquad \text{when } i \neq j, \tag{14}$$

where summation over k is taken from 1 to n.

A mode normalization condition to obtain unique mode shapes is taken as

$$\sum \gamma_k H_k \int_{X_{k-1}}^{X_k} W_{kj}^2\, \mathrm{d}X = 1. \tag{15}$$

3.5. MODE SHAPES

Since out of the four equations (11) only three are independent, three of them are solved first to get D_1 in terms of d_{41}. This is substituted in equations (9) to get D_2 and D_3 in terms of d_{41}. These are then substituted in solutions (5) to get the mode shapes as

$$W_k(X) = S_k(X)[e_{1k} \quad e_{2k} \quad e_{3k} \quad e_{4k}]' d_{41}; \qquad X_{k-1} \leqslant X \leqslant X_k, \qquad k = 1, 2, \ldots, n, \tag{16}$$

where

$$d = (s_{12} - s_{14})/(s_{13} - s_{11}), \qquad e_{11} = -d, \qquad e_{21} = -1, \qquad e_{31} = d, \qquad e_{41} = 1,$$
$$e_{ql} = d(c_{q3}^{(l)} - c_{q1}^{(l)}) + (c_{q4}^{(l)} - c_{q2}^{(l)}), \qquad q = 1, 2, 3, 4. \tag{17}$$

To get d_{41}, the suffix j of equation (15) is suppressed and $W_k(X)$ from equation (16) is substituted in it. It gives

$$d_{41}^2 = 1/\sum [F_k(X_k) - F_k(X_{k-1})], \tag{18}$$

where

$$F_k(X) = (\gamma_k H_k/4\omega_k)[f_{1k}\omega_k X + f_{2k} \sinh(2\omega_k X) + f_{3k} \sin(2\omega_k X)$$
$$+ f_{4k} \cosh(2\omega_k X) + f_{5k} \cos(2\omega_k X) + \cosh(\omega_k X)\{f_{6k} \sin(\omega_k X)$$
$$+ f_{7k} \cos(\omega_k X)\} + \sinh(\omega_k X)\{f_{8k} \sin(\omega_k X) + f_{9k} \cos(\omega_k X)\}], \tag{19}$$

$$f_{1k} = 2(e_{1k}^2 - e_{2k}^2 + e_{3k}^2 + e_{4k}^2), \qquad f_{2k} = e_{1k}^2 + e_{2k}^2, \qquad f_{3k} = e_{3k}^2 - e_{4k}^2, \qquad f_{4k} = 2e_{1k}e_{2k},$$
$$f_{5k} = -2e_{3k}e_{4k}, \qquad f_{6k} = 4(e_{1k}e_{3k} + e_{2k}e_{4k}), \qquad f_{7k} = 4(e_{2k}e_{3k} - e_{1k}e_{4k})$$
$$f_{8k} = 4(e_{1k}e_{4k} + e_{2k}e_{3k}), \qquad f_{9k} = 4(e_{1k}e_{3k} - e_{2k}e_{4k}). \tag{20}$$

4. FORCED MOTION ANALYSIS

A solution of the forced motion equations (2) subjected to the continuity conditions (6) and edge conditions (10) is assumed to be

$$W_k(X, T) = \sum W_{kj}(X) g_j(T); \qquad X_{k-1} \leqslant X \leqslant X_k, \qquad k = 1, 2, \ldots, n. \tag{21}$$

where the summation over j is from 1 to ∞. Substituting it in equations (2) and using equations (4), one gets

$$\sum \gamma_k H_k W_{kj}(g_{j,TT} + \Omega_j^2 g_j) = P_k(X, T). \tag{22}$$

Multiplying it by W_{ki} and using conditions (14) and (15), one gets

$$g_{j,TT} + \Omega_j^2 g_j = G_j(T), \tag{23}$$

where

$$G_j(T) = \sum \int_{X_{k-1}}^{X_k} P_k W_{kj}\, \mathrm{d}X. \tag{24}$$

The solution of equation (23) is

$$\Omega_j g_j(T) = \Omega_j g_j(0) \cos(\Omega_j T) + g_{j,T}(0) \sin(\Omega_j T) + \int_0^T G_j(\tau) \sin\{\Omega_j(T-\tau)\}\, \mathrm{d}\tau, \tag{25}$$

where

$$g_j(0) = \sum \gamma_k H_k \int_{X_{k-1}}^{X_k} W_k(X, 0) W_{kj}\, \mathrm{d}X,$$

$$g_{j,T}(0) = \sum \gamma_k H_k \int_{X_{k-1}}^{X_k} W_{k,T}(X, 0) W_{kj}\, \mathrm{d}X. \tag{26}$$

If the initial conditions are taken as $W_k(X, 0) = W_{k,T}(X, 0) = 0$, then

$$g_j(0) = g_{j,T}(0) = 0. \tag{27}$$

4.1. LOADING CONDITION

The following two types of external loads uniformly distributed over a portion of each plate element are taken:

4.1.1. *Constant load* (*CL*)

$$P_k(X, T) = P_0[U(X-\xi_k) - U(X-\eta_k)]U(T)/\sum(\eta_k - \xi_k)$$

$$X_{k-1} \leqslant \xi_k < \eta_k \leqslant X_k, \qquad k = 1, 2, \ldots, n, \tag{28}$$

where P_0 is the total load on the plate and U denotes unit step function.

$G_j(T)$, evaluated after substituting from equations (16) and (28) in equation (24), is substituted in equation (25) and the condition (27) is used to get

$$g_j(T) = P_j[1 - \cos(\Omega_j T)]/\Omega_j^2, \tag{29}$$

where

$$P_j = P_0 \sum [\phi_{kj}(\eta_k) - \phi_{kj}(\xi_k)] / \sum (\eta_k - \xi_k)$$

$$\phi_{kj}(X) = d_{41j}[e_{1kj} \sinh(\omega_{kj}X) + e_{2kj} \cosh(\omega_{kj}X) + e_{3kj} \sin(\omega_{kj}X) - e_{4kj} \cos(\omega_{kj}X)]/\omega_{kj}. \quad (30)$$

4.1.2. *Half sine pulse load* (*HL*)

$$P_k(X, T) = P_0[U(X - \xi_k) - U(X - \eta_k)]\{1 - U(T - t_1)\} \sin(\pi T/t_1) / \sum (\eta_k - \xi_k),$$

$$X_{k-1} \leqslant \xi_k < \eta_k \leqslant X_k, \qquad k = 1, 2, \ldots, n. \quad (31)$$

where t_1 is the duration of HL.

Proceeding as above one gets

$$g_j(T) = \begin{cases} P_j t_1[\pi \sin(\Omega_j T) - \Omega_j t_1 \sin(\pi T/t_1)]/[\Omega_j(\pi^2 - \Omega_j^2 t_1^2)], & \text{when } T < t_1 \\ 2P_j \pi t_1[\sin\{\Omega_j(T - t_1/2)\} \cos(\Omega_j t_1/2)]/[\Omega_j(\pi^2 - \Omega_j^2 t_1^2)], & \text{when } T \geqslant t_1. \end{cases} \quad (32)$$

The substitution of unique mode shapes W_{kj} given by equations (18) and (16) and $g_j(T)$ from equation (29) or (32) as the case may be gives the transverse deflection $W_k(X, T)$ for forced motion.

5. RESULTS AND DISCUSSION

The variations in breadths, thicknesses and densities of different plate elements are defined in such a way that the total breadth, average thickness and average density of the plate remain constant by taking $\alpha_k = a_k/a_1$, $\beta_k = h_k/h_1$ and $\delta_k = \rho_k/\rho_1$.
Now

$$\Sigma a_k = a \ \text{ or } \ a_1 \Sigma \alpha_k = a \ \text{ or } \ X_1 = 1/\Sigma \alpha_k \ \text{ and } \ X_k = X_1 \sum_{i=1}^{k} \alpha_i;$$

$$\Sigma a_k h_k = a h_a \ \text{ or } \ a_1 h_1 \Sigma \alpha_k \beta_k = a h_a \ \text{ or } \ H_1 = H_a/(X_1 \Sigma \alpha_k \beta_k), \qquad \text{and } \ H_k = H_1 \beta_k,$$

where

h_a is the average thickness of the plate and $H_a = h_a/a$.

$$\Sigma a_k h_k \rho_k = a h_a \rho_a \ \text{ or } \ \gamma_1 = H_a/(X_1 H_1 \Sigma \alpha_k H_k \delta_k) \ \text{ and } \ \gamma_k = \gamma_1 \delta_k.$$

Numerical results are computed for transverse deflection parameter $W_0 = (W_k \times 10^{-2}/P_0)_{X=0{\cdot}5}$ for a plate made up of three plate elements whose first and third elements are identical i.e., for $\alpha_3 = \beta_3 = \delta_3 = \varepsilon_3 = 1$, by taking $\nu_1 = \nu_2 = \nu_3 = 1{\cdot}3$, $H_a = 0{\cdot}05$ and $t_1 = 2\pi/\Omega_1$.

The frequencies Ω_j are computed by the bisection method up to an accuracy of five decimal places and the series of W_k (equation (21)) is summed up to the first ten terms which give an accuracy of at least four decimal places.

The graphs of W_0 versus T for CL are plotted in Figure 2 for various values of β_2 and α_2 and in Figure 3 for various values of δ_2 and ε_2. Figure 2(a) shows, when the breadth of the middle element is kept larger than the other two and its thickness is increased, the

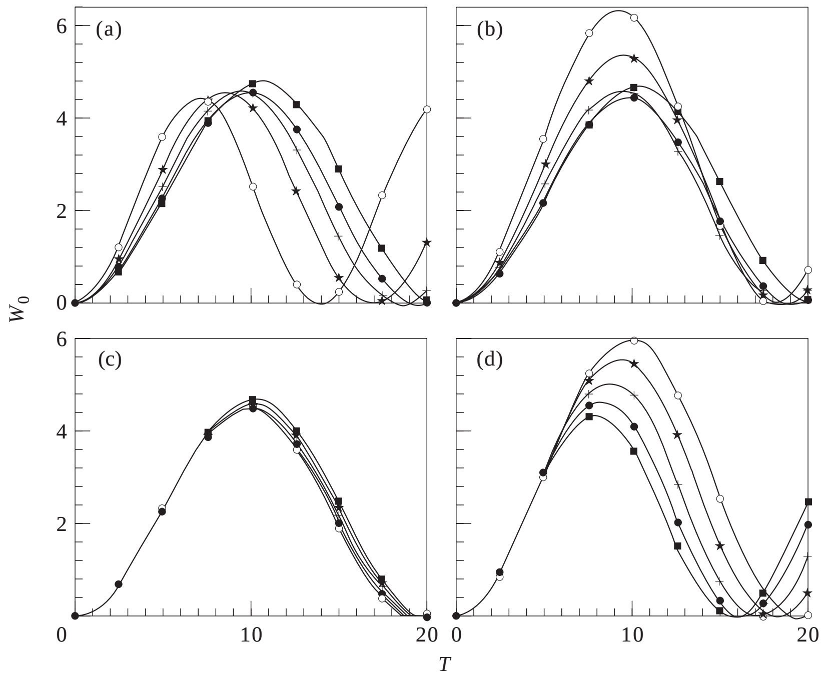

Figure 2. W_0 versus T for CL for various values of β_2 and α_2 ($\beta_2 = \alpha_2$): –○–, 0·4; –*–, 0·7; – + –, 1·0; –●–, 1·3; –■–, 1·6. (a) $\alpha_2 = 1{\cdot}4$; $\delta_2 = 1{\cdot}0$; $\varepsilon_2 = 1{\cdot}0$: (b) $\alpha_2 = 0{\cdot}6$; $\delta^2 = 1{\cdot}0$; $\varepsilon_2 = 1{\cdot}0$: (c) $\beta_2 = 1{\cdot}4$; $\delta_2 = 1{\cdot}0$; $\varepsilon_2 = 1{\cdot}0$: (d) $\beta_2 = 0{\cdot}5$; $\delta_2 = 1{\cdot}0$; $\varepsilon_2 = 1{\cdot}0$.

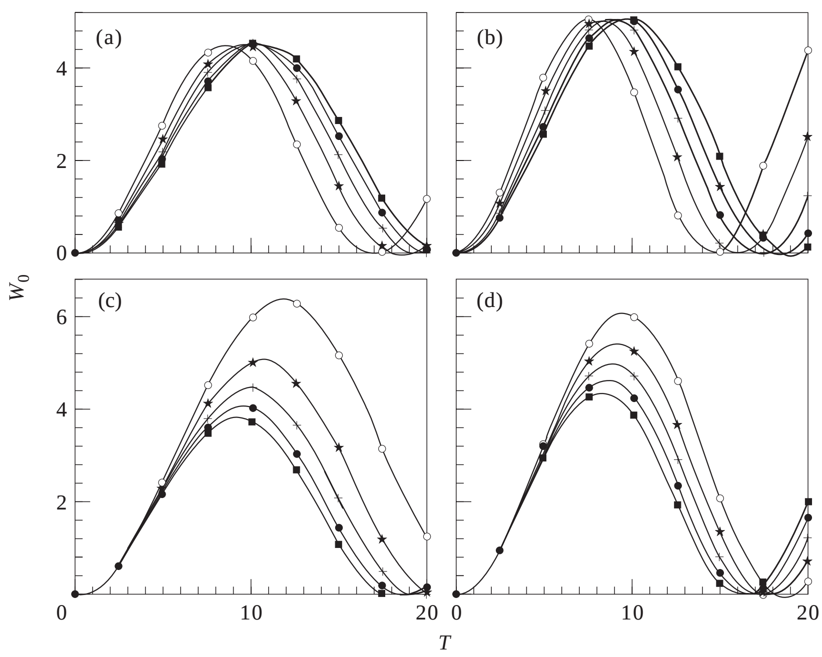

Figure 3. W_0 versus T for CL for various values of δ_2 and ε_2 ($\delta_2 = \varepsilon_2$): –○–, 0·4; –*–, 0·7; – + –, 1·0; –●–, 1·3; –■– 1·6. (a) $\beta_2 = 1{\cdot}4$; $\alpha_2 = 1{\cdot}0$: $\varepsilon_2 = 1{\cdot}0$; (b) $\beta_2 = 0{\cdot}6$; $\alpha_2 = 1{\cdot}0$; $\varepsilon_2 = 1{\cdot}0$: (c) $\beta_2 = 1{\cdot}4$; $\alpha_2 = 1{\cdot}0$; $\delta_2 = 1{\cdot}0$: (d) $\beta_2 = 0{\cdot}6$; $\alpha_2 = 1{\cdot}0$; $\delta_2 = 1{\cdot}0$.

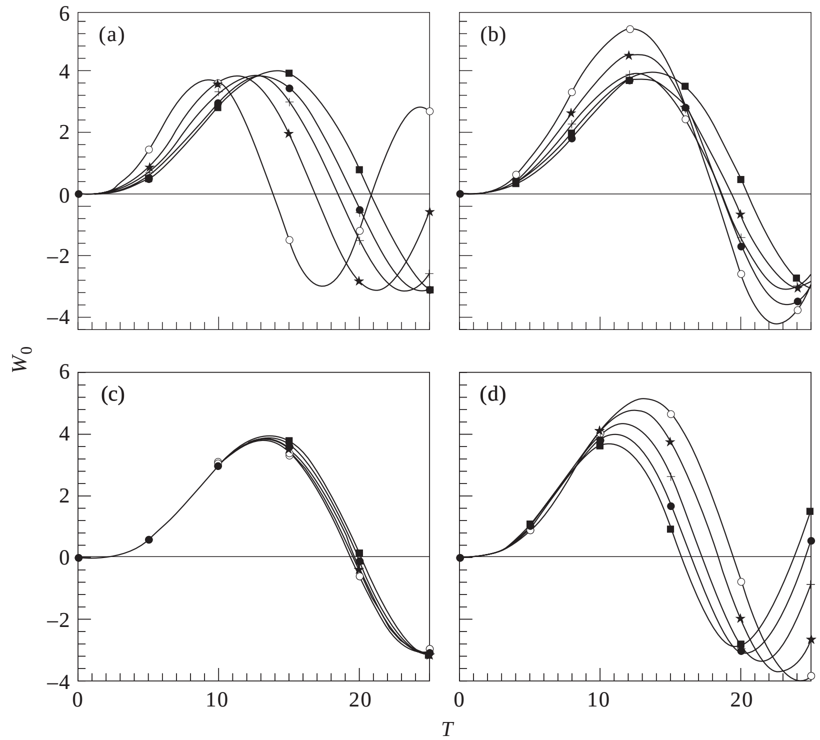

Figure 4. W_0 versus T for HL for various values of β_2 ans α_2. Keys for (a), (b), (c) and (d) as in Figure 2.

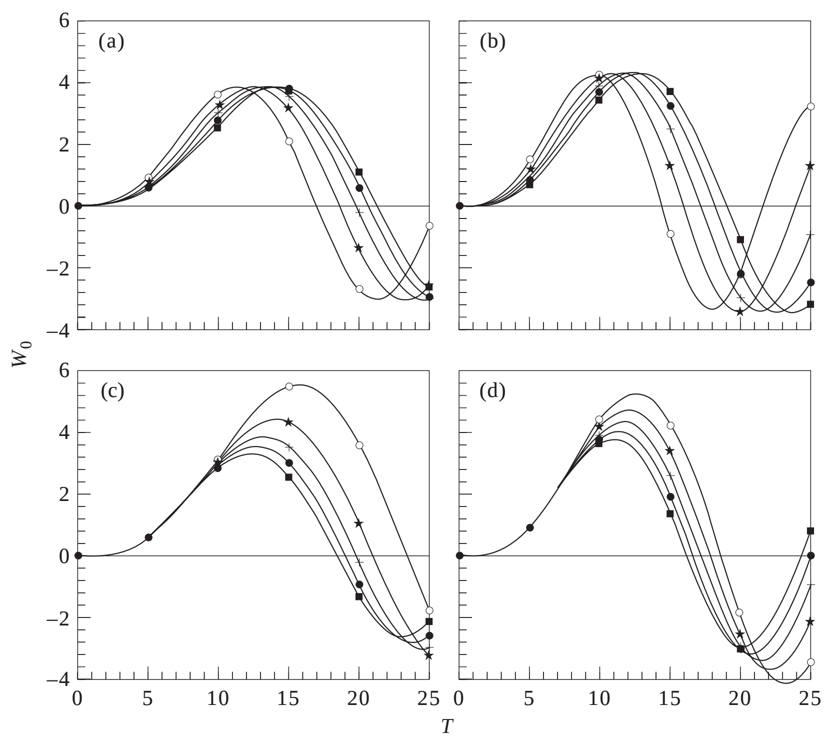

Figure 5. W_0 versus T for HL for various values of δ_2 and ε_2. Keys for (a), (b), (c) and (d) as in Figure 3.

time of attaining the first peak as well as the magnitude of W_0 at this first peak increases. Figure 2(b) shows, when the breadth of the middle element is kept smaller than the other two and its thickness is increased from a smaller value, the magnitude of W_0 at the first peak decreases and then increases for the maximum value of β_2. Figures 2(c) and 2(d) show, when the breadth of the middle element is increased, the magnitude of W_0 at the first peak increases if its thickness is kept larger than the other two but it decreases if the thickness is kept smaller. It is also seen that the magnitude of W_0 is hardly sensitive to the change in the breadth of the middle element when its thickness is kept larger than the other two. Figures 3(a) and 3(b) show that the magnitude of W_0 at the first peak remains unchanged with the increase in the density of the middle element but the time of attaining the first peak increases. Figures 3(c) and 3(d) show that the magnitude of W_0 at the first peak as well as the time of attaining the first peak decrease with the increase in the Young's modulus of the middle element.

The graphs of W_0 versus T for HL are plotted in Figures 4 and 5. The variations in W_0 are similar to its corresponding cases of CL except that here the time of attaining the first peak is longer but the value of W_0 at it is smaller. Here the peaks are seen alternatively on both sides of the z-axis.

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