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STABILITY ANALYSIS OF AN AXIALLY ACCELERATING STRING

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The dynamic response of an axially accelerating string is investigated. The time dependent velocity is assumed to vary harmonically about a constant mean velocity. Approximate analytical solutions are sought using two different approaches. In the first approach, the equations are discretized first and then the method of multiple scales is applied to the resulting equations. In the second approach, the method of multiple scales is applied directly to the partial differential system. Principal parametric resonances and combination resonances are investigated in detail. Stability boundaries are determined analytically. It is found that instabilities occur when the frequency of velocity fluctuations is close to two times the natural frequency of the constant velocity system or when the frequency is close to the sum of any two natural frequencies. When the velocity variation frequency is close to zero or to the difference of two natural frequencies, however, no instabilities are detected up to the first order of perturbation. Numerical results are presented for a band-saw and a threadline problem.

1. INTRODUCTION

Due to their technological importance, the dynamics of axially moving materials has received considerable attention from many researchers. Threadlines, high speed magnetic and paper tapes, strings, belts, band-saw blades, fibers, chains, beams and pipes transporting fluids are some of the technological examples. The vast literature on axially moving material vibrations is reviewed in the paper by Ulsoy *et al*. [1] and Wickert and Mote [2]. The basic characteristics of such systems include a transport velocity dependent natural frequency spectrum and the existence of a critical speed at which a divergence instability occurs [1]. Recently, Wickert and Mote [3] investigated the transverse vibrations of travelling strings and beams. They used travelling string eigenfunctions and introduced a convenient orthogonal basis suitable for discretization. An alternative complex form is also given by the same authors [4]. The effects of interspan coupling in dual-span axially moving materials are investigated in a number of papers by Ulsoy [5] and Al-Jawi *et al*. [6–8]. The vibration localization phenomenon is first introduced to the axially moving materials in the mentioned papers [6–8].

With few exceptions, most of these studies have addressed the constant axial transport velocity problem. However, technological devices are subject to accelerations and decelerations which may significantly alter the dynamic response. Miranker [9] was the first to derive the equations of motion for a string travelling with time dependent axial velocity.

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Later, Mote [10] investigated the problem of an axially accelerating string with harmonic excitation at one end; he replaced the variable coefficients by their time averaged values and investigated stability by Laplace transform techniques. Pakdemirli *et al*. [11] re-derived the equations of motion for an axially accelerating string using Hamilton's principle and numerically investigated the stability of the response using Floquet theory. A sinusoidal variation of the transport velocity, about a mean velocity of zero, is considered in the analysis. Pakdemirli and Batan [12] considered a different type of velocity variation, namely the periodic, constant acceleration–deceleration type.

In this study, we relax the constant velocity assumption by introducing a velocity function having small harmonic fluctuations about a constant mean velocity. This model better represents many real systems, since, in many applications, small variations in the velocity are likely to occur. Because the tension is velocity dependent, the velocity fluctuations also lead to tension fluctuations. In this sense, our study is closely related to a recent study by Mockensturm *et al*. [13], in which tension fluctuations for a constant velocity system were considered.

In this study, approximate analytical solutions are presented for the problem using the method of multiple scales (a perturbation technique). Two different approaches are used in finding the soloutions. In the first approach, the discretization–perturbation approach, the equation of motion is cast into a system of first order equations and suitable orthogonal basis functions are selected to discretize the equations. The formalism given by Wickert and Mote [4] is followed. The discretized equations are then solved by applying the method of multiple scales. In the second approach, the direct–perturbation approach, the method of multiple scales is applied directly to the partial differential system. This second method does not require transformation of the equations or the selection of an orthogonal basis. The analysis is straightforward and integrals similar to those in orthonormality conditions arise in finding the solvability conditions at higher orders of approximation. We believe that this second approach is new to the axially moving materials literature.

Recently, comparisons of these two methods for specific and more general problems have appeared in the literature. Nayfeh *et al*. [14] were the first to show that direct–perturbation methods yield better results for finite mode truncations and for systems having quadratic and cubic non-linearities. Later, Pakdemirli *et al*. [15] compared the results of both methods for a non-linear cable vibration problem and showed that the bifurcation and stability analysis might differ for both methods, with the direct perturbation method better representing the behavior of the real system. Instead of showing the discrepancies for specific problems. Pakdemirli [16] and Pakdemirli and Boyaci [17] used a generalized equation form with arbitrary quadratic and cubic non-linearities. Comparisons for finite mode truncations are presented in reference [16] and for infinite modes in reference [17]. The major conclusion is that the direct perturbation method yields more accurate results for finite mode truncations and for higher order perturbation schemes. However in other cases, such as in the analysis presented here, although the results are identical, the direct method has the advantage of being more straightforward. For a partial review of the literature on comparisons between the two methods, see reference [18].

In this study, we investigate the principal parametric resonances and combination resonances for any two modes. We find that for velocity fluctuation frequencies near twice any natural frequency, an instability region occurs whereas the frequencies close to zero, no instabilities are detected. For combination resonances, instabilities occur only for those of additive type. No instabilities are detected for difference type combination resonances in agreement with reference [13]. Boundaries separating stable and unstable regions are determined analytically. The formalism derived is applied to two specific technological problems of interest.

2. EQUATIONS OF MOTION

The system considered is a string or strip of length L moving with transport velocity $v^*(t^*)$. The equations of motion are derived in reference [11], using Hamilton's principle:

$$\rho A(\ddot{y}^* + \dot{v}^* y'^* + 2v^* \dot{y}'^*) + (\rho A v^{*2} - P)y''^* = 0, \qquad y^*(0, t^*) = y^*(L, t^*) = 0, \quad (1)$$

where ρ is the mass density, A is the cross-sectional area and y^* is the transverse displacement of the string. The dot denotes differentiation with respect to time and the prime denotes differentiation with respect to the spatial variable x. The dimensional quantities are represented by an asterisk. Following reference [11], we assume that the tension force varies with velocity according to the relation

$$P = P_0 + \eta \rho A v^{*2} \quad (2)$$

where P_0 is the initial tension and $0 \leqslant \eta \leqslant 1$. We also define a pulley support parameter $\kappa = 1 - \eta$. In constant displacement mechanisms, such as tapes and threadlines, κ can be taken as 1, whereas in constant tension mechanisms $\kappa = 0$. Inserting equation (2) into equation (1), we have

$$\rho A(\ddot{y}^* + \dot{v}^* y'^* + 2v^* \dot{y}'^*) + (\kappa \rho A v^{*2} - P_0)y''^* = 0. \quad (3)$$

Defining the non-dimensional quantities

$$x = x^*/L, \qquad y = y^*/L, \qquad t = (1/L)\sqrt{(P_0/\rho A)}\,t^*, \qquad v = v^*/\sqrt{P_0/\rho A} \quad (4)$$

and inserting into equation (3), we obtain

$$\ddot{y} + \dot{v}y' + 2v\dot{y}' + (\kappa v^2 - 1)y'' = 0, \qquad y(0, t) = y(1, t) = 0. \quad (5)$$

Note that the velocity is non-dimensionalized with respect to the wave velocity $\sqrt{P_0/\rho A}$.

We now assume a velocity function which allows for small harmonic variations about a non-dimensional mean velocity v_0 as follows:

$$v = v_0 + \varepsilon v_1 \sin \Omega t, \quad (6)$$

where ε is a small non-dimensional parameter ($\varepsilon \ll 1$), v_1 is of the same order as v_0 and Ω is the non-dimensional frequency of velocity variations. Note that this non-dimensional frequency is related to the dimensional one through the relation

$$\Omega^* = \frac{1}{L}\sqrt{\frac{P_0}{\rho A}}\,\Omega. \quad (7)$$

In reference [11], a special case of the above problem, in which the mean velocity $v_0 = 0$, has been investigated numerically.

3. PRINCIPAL PARAMETRIC RESONANCE

In this section, analytical solutions are presented using the method of multiple scales (a perturbation technique), [19, 20]. Two different approaches are considered in the analysis. In the first approach, the equations are discretized first and then perturbation is applied to the resulting equations (discretization–perturbation method). In the second approach, perturbation is applied directly to the partial differential system (direct–perturbation method). This second method is new in the axially moving materials research context. Parametric resonance of $\Omega \simeq 0$, $\Omega \simeq 2\omega_n$ and Ω away from 0 and $2\omega_n$ cases are investigated (ω_n being the natural frequency of a string travelling at constant velocity). It is found that only $\Omega \simeq 2\omega_n$ leads to instabilities and the boundaries separating stable and unstable regions are calculated analytically.

3.1. DISCRETIZATION–PERTURBATION METHOD

In this method, we first cast the equations in a convenient first order form and then discretize the equations using orthogonal basis functions. This method was first proposed by Wickert and Mote [3]. They adopted the method from the solutions presented by Meirovitch [21, 22] for gyroscopic systems. Later, Wickert and Mote [4] presented an alternative complex form of the discretization process. The idea is to use travelling string eigenfunctions instead of the stationary string eigenfunctions. It is demonstrated through numerical simulations that the usual choice of stationary string eigenfunctions has poor convergence properties [11, 12]. Taking only one mode of travelling string eigenfunction yields comparable results with those of four modes of stationary string eigenfunctions [13]. The convergence is superior since the physics of the problem involves motion which can be captured better through traveling eigenfunctions. This method has been well established and frequently used in the literature [23–25].

The travelling string eigenfunctions are not orthogonal. To employ the orthogonality conditions, the gyroscopic character of the equations should be first reflected by casting the equation to a suitable first order form. We first substitute equation (6) into equation (5), keep terms up to $O(\varepsilon)$ and obtain

$$\ddot{y} + 2v_0\dot{y}' + (\kappa v_0^2 - 1)y'' + \varepsilon\{2v_1 \sin \Omega t\dot{y}' + 2\kappa v_0 v_1 \sin \Omega t y'' + \Omega v_1 \cos \Omega t y'\} = 0. \quad (8)$$

Defining the operators

$$M = I, \qquad G = 2v_0 \frac{\partial}{\partial x}, \qquad K = (\kappa v_0^2 - 1)\frac{\partial^2}{\partial x^2},$$

$$L_1 = 2v_1 \frac{\partial}{\partial x}, \qquad L_2 = 2\kappa v_0 v_1 \frac{\partial^2}{\partial x^2}, \qquad L_3 = \Omega v_1 \frac{\partial}{\partial x} \quad (9)$$

and substituting into equation (8), we have

$$M\ddot{y} + G\dot{y} + Ky + \varepsilon\{\sin \Omega t(L_1\dot{y} + L_2 y) + \cos \Omega t(L_3 y)\} = 0. \quad (10)$$

The equations are cast into a first order equation as follows:

$$\mathbf{A}\dot{\mathbf{w}} + \mathbf{B}\mathbf{w} + \varepsilon\{\sin \Omega t\mathbf{C} + \cos \Omega t\mathbf{D}\}\mathbf{w} = \mathbf{0}, \quad (11)$$

where

$$\mathbf{w} = \begin{bmatrix} \dot{y} \\ y \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} L_1 & L_2 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & L_3 \\ 0 & 0 \end{bmatrix}. \quad (12)$$

The above form is convenient, especially for the discretization analysis. A base function having orthogonality properties with respect to matrices **A** and **B** is selected:

$$\Phi_n = \begin{bmatrix} \lambda_n \psi_n \\ \psi_n \end{bmatrix}, \quad (13)$$

where

$$\lambda_n = \mathrm{i}\omega_n, \qquad \psi_n = C_n\,\mathrm{e}^{\mathrm{i}\alpha_n x}\sin n\pi x, \tag{14}$$

$$\omega_n = \frac{n\pi(1-\kappa v_0^2)}{\sqrt{1-\kappa v_0^2+v_0^2}}, \qquad \alpha_n = \frac{n\pi v_0}{\sqrt{1-\kappa v_0^2+v_0^2}}. \tag{15}$$

ψ_n are the mode shapes corresponding to the travelling string with constant velocity v_0 and ω_n are the corresponding frequencies.

Following reference [4], we define the inner product

$$\langle \mathbf{w}_1, \mathbf{w}_2\rangle = \int_0^1 \mathbf{w}_1^{\mathrm{T}}\bar{\mathbf{w}}_2\,\mathrm{d}x, \tag{16}$$

With respect to this inner product, the Φ_n possess the following orthonormality properties:

$$\langle \mathbf{A}\Phi_n, \Phi_m\rangle = \delta_{nm}, \qquad \langle \mathbf{B}\Phi_n, \Phi_m\rangle = -\lambda_n\delta_{nm}, \tag{17}$$

where C_n is normalized such that

$$C_n = \frac{1}{n\pi\sqrt{1-\kappa v_0^2}}. \tag{18}$$

For discretization, we now assume a solution of the type

$$\mathbf{w} = \sum_{m=\mp 1,\mp 2,\ldots}^{\infty} \xi_m(t)\Phi_m(x) = \sum_{m=1,2,\ldots}^{\infty} \xi_m(t)\Phi_m(x) + \bar{\xi}_m(t)\bar{\Phi}_m(x), \tag{19}$$

where the overbar denotes the complex conjugate. Following references [3, 4, 13, 21–25], we assume that the eigenfunctions are complete, i.e., any arbitrary function in the domain of interest can be written in terms of the infinite sum of those eigenfunctions. Generally speaking, the eigenfunctions arising from solutions of physical systems are always complete. However, a more mathematical proof is beyond the scope of this work.

Substituting equation (19) into equation (11), taking the inner product with a vector Φ_n, we finally obtain

$$\dot{\xi}_n - \lambda_n\xi_n + \varepsilon\sum_{m=1,2,\ldots}^{\infty}\{\sin\Omega t(\xi_m\langle\mathbf{C}\Phi_m,\Phi_n\rangle + \bar{\xi}_m\langle\mathbf{C}\bar{\Phi}_m,\Phi_n\rangle) + \cos\Omega t(\xi_m\langle\mathbf{D}\Phi_m,\Phi_n\rangle$$
$$+\,\bar{\xi}_m\langle\mathbf{D}\bar{\Phi}_m,\Phi_n\rangle)\} = 0, \qquad n = 1, 2, 3, \ldots. \tag{20}$$

For a one-mode approximation (considering only the nth mode), the equations reduce to the following simple form:

$$\dot{\xi}_n - \lambda_n\xi_n + \varepsilon\{\sin\Omega t(\xi_n\langle\mathbf{C}\Phi_n,\Phi_n\rangle + \bar{\xi}_n\langle\mathbf{C}\bar{\Phi}_n,\Phi_n\rangle) + \cos\Omega t(\xi_n\langle\mathbf{D}\Phi_n,\Phi_n\rangle$$
$$+\,\bar{\xi}_n\langle\mathbf{D}\bar{\Phi}_n,\Phi_n\rangle)\} = 0. \tag{21}$$

To determine the principal instabilities, we apply the method of multiple scales to equation (21). We assume an expansion of the form

$$\xi_n(t;\varepsilon) = \xi_{n0}(T_0, T_1) + \varepsilon\xi_{n1}(T_0, T_1) + \cdots, \tag{22}$$

where $T_0 = t$ and $T_1 = \varepsilon t$ are the usual fast and slow time scales. The time derivatives are defined as

$$\mathrm{d}/\mathrm{d}t = D_0 + \varepsilon D_1 + \cdots, \qquad D_0 = \partial/\partial T_0, \qquad D_1 = \partial/\partial T_1. \tag{23}$$

Inserting equations (22) and (23) into equation (21) and separating terms at each order of ε, we have

$$O(1)\colon D_0\xi_{n0} - \mathrm{i}\omega_n\xi_{n0} = 0, \tag{24}$$

$$\begin{aligned} O(\varepsilon)\colon D_0\xi_{n1} - \mathrm{i}\omega_n\xi_{n1} = & -D_1\xi_{n0} - \sin\Omega T_0(\xi_{n0}\langle\mathbf{C}\Phi_n, \Phi_n\rangle + \bar{\xi}_{n0}\langle\mathbf{C}\bar{\Phi}_n, \Phi_n\rangle) \\ & - \cos\Omega T_0(\xi_{n0}\langle\mathbf{D}\Phi_n, \Phi_n\rangle + \bar{\xi}_{n0}\langle\mathbf{D}\bar{\phi}_n, \phi_n\rangle). \end{aligned} \tag{25}$$

The solution at $O(1)$ is

$$\xi_{n0} = A_n(T_1)\,\mathrm{e}^{\mathrm{i}\omega_n T_0}, \tag{26}$$

where $A_n(T_1)$ are the complex amplitudes slowly varying with time. Inserting equation (26) with the identities

$$\cos\Omega T_0 = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}\Omega T_0} + \mathrm{e}^{-\mathrm{i}\Omega t_0}), \qquad \sin\Omega T_0 = -\frac{\mathrm{i}}{2}(\mathrm{e}^{\mathrm{i}\Omega T_0} - \mathrm{e}^{\mathrm{i}\Omega T_0}) \tag{27}$$

into equation (25) and rearranging, we have

$$\begin{aligned} D_0\xi_{n1} - \mathrm{i}\omega_n\xi_{n1} = & -D_1 A_n\,\mathrm{e}^{\mathrm{i}\omega_n T_0} + A_n\,\mathrm{e}^{\mathrm{i}(\Omega+\omega_n)T_0}\left\{\frac{\mathrm{i}}{2}\langle\mathbf{C}\Phi_n, \Phi_n\rangle - \tfrac{1}{2}\langle\mathbf{D}\Phi_n, \Phi_n\rangle\right\} \\ & + \bar{A}_n\,\mathrm{e}^{\mathrm{i}(\Omega-\omega_n)T_0}\left\{\frac{\mathrm{i}}{2}\langle\mathbf{C}\bar{\Phi}_n, \Phi_n\rangle - \tfrac{1}{2}\langle\mathbf{D}\bar{\Phi}_n, \Phi_n\rangle\right\} \\ & + \bar{A}_n\,\mathrm{e}^{\mathrm{i}(\omega_n-\Omega)T_0}\left\{-\frac{\mathrm{i}}{2}\langle\mathbf{C}\Phi_n, \Phi_n\rangle - \tfrac{1}{2}\langle\mathbf{D}\Phi_n, \Phi_n\rangle\right\} \\ & + \bar{A}_n\,\mathrm{e}^{-\mathrm{i}(\Omega+\omega_n)T_0}\left\{-\frac{\mathrm{i}}{2}\langle\mathbf{C}\bar{\Phi}_n, \Phi_n\rangle - \tfrac{1}{2}\langle\mathbf{D}\bar{\Phi}_n, \Phi_n\rangle\right\}. \end{aligned} \tag{28}$$

Inspecting equation (28), we observe that three cases should be investigated separately.

3.1.1. *Case I*: Ω *away from* 0 *and* $2\omega_n$

For this case, elimination of secular terms yields

$$D_1 A_n = 0. \tag{29}$$

Therefore, up to $O(\varepsilon)$, we obtain constant amplitude solutions.

3.1.2. *Case II*: $\Omega \simeq 2\omega_n$

For this case, we first express the nearness of Ω to $2\omega_n$ by employing the relation

$$\Omega = 2\omega_n + \varepsilon\sigma, \tag{30}$$

where σ is a detuning parameter of $O(1)$. Substituting equation (30) into equation (28) and eliminating the secular terms yields

$$D_1 A_n + (k_1 - \mathrm{i}k_2)\bar{A}_n\,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{31}$$

where

$$k_1 = (\kappa v_1/4)\sin 2\alpha_n, \qquad k_2 = (\kappa v_1/4)(1 - \cos 2\alpha_n). \tag{32}$$

Note that α_n is defined in equation (15).

3.1.3. *Case III*: $\Omega \simeq 0$

Similar to the previous case, we express the nearness of Ω to zero as

$$\Omega = \varepsilon\sigma. \tag{33}$$

Substituting equation (33) into equation (28) and eliminating the secular terms, we have

$$D_1 A_n + (k_3 \cos \sigma T_1 + \mathrm{i}k_4 \sin \sigma T_1)A_n = 0, \tag{34}$$

where

$$k_3 = \frac{v_0 v_1 \Omega}{2(1 - \kappa v_0^2 + v_0^2)}, \qquad k_4 = \frac{n\pi v_0 v_1 [1 + \kappa(1 - \kappa v_0^2 + v_0^2)]}{(1 - \kappa v_0^2 + v_0^2)^{3/2}}, \tag{35}$$

3.2. DIRECT–PERTURBATION METHOD

In this alternative approach, we apply the method of multiple scales directly to the partial differential system. The advantage is that we do not need to cast the equations into a first order form and define orthonormality conditions. Conditions similar to those for orthonormality arise when finding the solvability conditions at higher orders of approximation in a straightforward manner. It is shown that the results obtained are identical with the discretization–perturbation case. However, for non-linear systems and for higher order perturbation schemes, it is well known that the direct–perturbation method yields more accurate results for finite mode truncations [14–18].

We rewrite the partial differential equation (8) as our starting point:

$$\ddot{y} + 2v_0\dot{y}' + (\kappa v_0^2 - 1)y'' + \varepsilon\{2v_1 \sin \Omega t \dot{y}' + 2\kappa v_0 v_1 \sin \Omega t y'' + \Omega v_1 \cos \Omega t y'\} = 0,$$
$$y(0, t) = y(1, t). \tag{36}$$

We assume an expansion of the form

$$y(x, t; \varepsilon) = y_0(x, T_0, T_1) + \varepsilon y_1(x, T_0, T_1) + \cdots \tag{37}$$

Substituting this expansion, together with the time derivatives defined in equation (23), into equation (36) and separating terms at each order of ε, we finally obtain

$$O(1)\colon D_0^2 y_0 + 2v_0 D_0 y_0' + (\kappa v_0^2 - 1)y_0'' = 0, \tag{38}$$

$$O(\varepsilon)\colon D_0^2 y_1 + 2v_0 D_0 y_1' + (\kappa v_0^2 - 1)y_1'' = -2D_0 D_1 y_0 - 2v_0 D_1 y_0' - 2v_1 \sin \Omega T_0 D_0 y_0'$$
$$- 2\kappa v_0 v_1 \sin \Omega T_0 y_0'' - \Omega v_1 \cos \Omega T_0 y_0'. \tag{39}$$

The solution at $O(1)$ is

$$y_0(x, T_0, T_1) = A_n(T_1)\, \mathrm{e}^{\mathrm{i}\omega_n T_0}\psi_n(x) + \bar{A}_n(T_1)\, \mathrm{e}^{-\mathrm{i}\omega_n T_0}\bar{\psi}_n(x), \tag{40}$$

where $\psi_n(x)$ and ω_n are given in equations (14) and (15). Substituting this solution into equation (39) and rearranging, we finally obtain

$$D_0^2 y_1 + 2v_0 D_0 y_1' + (\kappa v_0^2 - 1)y_1'' = D_1 A_n\, \mathrm{e}^{\mathrm{i}\omega_n T_0}(-2\mathrm{i}\omega_n \psi_n - 2v_0 \psi_n')$$

$$+ A_n\, \mathrm{e}^{\mathrm{i}(\Omega + \omega_n)T_0}\left[-v_1 \omega_n \psi_n' + \mathrm{i}\kappa v_0 v_1 \psi_n'' - \frac{\Omega v_1}{2}\, \psi_{\mathrm{n}}'\right]$$

$$+ \bar{A}_n \, e^{i(\Omega - \omega_n)T_0} \left[v_1 \omega_n \bar{\psi}'_n + i\kappa v_0 v_1 \bar{\psi}''_n - \frac{\Omega v_1}{2} \bar{\psi}'_n \right] + cc, \tag{41}$$

where cc denotes the complex conjugate of all preceding terms. Since all results are identical with the previous section, for illustration purposes, we only present results for the case of the frequency being approximately twice one of the natural frequencies of the system.

3.2.1. $\Omega \simeq 2\omega_n$

We again express the nearness of Ω to $2\omega_n$ by the relation

$$\Omega = 2\omega_n + \varepsilon\sigma. \tag{42}$$

Substituting equation (42) into equation (41) and arranging terms, we have

$$D_0^2 y_1 + 2v_0 D_0 y'_1 + (\kappa v_0^2 - 1) y''_1 = D_1 A_n \, e^{i\omega_n T_0}(-2i\omega_n \psi_n - 2v_0 \psi'_n)$$

$$+ \bar{A}_n \, e^{i\omega_n T_0} \, e^{i\sigma T_1} \left[v_1 \omega_n \bar{\psi}'_n + i\kappa v_0 v_1 \bar{\psi}''_n - \frac{\Omega v_1}{2} \psi'_n \right] + NST + cc, \tag{43}$$

where NST denotes non-secular terms.

We now assume a solution of the form

$$y_1(x, T_0, T_1) = Y(x, T_1) \, e^{i\omega_n T_0} + W(x, T_0, T_1) + cc, \qquad Y(0, T_1) = Y(1, T_1) = 0, \tag{44}$$

where $W(x, T_0, T_1)$ is the solution of the non-homogeneous equation having only non-secular terms at the right side. Therefore $W(x, T_0, T_1)$ exists, unique and free from secular terms. $Y(x, T_1)$ is determined by the equation

$$-\omega_n^2 Y + 2v_0 i\omega_n Y' + (\kappa v_0^2 - 1) Y'' = -2(i\omega_n \psi_n + v_0 \psi'_n) D_1 A_n$$

$$+ \left(v_1 \omega_n \bar{\psi}'_n + i\kappa v_0 v_1 \bar{\psi}''_n - \frac{\Omega v_1}{2} \bar{\psi}'_n \right) \bar{A}_n \, e^{i\sigma T_1}. \tag{45}$$

The homogeneous part of this equation possesses a non-trivial solution. For the non-homogeneous equation to possess a solution a solvability condition should be satisfied. (See details of finding solvability conditions in reference [19], chapter 15).

To find this solvability condition, we multiply equation (45) by an arbitrary function $u(x)$, integrate over the domain, and carry all differential operators from Y to $u(x)$ using integration by parts. The final result is

$$\int_0^1 \{ -\omega_n^2 u - 2v_0 i\omega_n u' + (\kappa v_0^2 - 1) u'' \} Y \, dx + (\kappa v_0^2 - 1) u Y' |_0^1 = \int_0^1 u(RHS) \, dx, \tag{46}$$

where RHS denotes the right side terms in equation (45). Since $u(x)$ is arbitrary, we now choose $u(x)$ so that the integral and the second term vanishes, leading to the differential system

$$-\omega_n^2 u - 2v_0 i\omega_n u' + (\kappa v_0^2 - 1) u'' = 0, \qquad u(0) = u(1) = 0. \tag{47}$$

This problem is actually the adjoint problem of the original equation. The solution is

$$u = \bar{\psi}_n(x) \tag{48}$$

where u turns out to be the complex conjugate of the eigenfunctions of the travelling string (see equation (14)).The solvability condition can now be written as

$$\int_0^1 \bar{\psi}_n(RHS)\,\mathrm{d}x = 0. \tag{49}$$

Therefore, the solvability condition requires that the complex conjugate of constant velocity eigenfunctions be orthogonal to the right side of equation (45). Note that this condition yields similar integrals with the orthonormality conditions given in the previous method. However, finding this solvability condition is straightforward and requires only the solution of the adjoint problem.

Inserting the appropriate terms in equation (49) and performing the integrals, we finally obtain the modulation equation for the complex amplitudes:

$$D_1 A_n + (k_1 - \mathrm{i}k_2)\bar{A}_n\,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{50}$$

where the coefficients are defined in equation (32). Equation (50) is exactly the same equation as given in equation (31) obtained using the discretization–perturbation method.

The cases $\Omega \simeq 0$ and Ω away from 0 and $2\omega_n$ yield also results identical to those of the previous section. For the sake of brevity, we refrain from presenting the calculations for those cases.

3.3. STABILITY ANALYSIS

We determine the stability of the amplitude modulation equations for the cases of $\Omega \simeq 0$ and $\Omega \simeq 2\omega_n$. We already found that when Ω is away from 0 and $2\omega_n$, the solutions are always bounded up to $O(\varepsilon)$.

3.3.1. *Case I*: $\Omega \simeq 0$

For this case, the complex amplitude modulation equation is found to be

$$D_1 A_n + (k_3 \cos\sigma T_1 + \mathrm{i}k_4 \sin\sigma T_1)A_n = 0, \tag{51}$$

where the k_3 and k_4 coefficients are given in equations (35). A direct integration yields

$$A_n = A_0 \exp\left(\frac{-k_3}{\sigma}\sin\sigma T_1 + \mathrm{i}\frac{k_4}{\sigma}\cos\sigma T_1\right). \tag{52}$$

It is evident that solutions are always bounded as $T_1 \to \infty$, because $-1 \leqslant \sin\sigma T_1 \leqslant 1$ and $-1 \leqslant \cos\sigma T_1 \leqslant 1$. Therefore, for very small frequencies we do not expect any instabilities up to $O(\varepsilon)$.

3.3.2. *Case III*: $\Omega \simeq 2\omega_n$

From equation (31) or (50), we write

$$D_1 A_n + (k_1 - \mathrm{i}k_2)\bar{A}_n\,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0. \tag{53}$$

First, we introduce the transformation

$$A_n = B_n\,\mathrm{e}^{\mathrm{i}\sigma T_1/2}. \tag{54}$$

Then we insert equation (54) into equation (53), and obtain

$$D_1 B_n + \frac{\mathrm{i}\sigma}{2}B_n + (k_1 - \mathrm{i}k_2)\bar{B}_n = 0. \tag{55}$$

Separating real and imaginary parts and calculating the eigenvalues of the coefficient matrix, we find

$$\lambda = \mp \tfrac{1}{2}\sqrt{-\sigma^2 + 4(k_1^2 + k_2^2)}. \tag{56}$$

In the interval $-2\sqrt{k_1^2 + k_2^2} < \sigma < 2\sqrt{k_1^2 + k_2^2}$ the response is unstable, whereas it is stable outside this region. Therefore, the stability boundaries are determined by

$$\sigma = \mp 2\sqrt{k_1^2 + k_2^2}. \tag{57}$$

Substituting for k_1 and k_2 from equations (32)–(57), and inserting the results into equation (42), we finally obtain

$$\Omega = 2\omega_n \mp \varepsilon v_1 \kappa \sin \alpha_n. \tag{58}$$

For constant tension mechanisms, $\kappa = 0$, and the instability regions close up to $O(\varepsilon)$. Hence, for this type of instability, decreasing κ increases the stability. When the mean velocity is zero, α_n is zero and again the instability regions close up to $O(\varepsilon)$. Further comments on the vanishing mean velocity case as well as comparisons with reference [11] are presented in section 5. When the amplitude of fluctuations v_1 increases, the stability regions widen, as expected. For constant displacement mechanisms, $\kappa = 1$, and equation (58) takes the form

$$\Omega = 2n\pi(1 - v_0^2) \mp \varepsilon v_1 \sin(n\pi v_0). \tag{59}$$

Plots of these stability boundaries will be presented in section 5 for band-saw and threadline problems.

4. COMBINATION RESONANCES

In this section, we assume that there are two dominant modes and investigate the combination resonances for the sum and difference of these modes with the frequency. Taking the nth and mth modes only, the discretized equations take the form

$$\begin{aligned}\dot{\xi}_n - \mathrm{i}\omega_n\xi_n + \varepsilon\{&\sin \Omega t(\xi_n\langle \mathbf{C}\Phi_n, \Phi_n\rangle + \bar{\xi}_n\langle \mathbf{C}\bar{\Phi}_n, \Phi_n\rangle + \xi_m\langle \mathbf{C}\Phi_m, \Phi_n\rangle \\ &+ \bar{\xi}_m\langle \mathbf{C}\bar{\Phi}_m, \Phi_n\rangle) + \cos \Omega t(\xi_n\langle \mathbf{D}\Phi_n, \Phi_n\rangle \\ &+ \bar{\xi}_n\langle \mathbf{D}\bar{\Phi}_n, \Phi_n\rangle + \xi_m\langle \mathbf{D}\Phi_m, \Phi_n\rangle + \bar{\xi}_m\langle \mathbf{D}\bar{\Phi}_m, \Phi_n\rangle)\} = 0,\end{aligned} \tag{60}$$

$$\begin{aligned}\dot{\xi}_m - \mathrm{i}\omega_m\xi_m + \varepsilon\{&\sin \Omega t(\xi_n\langle \mathbf{C}\Phi_n, \Phi_m\rangle + \bar{\xi}_n\langle \mathbf{C}\bar{\Phi}_n, \Phi_m\rangle + \xi_m\langle \mathbf{C}\Phi_m, \Phi_m\rangle \\ &+ \bar{\xi}_m\langle \mathbf{C}\bar{\Phi}_m, \Phi_m\rangle) + \cos \Omega t(\xi_n\langle \mathbf{D}\Phi_n, \Phi_m\rangle \\ &+ \bar{\xi}_n\langle \mathbf{D}\bar{\Phi}_n, \Phi_m\rangle + \xi_m\langle \mathbf{D}\Phi_m, \Phi_m\rangle + \bar{\xi}_m\langle \mathbf{D}\bar{\Phi}_m, \Phi_m\rangle)\} = 0.\end{aligned} \tag{61}$$

Assuming again approximate solutions of the form

$$\begin{aligned}\xi_n(t; \varepsilon) &= \xi_{n0}(T_0, T_1) + \varepsilon\xi_{n1}(T_0, T_1) + \dots, \\ \xi_m(t; \varepsilon) &= \xi_{m0}(T_0, T_1) + \varepsilon\xi_{m1}(T_0, T_1) + \dots\end{aligned} \tag{62}$$

and applying the method of multiple scales to the equations, performing the straightforward algebra and eliminating the secular terms, we finally obtain the following results for sum and difference type of combination resonances.

Case I: $\Omega \simeq \omega_m + \omega_n$

The frequency detuning is defined as

$$\Omega = \omega_m + \omega_n + \varepsilon\sigma \tag{63}$$

and the complex amplitude modulation equations are

$$D_1 A_n + \alpha_1 \bar{A}_m \,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \qquad D_1 A_m + \alpha_2 \bar{A}_n \,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{64, 65}$$

where

$$\alpha_1 = -\frac{\mathrm{i}}{2}\langle \mathbf{C}\bar{\Phi}_m, \Phi_n\rangle + \tfrac{1}{2}\langle \mathbf{D}\bar{\Phi}_m, \Phi_n\rangle, \qquad \alpha_2 = -\frac{\mathrm{i}}{2}\langle \mathbf{C}\bar{\Phi}_n, \Phi_m\rangle + \tfrac{1}{2}\langle \mathbf{D}\bar{\Phi}_n, \Phi_m\rangle. \tag{66}$$

Case II: $\Omega \simeq \omega_m - \omega_n$

Without loss of generality, we assume that $m > n$. The frequency detuning is defined as

$$\Omega = \omega_m - \omega_n + \varepsilon\sigma \tag{67}$$

and the complex amplitude modulation equations are

$$D_1 A_n + \alpha_3 A_m \,\mathrm{e}^{-\mathrm{i}\sigma T_1} = 0, \qquad D_1 A_m + \alpha_4 A_n \,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0, \tag{68, 69}$$

where

$$\alpha_3 = \frac{\mathrm{i}}{2}\langle \mathbf{C}\Phi_m, \Phi_n\rangle + \tfrac{1}{2}\langle \mathbf{D}\Phi_m, \Phi_n\rangle, \qquad \alpha_4 = -\frac{\mathrm{i}}{2}\langle \mathbf{C}\Phi_n, \Phi_m\rangle + \tfrac{1}{2}\langle \mathbf{D}\Phi_n, \Phi_m\rangle. \tag{70}$$

4.1. STABILITY BOUNDARIES

We determine the stability of the solutions for each case.

4.1.1. *Combination resonances of sum type*

For equations (64) and (65), we first introduce the transformation

$$A_n = B_n \,\mathrm{e}^{\mathrm{i}\sigma T_1/2}, \qquad A_m = B_m \,\mathrm{e}^{\mathrm{i}\sigma T_1/2} \tag{71}$$

and obtain

$$D_1 B_n + \frac{\mathrm{i}\sigma}{2} B_n + \alpha_1 \bar{B}_m = 0, \qquad D_1 B_m + \frac{\mathrm{i}\sigma}{2} B_m + \alpha_2 \bar{B}_n = 0. \tag{72, 72}$$

At this stage, we have two choices: (1) to determine stability from the complex equations above, or (2) to separate the equations into real and imaginary parts. The latter approach increases the algebra unnecessarily, hence we choose to determine stability from the complex amplitude modulation equations. Assuming that equations (72) and (73) possess solutions of the form

$$B_n = b_n \,\mathrm{e}^{\lambda T_1}, \qquad B_m = b_m \,\mathrm{e}^{\bar{\lambda}\mathrm{T}_1} \tag{74}$$

and substituting them into the equations, taking the complex conjugate of the second equation, we obtain for non-trivial solutions

$$\lambda = \mp\tfrac{1}{2}\sqrt{-\sigma^2 + 4\alpha_1\bar{\alpha}_2}. \tag{75}$$

Since $\alpha_1\bar{\alpha}_2 > 0$ always, the stability boundaries are determined by

$$\sigma = \mp 2\sqrt{\alpha_1\bar{\alpha}_2}. \tag{76}$$

Calculating α_1 and α_2 from equation (66), we obtain the explicit relations

$$\alpha_1\bar{\alpha}_2 = 4k_1k_2\sin^2\left(\frac{\alpha_n+\alpha_m}{2}\right), \qquad n+m \text{ even}, \tag{77}$$

$$\alpha_1\bar{\alpha}_2 = 4k_1k_2\cos^2\left(\frac{\alpha_n+\alpha_m}{2}\right), \qquad n+m \text{ odd}, \tag{78}$$

where

$$k_1 = \frac{nv_1[-(n-m)^2+(n+m)^2\kappa v_0^2]}{2(n+m)[-4nmv_0^2+(n-m)^2(1-\kappa v_0^2)]}, \tag{79}$$

$$k_2 = \frac{mv_1[-(n-m)^2+(n+m)^2\kappa v_0^2]}{2(n+m)[-4nmv_0^2+(n-m)^2(1-\kappa v_0^2)]}. \tag{80}$$

Therefore the stability regions are given by

$$\Omega = \omega_m+\omega_n \mp \varepsilon 4\sqrt{k_1k_2}\sin\left(\frac{\alpha_n+\alpha_m}{2}\right), \qquad n+m \text{ even}, \tag{81}$$

$$\Omega = \omega_m+\omega_n \mp \varepsilon 4\sqrt{k_1k_2}\cos\left(\frac{\alpha_n+\alpha_m}{2}\right), \qquad n+m \text{ odd}, \tag{82}$$

Note that, for $m=n$, equation (81) reduces to equation (58). Stability boundaries will be plotted in section 5 for band-saw and threadline examples.

4.1.2. *Combination resonances of difference type*

We first introduce the following transformation

$$A_n = B_n\,\mathrm{e}^{-\mathrm{i}\sigma T_1/2}, \qquad A_m = B_m\,\mathrm{e}^{\mathrm{i}\sigma T_1/2}, \tag{83}$$

and obtain, from equations (68) and (69),

$$D_1B_n - \frac{\mathrm{i}\sigma}{2}B_n + \alpha_3B_m = 0, \qquad D_1B_m + \frac{\mathrm{i}\sigma}{2}B_m + \alpha_4B_n = 0. \tag{84, 85}$$

We assume solutions of the form

$$B_n = b_n\,\mathrm{e}^{\lambda T_1}, \qquad B_m = b_m\,\mathrm{e}^{\lambda T_1}. \tag{86}$$

For non-trivial solutions,

$$\lambda = \mp\tfrac{1}{2}\sqrt{-\sigma^2+4\alpha_3\alpha_4}. \tag{87}$$

Calculating α_3 and α_4 from equation (70), we have

$$\alpha_3\alpha_4 = -4k_3k_4\sin^2\left(\frac{\alpha_m-\alpha_n}{2}\right), \qquad m-n \text{ even}, \tag{88}$$

$$\alpha_3\alpha_4 = -4k_3k_4\cos^2\left(\frac{\alpha_m-\alpha_n}{2}\right), \qquad m-n \text{ odd}, \tag{89}$$

where

$$k_3 = \frac{nv_1[(m^2+n^2)(1-\kappa v_0^2) + 2mn(1+\kappa v_0^2)]}{2(m-n)[(m+n)^2(1-\kappa v_0^2) + (4mnv_0^2)]}, \tag{90}$$

$$k_4 = \frac{mv_1[(m^2+n^2)(1-\kappa v_0^2) + 2mn(1+\kappa v_0^2)]}{2(m-n)[(m+n)^2(1-\kappa v_0^2) + (4mnv_0^2)]}, \tag{91}$$

Since $k_3k_4 > 0$ always for speeds below the critical speed, $v_0 \leqslant 1/\sqrt{\kappa}$, from equation (88) and (89) we conclude that $\alpha_3\alpha_4 < 0$ always. Hence, the eigenvalues are always imaginary. Therefore, no instabilities are detected for combination resonances of difference type up to $O(\varepsilon)$. This result is in agreement with the one given in Mockenstrum *et al*. [13] for similar problem. Note that in that analysis, only the first and second modes are considered whereas in our analysis, resonances of any arbitrary two modes are taken into account.

5. NUMERICAL EXAMPLES

In this section, we give numerical results for two examples of technological importance. We investigate the stability of a band-saw and a threadline. Since the stability boundaries are given analytically in the previous section, any other problem of interest can be examined in a straightforward manner. Before concluding the section, we also discuss results of reference [11] and make some comparisons.

5.3. BAND-SAW

Numerical data are given for a commercial band-saw in Table 1, taken from reference [26]. The wave velocity is $\sqrt{P_0/\rho A} = 90{\cdot}5327$ m/s. For an average speed of 50 m/s, the non-dimensional velocity is $v_0 = 0{\cdot}5523$. For $v_0 = 0{\cdot}5523$ and $\kappa = 0{\cdot}21$, the principal parametric instabilities ($\Omega \simeq 2\omega_n$) are plotted in Figure 1 for the first three natural frequencies. The instability region for the second natural frequency is so narrow that it appears as a line. For the same v_0 and κ values, the combination resonances of sum are plotted in Figure 2 for $\Omega \simeq \omega_1 + \omega_2$, $\omega_1 + \omega_3$, $\omega_2 + \omega_3$. The instability region is relatively large for $\Omega \simeq \omega_1 + \omega_3$ compared to other frequencies. Note that the frequencies are non-dimensional in all plots. A change to dimensional quantities are achieved through equation (7), where Ω^* is the dimensional frequency. The lowest dimensional frequencies around which instabilities occur are as follows:

principal parametric resonances, $\Omega^* \simeq 803, 1606, 2409, \ldots$ (rad/s);
combination resonances of sum, $\Omega^* \simeq 1204, 1606, 2007, \ldots$ (rad/s).

The above frequencies are very high for current operating mechanisms. However, for higher transport velocities v_0, the critical frequencies decrease.

TABLE 1

Parameter values for commercial band-saw [26]

Parameter	Value	Unit
P_0	25 900	N
ρ	7800	kg/m^3
A	$4{\cdot}0513 \times 10^{-4}$	m^2
κ	0·21	—
L	0·5953	m

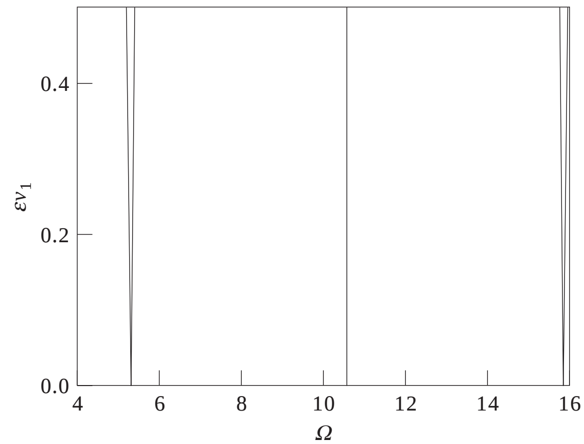

Figure 1. Stable and unstable regions for principal parametric resonances for a commercial band-saw: $\kappa = 0{\cdot}21$, $v_0 = 0{\cdot}5523$, $\Omega \simeq 2\omega_1$, $2\omega_2$, $2\omega_3$.

5.2. THREADLINE

The numerical data for this case are given in Table 2. The wave velocity is calculated to be 31·6228 m/s. For an average speed of 5 m/s, the non-dimensional velocity is 0·1581. For $v_0 = 0{\cdot}1581$ and $\kappa = 1$, the principal parametric stable-unstable regions are plotted in Figure 3 for the first three natural frequencies using equation (59). For the same v_0 and κ values, the combination resonances of sum are plotted in Figure 4. Conversion to dimensional frequencies yields the following:

principal parametric resonances, $\Omega^* \simeq 194, 388, 582, \ldots$ (rad/s);

combination resonances of sum, $\Omega^* \simeq 291, 388, 484, \ldots$ (rad/s).

Note that critical frequencies are much lower for this case, compared to the band-saw problem.

5.3. A NOTE ON THE BAND-SAW PROBLEM OF REFERENCE [11]

Reference [11] presents numerical results for stability of an accelerating string. The problem investigated is a special case of our problem where the mean velocity is zero. The

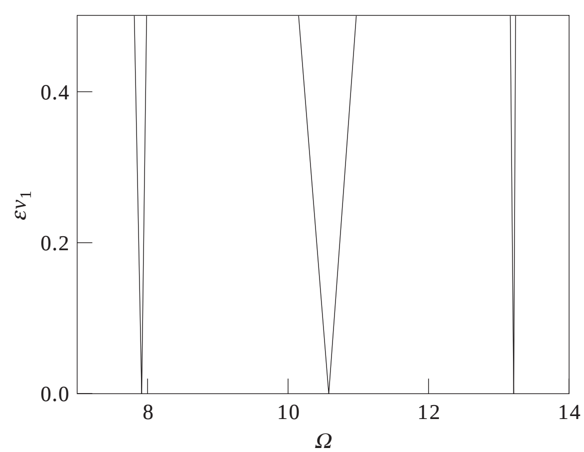

Figure 2. Stable and unstable regions for combination resonances of sum for a commercial band-saw: $\kappa = 0{\cdot}21$, $v_0 = 0{\cdot}5523$, $\Omega \simeq \omega_1 + \omega_2$, $\omega_1 + \omega_3$, $\omega_2 + \omega_3$.

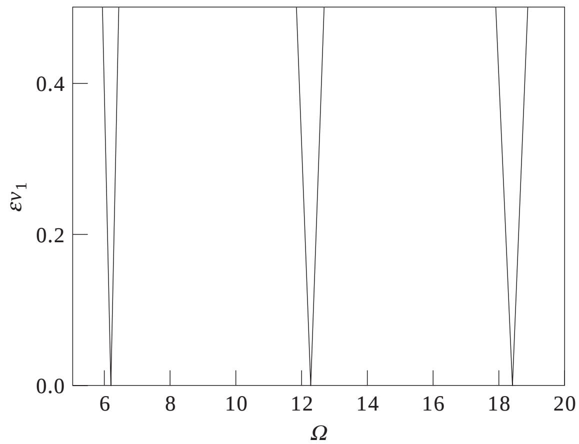

Figure 3. Stable and unstable regions for principal parametric resonances for a threadline: $\kappa = 1$, $v_0 = 0{\cdot}1581$, $\Omega \simeq 2\omega_1,\ 2\omega_2,\ 2\omega_3$.

amplitude of fluctuations, εv_1 in our notation, is equivalent to v_0 in reference [11]. Stationary string eigenfunctions are used in the discretization process. For vanishing mean velocity, our eigenfunctions also reduce to the stationary string eigenfunctions. Note that our analytical results are valid for small fluctuation amplitudes whereas numerical data are presented in reference [11] for large amplitudes also.

In our analysis, due to the excellent convergence properties of travelling string eigenfunctions, we consider one-term and two-term approximations. However, our one-term and two-term approximations are not the first and second mode approximations considered in reference [11]. Rather, they are any arbitrary one or two modes that are assumed to be dominant in the response of the system. When the mean velocity is zero, the eigenfunctions reduce to the stationary string eigenfunctions, which are well known for their poor convergence characteristics. However, for small amplitudes of fluctuations, the convergence might be better.

Our results can be compared to the numerical results for one-term and two-term approximations in reference [11]. To distinguish from this text, all references to the

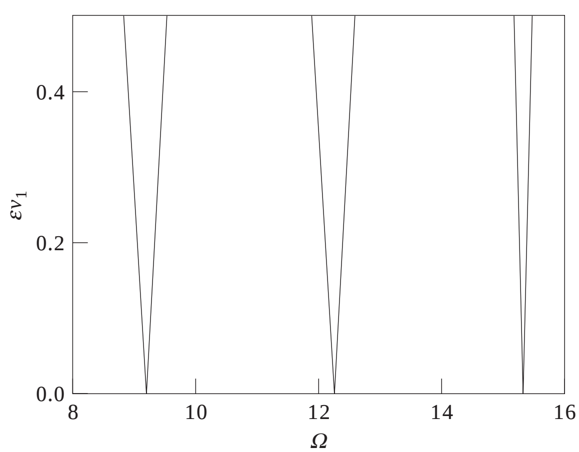

Figure 4. Stable and unstable regions for combination resonances for sum for a threadline: $\kappa = 1$, $v_0 = 0{\cdot}1581$, $\Omega \simeq \omega_1 + \omega_2,\ \omega_1 + \omega_3,\ \omega_2 + \omega_3$.

TABLE 2

Parameter values for a threadline

Parameter	Value	Unit
P_0	10	N
ρA	0·01	kg/m
κ	1	—
L	1	m

equations, tables and figures in reference [11] will be given with a "P" prefix. For the one-term approximation, we find that the instability regions close up to $O(\varepsilon)$ (see equation (58)). Therefore, for $v_0 = 0$, no instabilities are detected. This result is in agreement with reference [11]. For the one-term approximation (the first mode actually), equation (P17) is obtained using discretization. The v^2 term is of $O(\varepsilon^2)$, and hence should be neglected. The remaining terms produce periodic bounded solutions. If we carry out analysis to $O(\varepsilon^2)$, some of the weaker instabilities might also be detected as shown in Figure P2.

For two-term approximations, the stable and unstable points are shown in Figure P5. In the frequency range of $0 \leqslant \omega_0 \leqslant 50$, no instabilities were detected for small amplitudes of velocity. This result is in agreement with what we found here. We expect the lowest instability at $\Omega \simeq \omega_1 + \omega_2 \simeq 3\pi$ or $\Omega \simeq 1113$ in our analysis. The numerical calculation presented for this point reveals that there is indeed instability at that point (see Table P4).

A number of comments follows for section P4.3. In that section, numerical results of reference [11] are compared to the analytical results of reference [27]. To make comparisons, equation (P26) is cast into equation (P27). The choice of the perturbation parameter $\varepsilon = v_0/L$ is not good since it is dimensional. However, by choosing v_0 very small ($v_0 = 0{\cdot}01$), the pitfalls of this choice is avoided in some sense. The definition of $\mathbf{B}(0)$ in equation (P30) should not contain $\kappa v_0^2/L^2$ terms since they are of $O(\varepsilon^2)$. By the same token, the second matrix should be eliminated completely from equation (P31). This last elimination requires that $s = 1$ only (see reference [27]). Therefore from equation (P32), the critical points are the combination resonances of sum and difference in agreement with our analysis. We further found that combination resonances of difference would not yield to instability.

Natural frequencies in equation (P33) should be written as

$$\Omega_i^2 = \frac{P_0}{\rho A}\frac{\mathrm{i}^2\pi^2}{L^2}.$$

There is a misprint involving squaring in equation (P33) and $\kappa v_0^2/L^2$ should be neglected.

The numerical data of Tables P4 and P5, as well as all other data presented in the tables and figures of reference [11] are accurate. However, the determination of critical points where instability may occur using the analytical approach contains excess data in Tables P4 and P5. Table P4 should, for example, contain only the frequencies of 371 and 1113. Data 371 corresponds to $\Omega_2 - \Omega_1$ and is found to be stable numerically. This is in agreement with our analysis, since we also predict the difference type combination resonances do not lead to instabilities. Data 1113, as mentioned before, corresponds to the additive type combination resonances of the first and second modes which is unstable numerically. This result is also in agreement with our analytical approach. Consequently, the results presented here are in agreement with those in reference [11], but more complete.

6. CONCLUDING REMARKS

An analytical approach to the stability of a string travelling with a time dependent velocity is presented. The velocity function is assumed to be harmonically varying about a constant mean velocity. The influence of the small fluctuations on the stability of the system is investigated. The method of multiple scales (a perturbation technique) is used to calculate the boundaries separating stable and unstable regions. Two different approaches are presented in the analysis. In the first approach, the equations are discretized first and then the method of multiple scales is applied to the resulting equations. In the second approach, the method of multiple scales is directly applied to the partial differential system. Although both methods yield identical results, the latter approach is more straightforward. Principal parametric resonances and combination resonances of sum and difference type are considered in the analysis. It is found that instabilities arise when $\Omega \simeq 2\omega_n$ for a one-mode approximation and $\Omega \simeq \omega_m + \omega_n$ for a two-mode approximations. When $\Omega \simeq 0$ for a one mode approximation and $\Omega \simeq \omega_m - \omega_n$ for a two-mode approximations, however, no instabilities are detected up to the first order of approximation. The analysis is applied to a band-saw and a threadline problem. Boundaries separating stable and unstable regions are plotted for both cases. It is found that these types of instabilities do not occur unless the fluctuation frequencies are considerably larger compared to those of typical working mechanisms.

If the analysis is carried to higher orders of perturbations, the weaker resonances would appear at relatively low frequencies. However, these secondary resonances would not be as effective as the primary ones.

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