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CONVECTED ACOUSTIC WAVE MOTION ALONG A CAPILLARY DUCT WITH AN AXIAL TEMPERATURE GRADIENT

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An analytical solution is presented of the problem of isentropic acoustic wave motion in a circular capillary tube in the presence of both mean flow and a background axial temperature gradient. The main application area is the study of the acoustic effects of catalytic converters. The solution makes use of a series expansion and is valid for only low Mach numbers of mean flow and small relative changes in the background temperature. It is shown that, to the order of approximation used, the solution reduces to that of previous work in both the wide-tube limit and in the limit of zero temperature gradient. A temperature gradient of the magnitude relevant to catalytic converters is shown to have a very marked affect upon the wave attenuation.

1. INTRODUCTION

Catalytic converters are now commonplace components of the exhaust systems of modern road vehicles, with the prime purpose of reducing the levels of noxious gases emitted into the atmosphere. In addition, they have a small but noticeable effect upon the acoustic performance of the exhaust system and must therefore be included in any modelling of the entire system. Conventionally, a catalytic converter consists of a ceramic brick with a uniform set of small, parallel, open pores of square cross-section running along the length of the brick. A more recent alternative is to use a metallic rather than a ceramic substrate, for which the cross-section of the pores is not square. In either case, a washcoat is applied to the surface of the pores and then the catalyst is spread over the surface of the washcoat. The catalyst creates an exothermic reaction within the exhaust gas flowing through the pores as oxygen is stripped from the nitrous oxides and used to completely burn the carbon dioxide and the hydrocarbons.

From an acoustic viewpoint, therefore, the catalytic converter is an element within which the acoustic pulsations from the exhaust ports are convected by a mean flow through capillary tubes and are subjected to large increases in mean temperature resulting from the exothermic reaction. Experiments have shown that a temperature increase, of the order of 100°C, occurs within the first 20 mm of axial distance along the capillary pore, which typically has a total length of 150 mm. The temperature after the first 20 mm is virtually constant, as is the temperature throughout any further bricks downstream of the first. Thus the temperature gradient is very localized, but of such a magnitude that its effect may be significant.

Analytical solutions for the acoustic wave propagation through cylindrical capillary tubes in which the background medium is stationary and isothermal have been obtained

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for both isentropic [1] and non-isentropic [2] cases. Variational/numerical solutions in which the acoustic waves are convected by an isothermal, fully developed laminar mean flow, for both isentropic and non-isentropic cases, have been developed more recently [3–6]. In particular it has been shown that results for tubes of circular cross-section are a close approximation to those for tubes of other cross-sections with the same hydraulic mean radius, and that the effects of the radial velocity component are small.

This paper is concerned with convected acoustic wave motion through a capillary tube of circular cross-section in which the mean flow is laminar and fully-developed, and in which there is a large, constant gradient of background temperature. Radial velocity components are ignored. An analytical solution is developed for low-order expansions in both the Mach number and the temperature change parameter. The latter is defined as the temperature increase divided by twice the mean temperature. A catalytic converter typically operates at 1000°K with a temperature increase of 100°K over 20 mm. Thus although the temperature gradient is very large, the temperature change parameter is only 0·05. In contrast the mean flow can have a Mach number as high as 0·3 when the engine operates at maximum speed. It is shown in the paper, by comparison with numerical results for the isothermal case, that the small-order expansion remains reasonably accurate up to a Mach number of 0·1. It is deduced that temperature gradient effects within catalytic converters are well represented by the low-order solution but that the restriction on Mach number is too severe for some practical applications.

In the analytical solution developed in this paper use is made of the assumption that acoustic fluctuations occur isentropically. The heat release from the exothermic reaction in the catalytic converter is assumed to be steady-state and therefore does not influence this approximation, but viscous effects in the capillary pore clearly invalidate the assumption. Hence this analytical solution can be regarded only as a first approximation to the real problem. In the absence of mean flow and temperature gradient, the differences in acoustic propagation through capillary tubes for isentropic and non-isentropic cases can be seen by comparison of the results of Kerris [1] and Zwikker and Kosten [2]. Similar comparison with mean flow effects included is provided by the variational solution of Peat [3], while results with a temperature gradient but no mean flow have been obtained from the numerical solution of Peat and Kirby [7]. In all cases the attenuation is somewhat under-predicted by the use of the isentropic assumption.

A solution of the non-isentropic equations, especially one which is valid up to a steady flow Mach number of 0·3, is almost certain to involve an iterative numerical procedure. The analytical solution developed here will be useful both as a starting point for the iteration and for validation in the isentropic, low Mach number limit.

2. GOVERNING EQUATIONS

The equations of continuity, momentum and energy for axisymmetric flow of a homogeneous gas through a capillary duct, when the small radial velocity component is neglected, can be written as [3]

$$\partial\tilde{\rho}/\partial t + \tilde{u}\,\partial\tilde{\rho}/\partial x + \tilde{\rho}\,\partial\tilde{u}/\partial x = 0, \qquad \tilde{\rho}(\partial\tilde{u}/\partial t + \tilde{u}\,\partial\tilde{u}/\partial x) = -\partial\tilde{p}/\partial x + \mu/r\,\partial/\partial r(r\,\partial\tilde{u}/\partial r),$$
$$\partial\tilde{p}/\partial r = 0, \tag{1–3}$$

where $\tilde{\rho}$, $\tilde{p}$, $\tilde{u}$, and μ denote the density, pressure, axial velocity and absolute viscosity of the gas. x and r are the axial and radial co-ordinates respectively and is the time. The gas is assumed to obey the equation of state

$$\tilde{p} = \tilde{\rho} R_0 \tilde{T}, \tag{4}$$

where $\tilde{T}$ is the gas temperature and R_0 is the gas constant. It is assumed that the flow through the capillary duct is a superposition of a laminar steady flow and a small harmonic acoustic disturbance of radian frequency ω. A linear change in the steady-state temperature T_S is assumed to occur over a length $2L$ of duct, such that

$$T_S(\xi) = \bar{T}(1 + \tau\xi), \qquad -1 \leqslant \xi \leqslant 1, \tag{5}$$

where $\xi = x/L$, τ is a constant, the temperature change parameter, and an overbar is used to denote the value of a steady-state variable at the centre of the duct, $\xi = 0$. The fluid variables are now expanded as the sum of a steady state component and a small harmonic acoustic fluctuation, in the form

$$\tilde{\rho} = \bar{\rho}[(1 - \tau\xi) + \alpha\rho(\xi)\,\mathrm{e}^{\mathrm{i}\omega t}], \qquad \tilde{u} = \bar{a}[\bar{M}(1 + \tau\xi)f(\eta) + \alpha u(\xi, \eta)\,\mathrm{e}^{\mathrm{i}\omega t}], \tag{6, 7}$$

$$\tilde{p} = \bar{p}[1 + \bar{M}^2 g(\xi) + \alpha p(\xi)\,\mathrm{e}^{\mathrm{i}\omega t}], \qquad \tilde{T} = \bar{T}[(1 + \tau\xi) + \alpha T(\xi)\,\mathrm{e}^{\mathrm{i}\omega t}], \tag{8, 9}$$

where $\alpha \ll 1$, M is the steady flow Mach number and a is the speed of sound. The acoustic variables ρ, u, p and T are non-dimensional, as are the steady flow variables $g(\xi)$ and $f(\eta)$. The normalised radius $\eta = r/R$, where R is the radius of the capillary duct.

2.1. STEADY FLOW SOLUTION

It remains to be shown that the steady flow terms assumed in equations (6)–(9) are consistent with the governing equations (1)–(4). The exothermic reaction in the pore of the catalytic converter results in a steady rate of heat release, with an assumed distribution which gives rise to the linear change in the steady state temperature; thus the steady state form of the energy equation is redundant. The radial momentum equation (3) is clearly satisfied and the continuity equation (1) reduces to

$$(\mathrm{d}/\mathrm{d}\xi)(1 - \tau^2\xi^2) = 0, \tag{10}$$

which is satisfied if terms of $O(\tau^2) \ll 1$. The equation of state becomes

$$1 + \bar{M}^2 g(\xi) = 1 - \tau^2\xi^2 \tag{11}$$

and is satisfied only if $O(\tau^2) \ll 1$ and $O(\bar{M}^2 g(\xi)) \ll 1$. Finally the axial momentum equation reduces to

$$\tau f^2 = -\frac{1}{\gamma}\frac{\mathrm{d}g}{\mathrm{d}\xi} + \frac{(1 + \tau\xi)(1 + \tau\xi/2)}{R_e(R/L)}\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right), \tag{12}$$

where $R_e = \bar{\rho}\bar{M}\bar{a}R/\bar{\mu}$ is the mean Reynolds number based upon the duct radius. With reference to the properties of gases, it is assumed that the variation of absolute viscosity with temperature is of the form $\mu = \bar{\mu}(1 + \tau\xi/2)$ and that the ratio of specific heats is essentially invariant with temperature. Consider a series solution of equation (12) in terms of the temperature change parameter τ, with

$$f(\eta) = f_0(\eta) + \tau f_1(\eta) + \tau^2 f_2(\eta) + \cdots, \qquad g(\xi) = g_0(\xi) + \tau g_1(\xi) + \tau^2 g_2(\xi) + \cdots. \tag{13a, b}$$

Equation (12), written to zeroth order in τ, becomes

$$(R_e(R/L)/\gamma)(\mathrm{d}g_0/\mathrm{d}\xi) = (1/\eta)(\mathrm{d}/\mathrm{d}\eta)(\eta\,\mathrm{d}f_0/\mathrm{d}\eta) = \text{constant}. \tag{14}$$

The solution to equation (14) which satisfies the no-slip condition at the wall is simply that of Poiseuille flow, namely

$$f_0 = 2(1-\eta^2), \qquad \mathrm{d}g_0/\mathrm{d}\xi = -8\gamma[R_e(R/L)]. \tag{15a, b}$$

Solutions for $f_1(\eta)$ and $g_1(\xi)$ follow from the first order τ terms of equation (12) and the zeroth order solutions of equation (15), but it transpires that these are not relevant to the convected acoustic solutions to first order in $\bar{M}$ and τ and so will not be developed here. Finally, it follows from equation (15b) that $f_1(\eta)$ and $g_1(\xi)$ is of $O(1)$ for the size of capillary duct and operating conditions pertinent to catalytic converters, such that the steady flow solution is self-consistent to first order in $\bar{M}$ and τ.

2.2. ACOUSTIC EQUATIONS

The linearized acoustic equations follow from substitution of equations (6)–(9) into equations (1)–(3), with retention of only first-order terms in α, to give

$$\mathrm{i}k\rho + \bar{M}f(\eta)\,\partial\rho/\partial\xi - \tau u + (1-\tau\xi)\,\partial u/\partial\xi = 0, \tag{16}$$

$$\mathrm{i}k(1-\tau\xi)u + \bar{M}f(\eta)\,\partial u/\partial\xi = -\frac{1}{\gamma}\frac{\partial p}{\partial\xi} + \frac{k(1+\tau\xi/2)}{s^2}\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{\partial\eta}\right), \qquad \partial p/\partial\eta = 0, \tag{17, 18}$$

where the mean non-dimensional wavenumber $k=\omega L/\bar{a}$, and the mean shear wavenumber $s = R\sqrt{\bar{\rho}\omega/\bar{\mu}}$. In addition it is assumed that the acoustic disturbances are isentropic such that

$$p/\rho = \gamma(1+\tau\xi). \tag{19}$$

Solutions are sought for the case when both τ and $\bar{M}$ are small. Let $\tau = C_\tau\epsilon$ and $\bar{M} = C_M\epsilon$ where terms of $O[\epsilon^2]\ll 1$. The acoustic variables are written in series expansions of ϵ as

$$\rho = \mathrm{e}^{k\Gamma\xi}\{\rho_0 + \epsilon\rho_1(\xi) + O(\epsilon^2)\}, \tag{20}$$

$$u = \mathrm{e}^{k\Gamma\xi}\{u_0(\eta) + \epsilon u_1(\xi,\eta) + O(\epsilon^2)\}, \qquad p = \mathrm{e}^{k\Gamma\xi}\{p_0 + \epsilon p_1(\xi) + O(\epsilon^2)\}, \tag{21, 22}$$

Γ being a dimensionless axial wavenumber.

2.2.1. *Acoustic equations of zeroth order in* ϵ

When the series expansions of equations (20)–(22) are substituted into equations (16), (17) and (19) and only the terms of zeroth order in ϵ are retained, then the continuity and axial momentum equations become

$$U = -i/\Gamma^2 \qquad \text{and} \qquad (1/s^2\eta)(\mathrm{d}/\mathrm{d}\eta)(\eta\,\mathrm{d}U/\mathrm{d}\eta) - \mathrm{i}U = 1, \tag{23, 24}$$

respectively, where equation (19) has been used to eliminate the density and $(u_0/p_0) = (\Gamma/\gamma)U$. The reduced form of equation (24) is Bessel's equation and it is readily observed that a particular solution to the complete equation is $U=\mathrm{i}$, and hence it follows that the full solution which satisfies the no-slip boundary condition, $U(1)=0$, is

$$U = \mathrm{i}[1 - \mathrm{J}_0(\mathrm{i}^{3/2}s\eta)/\mathrm{J}_0(\mathrm{i}^{3/2}s)]. \tag{25}$$

The continuity equation (23) can only be satisfied in the integral sense. The notation $\langle f(\eta)\rangle$ is used for the cross-sectionally averaged value of a general function $f(\eta)$ such that

$$\langle U\rangle = 2\mathrm{i}\int_0^1 \eta\left[1 - \frac{\mathrm{J}_0(\mathrm{i}^{3/2}s\eta)}{\mathrm{J}_0(\mathrm{i}^{3/2}s)}\right]\mathrm{d}\eta = \mathrm{i}\left[1 - \frac{2\mathrm{J}_1(\mathrm{i}^{3/2}s)}{\mathrm{i}^{3/2}s\mathrm{J}_0(\mathrm{i}^{3/2}s)}\right] = -\mathrm{i}\,\frac{\mathrm{J}_2(\mathrm{i}^{3/2}s)}{\mathrm{J}_0(\mathrm{i}^{3/2}s)}, \tag{26}$$

and hence equation (23) is satisfied when

$$\Gamma^2 = \mathrm{J}_0(\mathrm{i}^{3/2}s)/\mathrm{J}_2(\mathrm{i}^{3/2}s). \tag{27}$$

Thus the zeroth order solution is the isentropic solution of Kerris [1], as one would expect.

2.2.1. *Acoustic equations of first order in* ϵ

When the series expansions of equations (20)–(22) are substituted into equations (16), (17) and (19) and only the terms of first order in ϵ are retained, then the continuity and axial momentum equations become

$$\frac{\mathrm{i}k}{\gamma}\left[\frac{p_1}{p_0} - C_\tau \xi\right] + \left[k\Gamma + \frac{\partial}{\partial \xi}\right]\left(\frac{u_1}{p_0}\right) = C_\tau(1 + k\Gamma\xi)\left(\frac{u_0}{p_0}\right) - \frac{C_M f_0 k\Gamma}{\gamma} \tag{28}$$

and

$$\frac{1}{s^2\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{u_1}{p_0}\right)\right] - \mathrm{i}\left(\frac{u_1}{p_0}\right) = \frac{1}{\gamma}\left[\Gamma + \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}\xi}\right]\left(\frac{p_1}{p_0}\right) + [C_M f_0 \Gamma - \mathrm{i}C_\tau \xi]\left(\frac{u_0}{p_0}\right) - C_\tau\frac{\xi}{2}\left[\frac{\Gamma}{\gamma} + \mathrm{i}\left(\frac{u_0}{p_0}\right)\right], \tag{29}$$

respectively, where once again equation (19) has been used to eliminate the density. Solutions of the form

$$p_1/p_0 = A + B\xi + C\xi^2, \qquad u_1/p_0 = [U_A(\eta) + \xi U_B(\eta) + \xi^2 U_C(\eta)](\Gamma/\gamma) \tag{30a, b}$$

are sought. Substitution of these expressions into equations (28) and (29), followed by separation of the zeroth, first and second order terms in ξ, leads to integrated continuity equations of the form

$$\langle U_A\rangle + \langle U_B\rangle/(k\Gamma) + C_M/\Gamma - C_\tau/\langle U\rangle/(k\Gamma) = -\mathrm{i}A/\Gamma^2, \tag{31}$$

$$\langle U_B\rangle + 2\langle U_C\rangle/(k\Gamma) - C_\tau\langle U\rangle = -\mathrm{i}(B - C_\tau)/\Gamma^2, \qquad \langle U_C\rangle = -\mathrm{i}C/\Gamma^2, \tag{32, 33}$$

and momentum equations of the form

$$\frac{1}{s^2\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}U_A}{\mathrm{d}\eta}\right) - \mathrm{i}U_A = A + B/(k\Gamma) + 2C_M\Gamma(1-\eta^2)U, \tag{34}$$

$$\frac{1}{s^2\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}U_B}{\mathrm{d}\eta}\right) - \mathrm{i}U_B = B + \frac{2C}{k\Gamma} - \frac{C_\tau}{2} - \frac{3}{2}iC_\tau U, \qquad \frac{1}{s^2\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}U_C}{\mathrm{d}\eta}\right) - \mathrm{i}U_C = C. \tag{35, 36}$$

From the similarity of equations (33) and (36) with equations (23) and (24) respectively, and the fact that all velocity components are subject to the same no-slip boundary condition at the wall, it follows from equation (25) that

$$U_C = CU = \mathrm{i}C[1 - \mathrm{J}_0(\mathrm{i}^{3/2}s\eta)/\mathrm{J}_0(\mathrm{i}^{3/2}s)]. \tag{37}$$

The reduced forms of equations (34) and (35) are simply Bessel's equation. The general solution of these equations then follows if one can determine a particular solution of the complete equations. It may be verified that complete solutions of equations (34) and (35) which satisfy the no-slip boundary condition are

$$U_A = [A + B/(k\Gamma) - 8C_M\Gamma/s^2]U - 2C_M\Gamma(1-\eta^2) + C_M u_M \tag{38}$$

and

$$U_B = [B + 2C/(k\Gamma) + C_\tau]U + C_\tau u_\tau, \tag{39}$$

respectively, where

$$u_M = \frac{\Gamma}{3\mathrm{J}_0(\mathrm{i}^{3/2}s)}\left\{\left(3 + \frac{2\mathrm{i}}{s^2} - \eta^2\right)\mathrm{i}^{3/2}s\eta\,\mathrm{J}_1(\mathrm{i}^{3/2}s\eta) - \eta^2\mathrm{J}_0(\mathrm{i}^{3/2}s\eta)\right.$$
$$\left. - \frac{\mathrm{J}_0(\mathrm{i}^{3/2}s\eta)}{\mathrm{J}_0(\mathrm{i}^{3/2}s)}\left[\left(2 + \frac{2\mathrm{i}}{s^2}\right)\mathrm{i}^{3/2}s\,\mathrm{J}_1(\mathrm{i}^{3/2}s) - \mathrm{J}_0(\mathrm{i}^{3/2}s)\right]\right\} \tag{40}$$

and

$$u_\tau = \frac{3s}{4\mathrm{i}^{3/2}\mathrm{J}_0(\mathrm{i}^{3/2}s)}\left[\frac{\mathrm{J}_1(\mathrm{i}^{3/2}s)}{\mathrm{J}_0(\mathrm{i}^{3/2}s)}\,\mathrm{J}_0(\mathrm{i}^{3/2}s\eta) - \eta\,\mathrm{J}_1(\mathrm{i}^{3/2}s\eta)\right]. \tag{41}$$

It follows from equations (23) and (38)–(41) that

$$\langle U_A\rangle = -\mathrm{i}[A + B/(k\Gamma) - 8C_M\Gamma/s^2]/\Gamma^2 + C_M(\langle u_M\rangle - \Gamma) \tag{42}$$

and

$$\langle U_B\rangle = -\mathrm{i}[B + 2C/(k\Gamma) + C_\tau]/\Gamma^2 + C_\tau\langle u_\tau\rangle \tag{43}$$

where

$$\langle u_M\rangle = (\Gamma/3)[4(1 + 6\mathrm{i}/s^2)/\Gamma^2 + (\mathrm{i}s^2 - 1)(1 + 1/\Gamma^2)^2 - 2(1 + 1/\Gamma^2)] \tag{44}$$

and

$$\langle u_\tau\rangle = (3/2)[-\mathrm{i}/\Gamma^2 + s^2(1 + 1/\Gamma^2)^2/4]. \tag{45}$$

The integrated continuity equations (31) and (32) can then be used to find

$$B = -(\mathrm{i}k\Gamma^2C_M/2)[1 + 8\mathrm{i}/s^2 - \Gamma^2 + \Gamma\langle u_M\rangle] + (C_\tau/4)[1 - \mathrm{i}\Gamma^2\langle u_\tau\rangle] \tag{46}$$

$$C = -(k\Gamma C_\tau/4)[1 + \mathrm{i}\Gamma^2\langle u_\tau\rangle] \tag{47}$$

and A is indeterminate. It may be noted from equations (22) and (30a) that A serves only to adjust the absolute value of acoustic pressure at $\xi = 0$ and may be taken to be zero if p_0 is regarded as $p|_{\xi=0}$ in all cases.

2.2.1. *Complete solution of the acoustic equations*

The solution for the acoustic pressure, complete to first order in both $\bar{M}$, and τ, follows from equations (22), (30a), (46) and (47) and is

$$p = p_0\,\mathrm{e}^{k\Gamma\xi}\left\{1 + \xi\left[\frac{\tau}{4}(1 - \mathrm{i}\Gamma^2\langle u_\tau\rangle) - \frac{\mathrm{i}k\Gamma^2\bar{M}}{2}\left(1 + \frac{8\mathrm{i}}{s^2} - \Gamma^2 + \Gamma\langle u_M\rangle\right)\right]\right.$$

$$\left. - \xi^2\frac{k\Gamma\tau}{4}(1 + \mathrm{i}\Gamma^2\langle u_\tau\rangle)\right\}. \tag{48}$$

3. REDUCED AND GENERAL SOLUTIONS

It is shown below that under various simplifying conditions the general solution outlined above reduces to established results.

3.1. WIDE TUBE LIMIT, $s \to \infty$

It follows from equation (27) that

$$\lim_{s\to\infty} \Gamma^2 = \lim_{s\to\infty} [\mathrm{J}_0\mathrm{i}^{3/2}s/\mathrm{J}_2\mathrm{i}^{3/2}s] = -1 \tag{49}$$

and hence $\Gamma = \pm\mathrm{i}$. Thus in the wide-tube limit the zeroth order solution in $\bar{M}$ and τ, that is in the absence of mean flow and temperature gradient effects, follows from equation (48) and is $p/p_0 = \exp(\pm\mathrm{i}k\xi)$, entirely as one would expect.

Since $\Gamma^2 = -1$ in the wide-tube limit, $s \to \infty$, it follows that $\langle u_M\rangle$ and $\langle u_\tau\rangle$are indeterminable from equations (44) and (45). However equations (23) and (24) both imply that $\lim_{s\to\infty}\langle U\rangle = \mathrm{i}$, hence it follows from equation (35) that $\lim_{s\to\infty}\langle U_B\rangle = \mathrm{i}(\mathrm{B} + 2\mathrm{C}/\mathrm{k}\Gamma + \mathrm{C}_\tau)$ and this expression may be compared with equation (43) to show that $\lim_{s\to\infty}\langle u_\tau\rangle = 0$. It follows from equation (45) that $\lim_{s\to\infty}[s^2(1 + \Gamma^2)^2] = -4\mathrm{i}$ and hence from equation (44) that $\lim_{s\to\infty}\langle u_M\rangle = 0$. Hence in the wide-tube limit, equation (48) reduces to

$$\lim_{s\to\infty}(p/p_0) = \mathrm{e}^{\pm\mathrm{i}k\xi}\{1 + \xi(\tau/4 + \mathrm{i}k\bar{M}) \mp \xi^2(ik\tau/4)\}. \tag{50}$$

In the absence of a temperature gradient, such that $\tau = 0$, this equation reduces to the correct solution for convected plane wave motion, to the order for which it is valid, since

$$\lim_{s\to\infty}(p/p_0) = \mathrm{e}^{\pm\mathrm{i}k\xi}(1 + \mathrm{i}k\bar{M}\xi) + O[\bar{M}^2] = \mathrm{e}^{\mathrm{i}k\xi(\pm 1 + \bar{M})} + O[\bar{M}^2] = \mathrm{e}^{\pm\mathrm{i}k\xi/(1\mp\bar{M})} + O[\bar{M}^2]. \tag{51}$$

In a similar manner, it follows that the more general solution (50) can be written as

$$\lim_{s\to\infty}(p/p_0) = \mathrm{e}^{\pm\mathrm{i}k\xi/(1\mp\bar{M})}(1 + \xi\tau/4 \mp \mathrm{i}\xi^2 k\tau/4) + O[\bar{M}^2, \bar{M}\tau, \tau^2] \tag{52}$$

which, to the relevant order, is identical to the solution for planar, isentropic acoustic waves in a duct with mean flow and a linear temperature gradient, Peat [8]. In order to make this comparison it should be noted that in reference [8] a negative temperature gradient was considered such that τ was of the opposite sign to that used here, and that A serves only to adjust the absolute acoustic pressure and has been taken to be zero here.

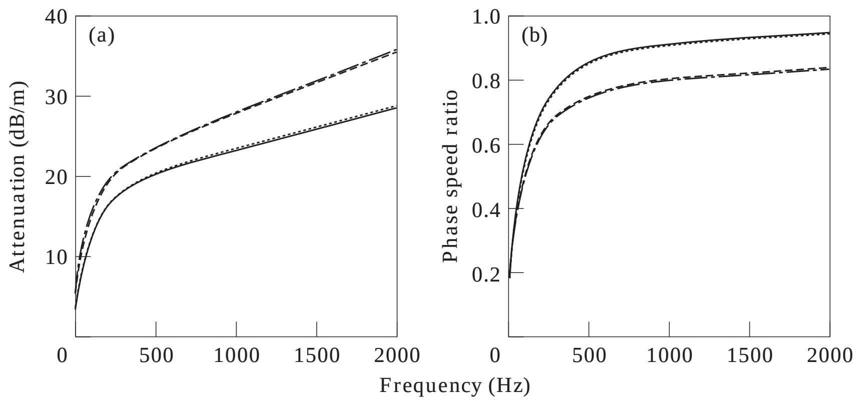

Figure 1. Comparison of analytical and numerical results for $\bar{M} = 0{\cdot}05$, $\tau = 0$. Forward going waves: ——, analytical; ······, numerical. Backward going waves: – – – –, analytical; — – — –, numerical.

3.2. CAPILLARY TUBE, ZERO TEMPERATURE GRADIENT

When $\tau = 0$ it follows from equation (48) that $p = p_0 \exp(\hat{\Gamma} x)$ where

$$\hat{\Gamma} = (\omega\Gamma/\bar{a})\left[1 - \frac{\mathrm{i}\Gamma\bar{M}}{2}\left(1 + \frac{8\mathrm{i}}{s^2} - \Gamma^2 + \Gamma\langle u_M\rangle\right)\right] + O[\bar{M}^2]. \tag{53}$$

Results from this expression can be compared with numerical results for convected, isentropic waves, evaluated by the more general procedure given by Jeong and Ih [6]. The numerical results are valid for all Mach numbers, so long as the mean flow is essentially incompressible, and hence can be used to determine the result of ignoring terms of $O[\bar{M}^2]$ in the present analysis. Figures 1 and 2 give comparisons of analytical and numerical results of attenuation and phase speed ratio for Mach numbers of 0·05 and 0·1 respectively, where *attenuation* $= 8{\cdot}686\,|\mathrm{Re}\{\bar{\Gamma}\}|$ (dB/m) and *phase speed ratio* $= (\omega/\bar{a})/|\mathrm{Im}\,\{\bar{\Gamma}\}|$. It may be observed that the analytical results are very accurate for $\bar{M} = 0{\cdot}05$ and reasonably so for $\bar{M} = 0{\cdot}1$. Since the full analysis involves the same order expansion for both $\bar{M}$ and τ, the accuracy of the analytical results in the presence of mean flow and a temperature gradient may be inferred from these results.

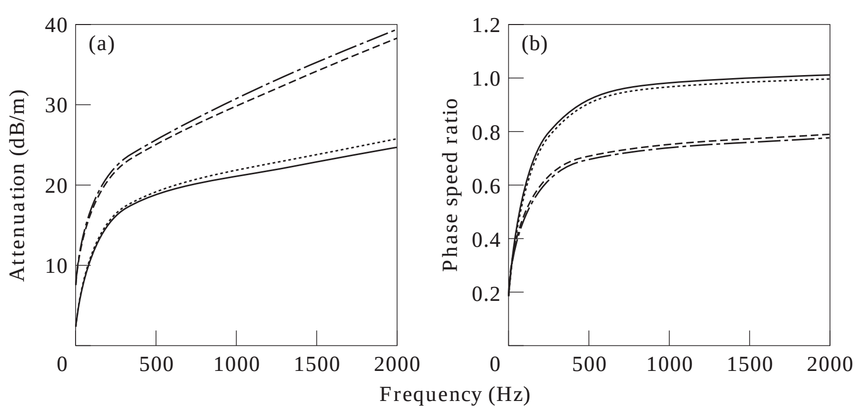

Figure 2. Comparison of analytical and numerical results for $\bar{M} = 0{\cdot}1$, $\tau = 0$. Forward going waves: ——, analytical; ······, numerical. Backward going waves: – – – –, analytical; — – — –, numerical.

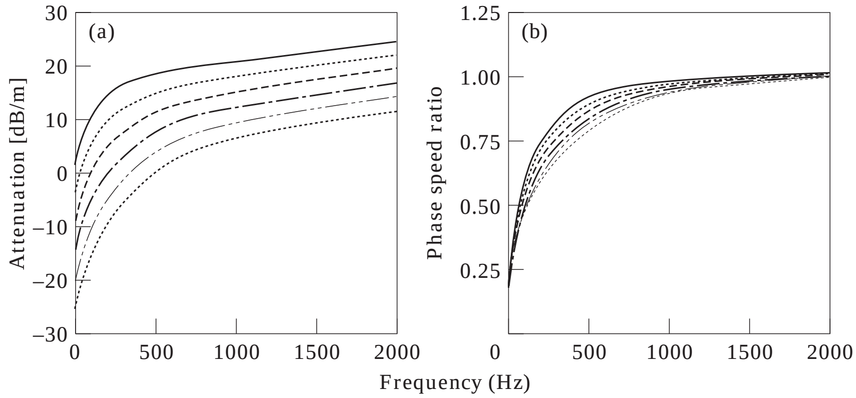

Figure 3. The effect of τ on the the propagation of forward going waves; $\bar{M} = 0{\cdot}1$, $\xi = 0$. ——, $\tau = 0$; ⋯⋯; $\tau = 0{\cdot}01$; – – –, $\tau = 0{\cdot}02$; — — —, $\tau = 0{\cdot}03$; –·–·–·– $\tau = 0{\cdot}04$; ----, $\tau = 0{\cdot}05$.

3.3. CAPILLARY TUBE, GENERAL CASE

In this section results for the general solution, equation (48), are discussed. All of the results given have been evaluated with values for geometry and fluid variables for typical operating conditions of a catalytic converter, namely $R = 0{\cdot}5$ mm, $L = 10$ mm and $\bar{T} = 1000^\circ$K. To aid physical understanding, results are shown in terms of cyclic frequency rather than shear wavenumber. It may be noted that a frequency of 2 kHz corresponds to a shear wavenumber of 5·17 for the particular duct and operating conditions assumed.

In the general case when $\tau \neq 0$, equation (48) can again be written in the form $p = p_0 \exp(\hat{\Gamma} x)$ where now

$$\hat{\Gamma} = (\omega/\Gamma/\bar{a})\left[1 - \frac{\mathrm{i}\Gamma\bar{M}}{2}\left(1 + \frac{8\mathrm{i}}{s^2} - \Gamma^2 + \Gamma\langle u_M\rangle\right) - \xi\,\frac{\tau}{4}(1 + \mathrm{i}\Gamma^2\langle u_\tau\rangle)\right]$$

$$+ \frac{\tau}{4L}(1 - \mathrm{i}\Gamma^2\langle u_\tau\rangle) + O(\bar{M}^2, \bar{M}\tau, \tau^2). \qquad (54)$$

It is observed that a temperature gradient has a two-fold effect upon the attenuation and phase speed. The first is a term with the usual frequency dependence but also an axial dependence, whereas the second is predominantly constant, the only frequency dependence coming through that of Γ, and becomes more dominant the shorter the duct length $2L$ over which the temperature increase occurs. Figures 3 and 4 illustrate the effect of temperature gradient upon attenuation and phase speed for forward-going and backward-going waves respectively, at the central duct position $\xi = 0$, for $\bar{M} = 0{\cdot}1$. The effect upon the attenuation in particular is very marked, with a decrease in attenuation for waves propagating in the direction of the temperature increase and *vice versa*. Indeed at low frequencies there is an increase of wave amplitude with distance, or negative attenuation, presumably as energy is fed into the acoustic wave from the increase in the background temperature and dominates over the viscous dissipation.

Figure 5 shows that there is no observable axial variation in attenuation of both forward and backward-going waves, even in the extreme case of $\tau = 0{\cdot}05$. Results for the phase speed ratio, not shown, also indicate negligible axial variation.

An expression for the axial velocity ratio, $|u/p_0|$, can be found from equations (21), (25), (30b) and (38) to (47). At the duct centre point, $\xi = 0$, and for fixed Mach number $\bar{M}$ and frequency, the results indicate negligible variation of axial velocity ratio with temperature

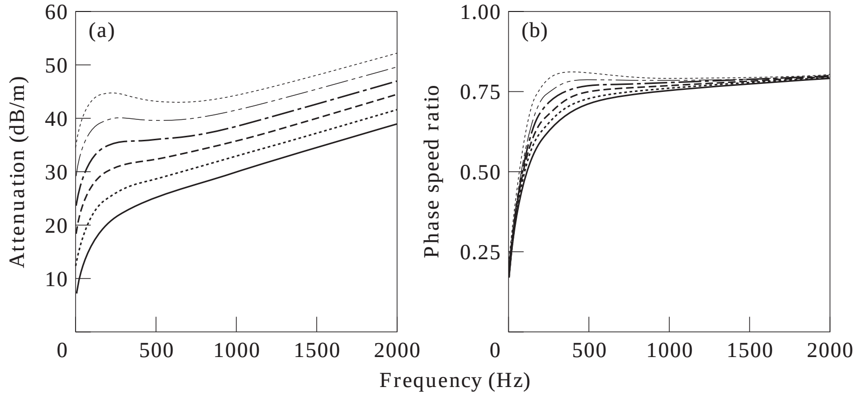

Figure 4. The effect of τ on the the propagation of backward going waves; $\bar{M} = 0{\cdot}1$, $\xi = 0$. ——, $\tau = 0$; ······; $\tau = 0{\cdot}01$; – – –, $\tau = 0{\cdot}02$; — — — –, $\tau = 0{\cdot}03$; –·–·–·– $\tau = 0{\cdot}04$; ----, $\tau = 0{\cdot}05$.

change parameter τ; hence they are not shown. Velocity profiles for different frequencies at $\xi = 0$ in the case $\xi = 0{\cdot}05$ and $\bar{M} = 0{\cdot}1$ are shown in Figure 6 and, as just remarked, would be almost identical for any value of τ. They confirm the findings of Jeong and Ih [6], firstly with respect to variation from a parabolic profile with increasing frequency and secondly that the velocity profiles of waves propagating against the mean flow are flattened and distorted more from a parabolic profile than waves being convected along with the mean flow. At different axial positions the magnitude of the velocity profiles may change with parameter τ, as discussed below, but the shape of the profiles remains effectively unchanged.

The axial dependence of the velocity profile at a frequency of 2 kHz for the case of $\tau = 0{\cdot}05$ and $\bar{M} = 0{\cdot}1$ is shown in Figure 7. There is a noticeable change in amplitude with distance for the backward wave but not the forward wave, and the shape of the profile is almost unchanged in both cases. The background density decreases in response to the temperature increase which causes a dilation and velocity increase of the forward acoustic wave. In the results shown this effect is just balanced by the reduction in amplitude of the acoustic velocity caused by viscous dissipation. The backward going wave suffers a decrease in amplitude from both effects as it travels in the direction of decreasing background temperature.

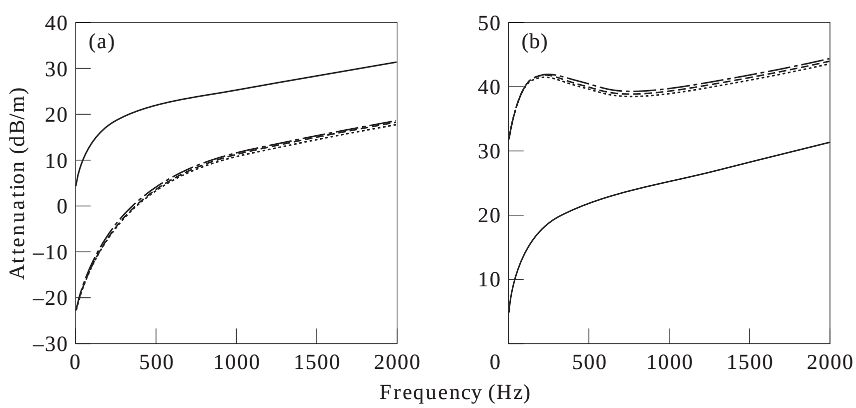

Figure 5. Variation of attenuation with axial distance; $\bar{M} = 0{\cdot}1$, $\tau = 0{\cdot}05$, 2 kHz. (a) Forward going waves, (b) backward going waves. ——, $\tau = 0$; ······, $\xi = -1$; – – – –, $\xi = 0$; — — — –, $\xi = +1$.

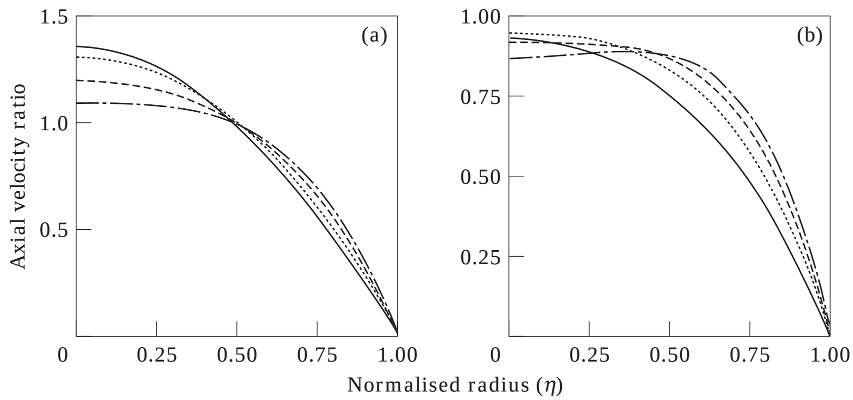

Figure 6. Variation of axial velocity profile with frequency; $\bar{M} = 0{\cdot}1$, $\tau = 0{\cdot}05$, $\xi = 0$ (a) Forward going waves; (b) backward going waves. ——, 0·5 kHz; ······, 1 kHz; – – –, 1·5 kHz; — – —, 2 kHz.

4. CONCLUSIONS

An analytical solution for isentropic acoustic propagation along a capillary tube with low order steady flow convection and change of steady state temperature has been developed. This general solution has been shown to simplify to previous solutions in the wide tube limit, both with and without mean flow and temperature gradient effects. The solution has also been compared with numerical results for the case of mean flow convection but zero temperature gradient. The comparison was very good at low Mach numbers and reasonable up to $\bar{M} = 0{\cdot}1$. Since the same expansion was used for both Mach number and temperature change parameter τ it may be concluded that the analysis is also accurate for $\tau \leqslant 0{\cdot}1$.

The analytical solution has been used to obtain results for acoustic propagation in a pore of a catalytic converter at typical operating conditions. The solution fully encompasses the temperature gradient effects, since τ is typically 0·05 or less, but is too restrictive on Mach number which can be as high as 0·3. It has been shown that the temperature gradient has a very marked affect upon attenuation of the pressure wave and a noticeable affect upon the phase speed. There is negligible change of either along the length of the duct. The

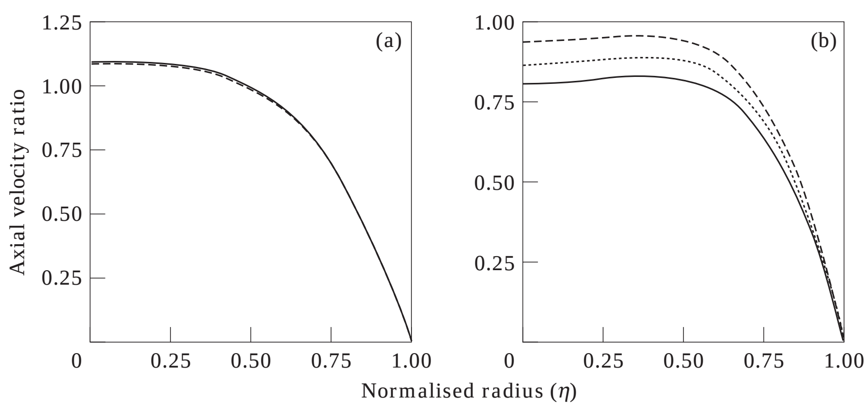

Figure 7. Variation of axial velocity profile with axial distance; $\bar{M} = 0{\cdot}1$, $\tau = 0{\cdot}05$, 2 kHz. (a) Forward going waves, (b) Backward going waves; ——, $\xi = -1$; ······, $\xi = 0$; – – –, $\xi = +1$.

temperature gradient occurs over a small distance, however, and so the overall affect upon acoustic performance of a catalytic converter will only be small.

The shape of the axial velocity profile is virtually unaffected by a temperature gradient. The magnitude of the axial velocity of a forward going wave has an increase with distance as the steady state density decreases, in addition to the normal decrease with distance due to viscous dissipation. The magnitude of the backward going wave is decreased by both effects.

The assumption that acoustic fluctuations occur isentropically in a capillary duct within which viscous effects are important is clearly unsatisfactory and will lead to an under-prediction of the attenuation. It is necessary to extend the analysis given in this paper to consider both non-isentropic effects and higher order Mach number of the mean flow. Such an analysis would almost certainly be numerical and iterative, for which the analytical solution presented here would be useful both as a first approximation in the general case, and for validation in the isentropic, low Mach number limit.

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