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CONDITIONS FOR BIFURCATION OF A CANTILEVER BEAM SUBJECTED TO GENERALIZED FOLLOWER LOADS: GEOMETRICALLY EXACT APPROACH

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1. INTRODUCTION

The cantilever beam subjected to generalized follower loads (a linear combination of conservative and non-conservative loads) is a classical problem of stability in solid mechanics. In this brief note the conditions under which static bifurcation can exist are derived using a geometrically exact beam theory. The conditions predicted by the linearized theory are proved to carry over exactly to the non-linear theory.

Consider the cantilever beam subjected to end loads, parameterized by η, as shown in Figure 1. For $\eta = 0$ the load remains oriented along the initial axis of the beam (Euler's elastica). For $\eta = 1$, the load is a pure follower force (Beck's beam). It is well-known that Euler's elastica exhibits static bifurcations at the loads $P_n = (2n - 1)^2\pi^2 EI/4\ell^2$ and that the linearized Beck's beam does not exhibit static bifurcation. For Beck's beam instability occurs under dynamic perturbation, often referred to as flutter instability. The study of the linearized Beck's problem has helped advance stability analysis beyond Euler's method of adjacent equilibrium to modern methods of dynamic perturbation and the study of non-conservative mechanical systems in general. Many researchers have studied various aspects of the discretized versions of the Beck's beam (for a comprehensive literature survey, see reference [1]). Recently, Zuo and Schreyer [2] have conducted a linear divergence (static bifurcation) and flutter instability study of the continuous model of the beam subjected to generalized follower loads. The linearized equations predict that static bifurcation exists only for $\eta \leqslant 1/2$ and that the instability changes from static bifurcation to flutter instability at $\eta = 1/2$ [3].

Plaut [4] has conducted a comprehensive study of non-linear postbifurcation for discrete, non-conservative elastic systems that exhibit static bifurcation. He found that for coincident critical points, at which instability transits from static bifurcation to flutter, the reduction in the critical load is proportional to the one-third power of the initial imperfection, which is more sensitive than the one-half power law discovered by Koiter for conservative systems that exhibit unstable post-bifurcation response. Kounadis *et al*. [5] presented a non-linear bifurcation analysis for a simple two-bar frame under a follower force by keeping the quadratic terms in the kinematic relations. They found that the bifurcation load coincides with that predicted by the linear analysis [6].

In this brief note the conditions under which a cantilever beam subjected to generalized follower loads will exhibit static bifurcation is examined, within the framework of a geometrically exact theory. It is shown here that the range of η for which static bifurcation can occur is exactly the same for the geometrically exact theory and the linearized theory. The novelty of the present formulation is that we avoid making an appeal to the associated linearized problem and thereby produce a stronger result.

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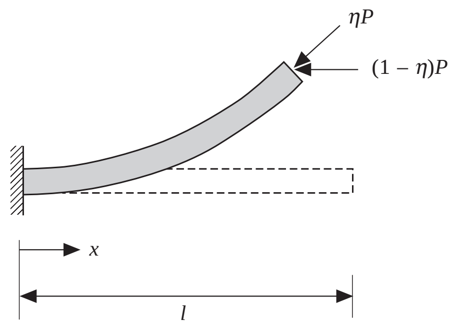

Figure 1. Cantilever beam under generalized follower load.

2. GOVERNING EQUATIONS FOR GEOMETRICALLY EXACT BEAMS

Consider a beam fixed at one end and loaded at the other end by a parameterized combination of horizontal and follower forces as shown in Figure 1. The equilibrium equations of the geometrically-exact beam are (see, for example, reference [7])

$$M' + V(1+u') - Hw' = 0, \qquad H = -\eta P \cos\theta_\ell - (1-\eta)P, \qquad V = -\eta P \sin\theta_\ell, \quad (1\text{–}3)$$

where M is the bending moment, H is the component of the resultant force oriented along the initial axis of the beam, V is the component of the resultant oriented perpendicular to the initial axis of the beam as shown in Figure 2, u is the displacement along the initial axis of the beam, w is the displacement transverse to the initial axis of the beam and θ is the rotation of the cross section relative to the initial orientation of the beam. The rotation at the end of the beam is denoted $\theta_\ell \equiv \theta(\ell)$.

Assume that the beam is inextensible and that the bending moment accrues according to the linear constitutive relationship $M = EI\theta'$, where EI is the bending stiffness of the beam. Assuming zero axial and shear deformation gives two constraint equations relating u and w to the rotation θ:

$$w' = \sin\theta, \qquad 1 + u' = \cos\theta. \tag{4, 5}$$

Substitution of equations (2) to (5) into equation (1) yields the governing equation in terms of the rotation θ. Letting $a^2 \equiv P/EI$ and noting that $\cos\theta_\ell \sin\theta - \sin\theta_\ell \cos\theta = \sin(\theta - \theta_\ell)$, one obtains the equation

$$\theta'' + (1-\eta)a^2 \sin\theta + \eta a^2 \sin(\theta - \theta_\ell) = 0, \tag{6}$$

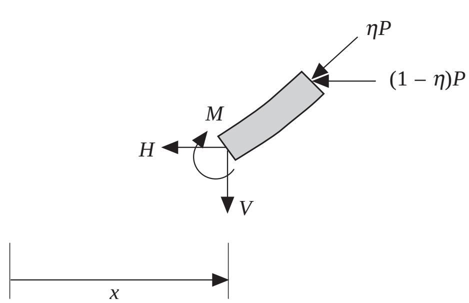

Figure 2. Equilibrium of a segment.

with boundary conditions

$$\theta(0) = 0, \qquad \theta'(\ell) = 0. \tag{7}$$

2.1 *Remark*

Note that the loading is conservative only when $\eta = 0$, wherein equation (6) reduces to the famous Euler elastica model:

$$\theta'' + \alpha^2 \sin\theta = 0. \tag{8}$$

The case $\eta = 1$ corresponds to a pure follower force (Beck's problem). By letting $\beta = \theta - \theta_\ell$, the equation governing Beck's problem takes the form

$$\beta'' + \alpha^2 \sin\beta = 0, \tag{9}$$

with boundary conditions $\beta(\ell) = 0$ and $\beta'(\ell) = 0$. This equation is interesting in that it is a terminal value problem (in fact, it is the classical pendulum problem with terminal values rather than initial values) and uniqueness theory of initial value problems shows that the solution $\beta(x) = 0$ is unique (see, for example, reference [8]). Clearly, vanishing of β implies vanishing of θ for all x. Therefore, the unbent configuration is the only possible equilibrium configuration for the Beck's problem. The above remark was first proved by Antman [9].

3. GENERALIZED BECK'S PROBLEM

Obviously, $\theta(x) = 0$ (the so-called trivial solution) is a solution to equation (6). One now asks the question: under what conditions can equation (6) admit a non-trivial solution? Let $\phi(x)$ be a positive definite function, analogous to a Lyapunov function for a dynamical system, defined as

$$\phi(x) \equiv \tfrac{1}{2}(\theta')^2 + \alpha^2(1-\eta)(1-\cos\theta) + \alpha^2\eta[1-\cos(\theta-\theta_\ell)], \tag{10}$$

Since the cosine function is never greater than 1, it is clear that $\phi(x)$ is positive for values of $\eta \in [0, 1]$. It follows that

$$\phi'(x) = [\theta'' + (1-\eta)\alpha^2 \sin\theta + \eta\alpha^2 \sin(\theta-\theta_\ell)]\theta'. \tag{11}$$

Using equation (6) we can conclude that $\phi'(x) = 0$, or $\phi(x) = C$, a constant. The constant C is determined by the boundary conditions (7). To wit,

$$\phi(0) = \tfrac{1}{2}(\theta_0')^2 + \eta a^2(1-\cos\theta_\ell) = C, \qquad \phi(\ell) = (1-\eta)a^2(1-\cos\theta_\ell) = C, \tag{12, 13}$$

where $\theta_0' = \theta'(0)$ is proportional to the bending moment at the fixed end. Equating equations (12) and (13) yields the condition

$$\tfrac{1}{2}(\theta_0')^2 = \alpha^2(1-2\eta)(1-\cos\theta_\ell). \tag{14}$$

One can observe that real values of θ_o' exist only if $\eta \leqslant 1/2$ or if $\theta_\ell = 0$. Therefore, if $\eta \leqslant 1/2$ then bifurcation can occur, but if $\eta > 1/2$ it cannot. If $\eta > 1/2$ one must have $\theta_\ell = 0$, implying that $C = 0$ and hence that $\phi(x) = 0$. Since $\phi(x)$ is strictly positive one concludes that $\theta(x) = 0$, that is, only the trivial solution exists. When $\eta > 1/2$ one would expect instability by flutter. The dynamic perturbation method must be employed to study the flutter instability [2].

4. CONCLUSIONS

In this short note we have presented a simple proof that a geometrically exact, inextensible, elastic cantilever beam subjected to a generalized follower load cannot exhibit static bifurcation for $\eta > 1/2$. This result strengthens the corresponding conclusion from the linearized theory.

REFERENCES

1. Z. P. BAZANT and L. CEDOLIN 1991 *Stability of Structures. Elastic, Inelastic, Fracture, and Damage Theories*. New York: Oxford University Press.
2. Q. H. ZUO and H. L. SCHREYER 1996 *International Journal of Solids and Structures* **33**, 1355–1367. Flutter and divergence instability of nonconservative beams and plates.
3. Y. G. PANOVKO and I. I. GUBANOVA 1965 *Stability and Oscillation of Elastic Systems, Paradoxes, Fallacies and New Concepts*. New York: Consultants Bureau Enterprises.
4. R. H. PLAUT 1976 *Journal of Structural Mechanics* **4**, 395–416. Postbuckling analysis of nonconservative elastic systems.
5. A. N. KOUNADIS, J. GIRI and G. I. SIMITSES 1978 *Journal of Applied Mechanics* **45**, 426–428. Divergence buckling of a simple frame subject to a follower force.
6. D. E. PANAYOTOUNAKOS and A. N. KOUNADIS 1979 *Journal of Sound and Vibration* **64**, 179–186. Elastic stability of a simple frame subjected to a circulatory load.
7. K .D. HJELMSTAD 1997 *Fundamentals of Structural Mechanics*. Upper Saddle River, NJ: Prentice-Hall.
8. F. BRAUER and J. A. NOHEL 1969 *The Qualitative Theory of Ordinary Differential Equations*. New York: Dover Publications.
9. S. S. ANTMAN 1995 *Nonlinear Problems of Elasticity*. New York: Springer-Verlag.