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TORSIONAL VIBRATION ANALYSIS OF SUSPENSION BRIDGES WITH GRAVITATIONAL STIFFNESS

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An analytical method for determining natural frequencies and mode shapes of torsional vibration of suspension bridges is developed by using the linearized deflection theory. This method takes into account the effect of gravitational stiffness due to dead loads of the stiffening girders, and its effect on natural frequencies of torsional vibration is investigated. A numerical example using data of an actual long-span suspension bridge is presented, and the computed results are given in tabular form.

1. INTRODUCTION

Suspension bridges with great span length are very flexible compared with other types of bridge structures, and as a result their dynamic behavior under oscillations often constitutes serious problems. When the suspension bridge is subjected to aerodynamic forces, earthquake ground motion and unsymmetric traffic loads, torsional vibration of the suspension bridge may be produced. Each of these dynamic loads produces vibrational torque of the deck about the bridge's longitudinal axis together with opposed phase vertical vibration of two cables. As a prerequisite to further investigation of wind- and earthquake-resistant designs of suspension bridges, it is necessary to accurately determine dynamic characteristics such as the natural frequencies and mode shapes.

Free torsional vibrations of suspension bridges have been analyzed by the application of an established method based on the linearized deflection theory [1–5]. This theory is the most convenient analytical procedure in order to understand the structural behavior of suspension bridges. It is still suitable and practical to calculate natural frequencies and mode shapes of suspension bridges at the preliminary design stage. In recent years, Abdel-Ghaffar [6–8] developed a method of analyzing free vertical, torsional and lateral vibrations of suspension bridges through a digital computer and a finite element approach. Computer methods using discrete structural elements are available for that various complexities such as tower movement, hanger extension, different stiffening structures (plate–girder type and truss type), and so forth can be taken into account. However, it requires the use of a large computer to solve the eigenvalue problem for large order matrices.

The purpose of this study is to investigate the effect of gravitational stiffness of stiffening girders upon free torsional vibration of suspension bridges. Although many papers dealing with free torsional vibrations of suspension bridges have been published, there have been comparatively few studies on gravitational stiffness due to the deck dead load. Only a few studies have been made on suspension bridges in recent decades. The effect of gravity

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stiffness characteristics on deflections and bending moments of multispan suspension bridges based on the deflection theory was studied by Jennings [9]. Woude [10] presented a procedure for finding natural modes and frequencies of a simple span suspension bridge with straight backstays. Yamaguchi *et al*. [11] theoretically investigated the effect of gravitational stiffness on the fundamental frequency of torsional vibration with a parameter of span length.

In the following study, the differential equation of motion governing torsional vibration of suspension bridges, including the effect of gravitational stiffness of stiffening girders, will be proposed. A procedure of free torsional vibration analysis is elucidated by the exact method [12] based on the solutions of the fourth order differential equation of motion. The analytical method which takes into account warping of the bridge deck cross-section, support conditions of the stiffening girder and the effect of flexural stiffness of the towers is designed to determine a sufficient number of natural frequencies and mode shapes of torsional vibration. This method enables an accurate vibration analysis from lower modes to higher modes. Furthermore, a numerical example which uses the data of an actual long-span suspension bridge is presented to demonstrate the applicability of the analysis developed here and to investigate the dynamic characteristics of suspension bridges.

2. BASIC ASSUMPTIONS

In analyzing free torsional vibration of suspension bridges, it is assumed that the hangers are considered to be inextensible, the dead-load curve of the cable forms a parabola, and the deformations of bridges are small. The torsional and warping stiffness, and the dead load of the stiffening girders (or trusses) are assumed constant throughout each span, although they may vary in the different spans. The original shape of every cross-section is unaltered during vibration (i.e., no distorsional deformation), but the section may undergo an out-of-plane deformation (warping). The additional horizontal component of cable tensions and additional axial forces in the towers due to torsional vibration of the suspension bridge are small in comparison with those due to the dead load. These assumptions permit the analysis to be based on a linear differential equation.

The geometry and co-ordinate systems of multispan suspension bridges with different types of stiffening girders are illustrated in Figure 1. The boundary conditions of stiffening girders may be either discontinuous or continuous supports at the towers. Thus, the bridges are divided into two different types of stiffening girders according to their characteristics. Figure 1(a) shows a hinged suspension bridge (hinged-span type bridge). This structural type is widely used and the hinged supports are located in the towers where they form a pair on either side. Figure 1(b) shows a continuous suspension bridge (continuous-span type bridge). The stiffening girder is continuously supported over all spans.

The connective conditions of the tower cable saddles are assumed to be as follows. In the first place, the additional horizontal component of the cable tension H_p is assumed to be the same on both sides of the tower in all spans of the cable (roller connection) as shown in Figure 2(a). This presupposes that the tower cable saddles are free to move horizontally upon roller nests and there is no tower resistance to displacement at the tower top. Secondly, the additional horizontal component of the cable tension $H_{p,i}$ on both sides of the tower may differ slightly from the friction forces at the tower cable saddles (hinged connection). This structural configuration means that the horizontal movement at the tower top is accompanied by a horizontal component of the force between the cable and the tower. The effect of the horizontal movement is considered in the cable equation which relates the elastic stretching of the cable to geometric displacement. Figures 2(b) and 2(c)

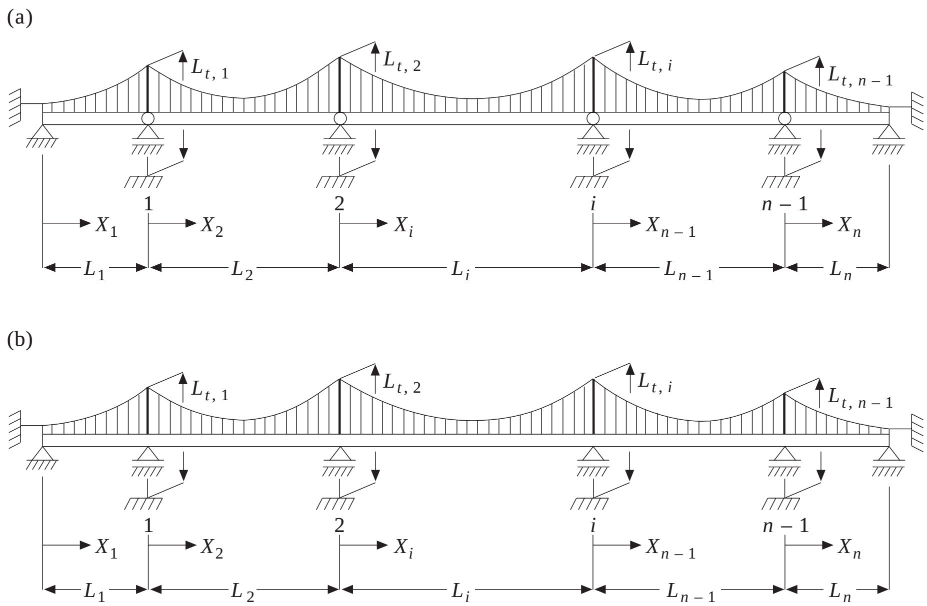

Figure 1. Geometry and co-ordinate system of multispan suspension bridge with different types of stiffening girders. (a) hinged-span type; (b) continuous-span type.

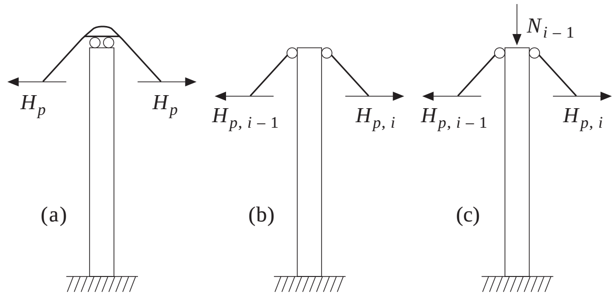

Figure 2. Cable support conditions at tower top. (a) roller connection: (b) hinged connection ($N = 0$); (c) hinged connection ($N \neq 0$).

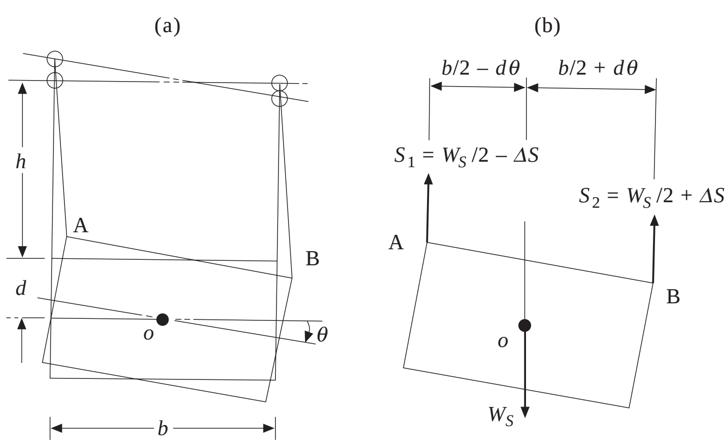

Figure 3. Torsional deformation of suspension bridge with deck of finite depth. (a) geometry; (b) cable tensions and dead load of stiffening girder.

show the cases of both neglecting and considering the effect of the axial force N_i acting on the tower top, respectively.

3. GRAVITATIONAL STIFFNESS

The torsional deformation of the deck cross-section which is symmetric in the center of the section is shown in Figure 3. The vertical displacement v_1 and v_2 of cables resulting from torsional deformation are given in terms of the vibrational angle of twist θ

$$v_{1,2} = \mp(b/2)\theta, \tag{1}$$

where b is the width of the deck cross-section.

The differential equation of the torsional vibration can be formulated by applying a flexible cable theory which takes into account geometrical deformation effects. The elastic cable reaction can be described by the following relationship [4]

$$S_i + w_c = -H_i(\mathrm{d}^2y/\mathrm{d}x^2 + \partial^2 v_i/\partial x^2), \qquad i = 1, 2 \tag{2}$$

$$H_{1,2} = H_w \mp H_p, \tag{3}$$

where H_i is the horizontal component of cable tension, which consists of the initial dead load horizontal component of cable tension H_w and the additional increment of cable tension H_p. S_i is the hanger tension, w_c is the dead load of one cable per unit length, and y is the dead load curve of the cable.

Now, if the cable is perfectly flexible and inextensible, the equilibrium equation of the cable curve is written accurately as

$$H_w\, \mathrm{d}^2y/\mathrm{d}x^2 = -(w_s/2 + w_c), \tag{4}$$

where w_s is the dead load of the stiffening girder per unit length. Substituting equations (1), (3) and (4) into equation (2), and the quadratic differential term is assumed to be negligibly small. The hanger tension S_i is expressed as follows:

$$S_{1,2} = w_s/2 \pm (H_w b/2)(\partial^2\theta/\partial x^2) + H_p \mathrm{d}^2y/\mathrm{d}x^2). \tag{5}$$

From consideration of the equilibrium of torsional deformation as illustrated in Figure 3, the torsional moment M_t is given as

$$M_t = S_1(b/2 - d\theta) - S_2(b/2 + d\theta), \tag{6}$$

where d is the vertical distance between the hanger connection point and the center of the cross-section (half depth of the deck). Substituting equation (5) into equation (6), the righting moment due to torsional deformation can be obtained as follows:

$$M_t = H_w(b^2/2)(\partial^2\theta)/\partial x^2) + H_p b\, \mathrm{d}^2y/\mathrm{d}x^2 - w_s d\theta. \tag{7}$$

The third term $w_s d\theta$ on the right side of the above equation is the gravitational stiffness due to the dead load of the stiffening girder. The effects of gravitational stiffness on the natural frequencies of torsional vibration of suspension bridges are discussed in detail below.

4. ANALYSIS OF FREE TORSIONAL VIBRATION

4.1. GOVERNING EQUATION OF MOTION AND GENERAL SOLUTIONS

By considering the righting moment M_t expressed in equation (7), the linealized differential equation of motion governing torsional vibration of the ith span of a multispan suspension bridge is given as follows [4, 7]:

$$E_i I_{wi} \frac{\partial^4 \theta_i}{\partial x_i^4} - \left(G_i J_i + H_w \frac{b_i^2}{2} \right) \frac{\partial^2 \theta_i}{\partial x_i^2} + \frac{w_i b_i}{H_w} H_{p,i} + w_{si} d_i \theta_i + I_{mi} \frac{\partial^2 \theta_i}{\partial t^2} = 0, \qquad \text{for } i = 1, 2, \ldots n, \tag{8}$$

where E_i is the modulus of elasticity, I_{wi} is the warping constant of the cross-section, θ_i is the torsional displacement, x_i is the co-ordinate system of the stiffening girder, G_i is the shear modulus, J_i is the torsion constant, b_i is the width of the deck, w_i is the dead weight of the bridge per unit length, w_{si} is the dead load of the stiffening girder per unit length, $H_{p,i}$ is the additional horizontal component of cable tension, d_i is the half depth of the deck, I_{mi} is the equivalent mass polar moment of inertia of the bridge cross-section, t is the time, and the subscript i presents the various properties on the ith span.

There are two independent variables in equation (8) concerning the time t and the distance x_i along the stiffening girder. The solutions can be written in the following well-known form

$$\theta_i = \overline{\theta}_i(x_i)\exp(\mathrm{j}\omega t), \qquad H_{p,i} = \overline{H}_{p,i} = \exp(\mathrm{j}\omega t), \tag{9}$$

in which

$$\overline{\theta}_i(x_i) = \exp(\lambda_i x_i / L_i), \tag{10}$$

and $\overline{H}_{p,i}$ is the amplitude of $H_{p,i}$, ω is the natural circular frequency of the suspension bridge, $\mathrm{j} = \sqrt{-1}$, and λ_i is the undetermined parameter. The following characteristic equation is obtained by substituting equations (9) and (10) into equation (8)

$$E_i I_{wi} \left(\frac{\lambda_i}{L_i} \right)^4 - \left(G_i J_i + H_w \frac{b_i^2}{2} \right) \left(\frac{\lambda_i}{L_i} \right)^2 + \frac{w_i b_i}{H_w} \overline{H}_{p,i} - (I_{mi}\omega^2 - w_{si} d_i)\overline{\theta}_i = 0. \tag{11}$$

The solution to the above equation can be expressed by using the following trigonometric and hyperbolic functions:

$$\overline{\theta}_i(x_i) = A_i \cos\frac{\mu_i x_i}{L_i} + B_i \sin\frac{\mu_i x_i}{L_i} + C_i \cosh\frac{\upsilon_i x_i}{L_i} + D_i \sinh\frac{\upsilon_i x_i}{L_i} + \frac{w_i b_i \overline{H}_{p,i}}{H_w(I_{mi}\omega^2 - w_{si} d_i)}, \tag{12}$$

in which

$$\mu_i = L_i \sqrt{((2G_i J_i + H_w b_i^2)/2E_i I_{wi})(Z_i - 1)}, \qquad \upsilon_i = L_i \sqrt{((2G_i J_i + H_w b_i^2)/2E_i I_{wi})(Z_i + 1)},$$
$$Z_i = \sqrt{1 + 16 E_i I_{wi}(I_{mi}\,\omega^2 - w_{si} d_i)/(2G_i J_i + H_w b_i^2)^2}, \tag{13}$$

and the unknown constants A_i, B_i, C_i and D_i are determined by the boundary conditions of the stiffening girders.

4.2. CABLE EQUATIONS

The cable equation provides a compatibility condition relating the changes which occur in the cable tension, to the changes in cable geometry, when the cable is displaced in-plane from its original equilibrium position. The different connection types of cable supports at the tower top are considered in the cable equation, which relates the elastic stretching of the cable to geometric displacement [13, 14].

In the first place, the roller connection as shown in Figure 2(a) is considered. The horizontal components of the cable tension H_w and H_p must be identical on both sides of the tower in all spans of the cable. The cable equation in the case of the roller connection is

$$\frac{L_c}{E_c A_c} H_p - \sum_{i=1}^{n} \frac{w_i b_i}{2H_w} \int_0^{L_i} \theta_i \, \mathrm{d}x_i = 0, \qquad L_c = \sum_{i=1}^{n} L_{c,i} = \sum_{i=1}^{n} \int_0^{L_i} \left(\frac{\mathrm{d}s_i}{\mathrm{d}x_i}\right)^3 \mathrm{d}x_i, \tag{14}$$

where E_c is the modulus of elasticity of the cable, $A_c$is the cross-sectional area of the cable, L_c and $L_{c,i}$ are virtual lengths of the cable, $\mathrm{d}s_i$ is the length of the cable element in the ith span, and n is the number of spans.

As the second alternative, the hinged connection as shown in Figures 2(b) and 2(c) is considered. This rocked type with a pin-bearing at the tower top provides one of the simplest and safest constructions, but the friction forces at the fixed tower cable saddles are so high that the tower tops move in unison with the adjacent cables. If the tower is flexible such as in the cantilever beam, then a horizontal displacement at the tower top is produced. The cable equations for the n-span suspension bridge are given as follows:

$$\frac{L_{c,i}}{E_c A_c} H_{p,i} - \frac{w_i b_i}{2H_w} \int_0^{L_i} \theta_i \, \mathrm{d}x_i = -\delta_i^l + \delta_i^r, \tag{15}$$

where δ_i^l and δ_i^r are horizontal displacements of the tower top at the left and right ends of the ith span, respectively. These displacements δ_i^l and δ_i^r can be expressed as the product of the elastic resistance of the tower and the difference $H_{p,i-1} - H_{p,i}$ of the additional horizontal component of the cable tension caused by inertia forces. In the case where the effect of the axial force N_i is neglected, these displacements are expressed by the following:

$$\delta_i^l = -(L_{t,i-1}^3/3E_t I_{t,i-1})(H_{p,i-1} - H_{p,i}), \qquad \delta_i^r = -(L_{t,i}^3/3E_t I_{t,i})(H_{p,i} - H_{p,i+1}), \tag{16}$$

where $E_t I_{t,i}$ is the average flexural stiffness of the ith tower, and $L_{t,i}$ is the height of the ith tower. In the case where the effect of the axial force N_i is considered, equation (16) may be written as

$$\delta_i^l = -(L_{t,i-1}/N_{i-1})(T_{i-1} - 1)(H_{p,i-1} - H_{p,i}), \qquad \delta_i^r = -(L_{t,i}/N_i)(T_i - 1)(H_{p,i} - H_{p,i+1}), \tag{17}$$

in which the coefficients T_i are defined as

$$T_i = \tan(k_i L_{t,i})/k_i L_{t,i}, \qquad k_i = \sqrt{N_i/E_t I_{t,i}}. \tag{18}$$

4.3. FREQUENCY EQUATION

The boundary conditions at the supports of the stiffening girder of multispan suspension bridges are summarized in Table 1. The following matrix notation may be obtained by substituting equation (12) into the equations of the boundary conditions:

$$\mathbf{Aa} = \mathbf{Hh}, \tag{19}$$

in which

TABLE 1

Boundary conditions of stiffening girders

Supports	Hinged-span type	Continuous-span type
Left end; $x_1 = 0$ for $i = 1$	$\bar{\theta}_1(0) = 0$ $E_1 I_{w1} \bar{\theta}_1''(0) = 0$	$\bar{\theta}_1(0) = 0$ $E_1 I_{w1} \bar{\theta}_1''(0) = 0$
Intermediate supports; $x_{i-1} = L_{i-1}$ and $x_i = 0$, for $i = 2, 3, \ldots, n$	$\bar{\theta}_{i-1}(L_{i-1}) = 0$ $E_{i-1} I_{w,i-1} \bar{\theta}_{i-1}''(L_{i-1}) = 0$ $\bar{\theta}_i(0) = 0$ $E_i I_{wi} \bar{\theta}_i''(0) = 0$	$\bar{\theta}_{i-1}(L_{i-1}) = 0$ $\bar{\theta}_{i-1}'(L_{i-1}) = \bar{\theta}_i'(0)$ $\bar{\theta}_i(0) = 0$ $E_{i-1} I_{w,i-1} \bar{\theta}_{i-1}''(L_{i-1}) = E_i I_{wi} \bar{\theta}_i''(0)$
Right end; $x_n = L_n$ for $i = n$	$\bar{\theta}_n(L_n) = 0$ $E_n I_{wn} \bar{\theta}_n''(L_n) = 0$	$\bar{\theta}_n(L_n) = 0$ $E_n I_{wn} \bar{\theta}_n''(L_n) = 0$

$$\mathbf{a} = \{A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, \ldots, A_n, B_n, C_n, D_n\}^{\mathrm{T}},$$

$$\mathbf{h} = \{\bar{H}_{p,1}, \bar{H}_{p,2}, \ldots, \bar{H}_{p,n}, \}^{\mathrm{T}}, \tag{20}$$

and the orders of the coefficients matrices **A** and **H** are $4n \times 4n$ and $4n \times n$, respectively. The vectors **a** and **h** are unknown constants of the orders $4n \times 1$ and $n \times 1$, respectively.

Next, the cable equations expressing the relationship between torsional amplitude of vibration $\bar{\theta}_i(x_i)$ and the cable tension $\bar{H}_{p,i}$ are applied to determine the natural circular frequency ω. By substituting equation (12) into either equation (14) or equation (15), the relation between the vectors **a** and **h** can also be obtained as follows:

$$\mathbf{Ga} = \mathbf{Eh}. \tag{21}$$

The coefficients matrices **G** and **E** are of the orders $n \times 4n$ and $n \times n$, respectively. It should be pointed out that the vector **h** in equation (20) will be reduced to a single element for the roller cable connection at the tower top, because the unknown constant H_p in equation (14) is only a term throughout each span. Then, the matrices **H** and **G** and the square matrix **E** are reduced to a column matrix of order $4n$ and a row matrix of order 4n and a single element, respectively.

The homogeneous equation on the vector **a** is obtained by eliminating the vector **h** from equations (19) and (21), as follows

$$(\mathbf{A} - \mathbf{HE}^{-1}\mathbf{G})\mathbf{a} = \mathbf{0}. \tag{22}$$

The solution of equation (22) is possible only when the determinant of the coefficient matrix vanishes, as,

$$\det|\mathbf{A} - \mathbf{HE}^{-1}\mathbf{G}| = 0. \tag{23}$$

This equation is the frequency equation of suspension bridges and is a transcendental equation on the natural circular frequencies ω. The roots may be computed by applying the Regula–Falsi method [15, 16] and by using a high-speed digital computer. Furthermore, the relative magnitude of the vectors **a** and **h** can be found from equations (22) and (21), respectively.

5. NUMERICAL RESULTS

A numerical example is presented to demonstrate the effectiveness of the analytical method developed here and to describe some dynamic characteristics of torsional vibrations of suspension bridges. Computations using data from the Innoshima Suspension Bridge located between Honshu and Shikoku in Japan provide the basis for this example. The geometry of the bridge and the structural properties necessary for a torsional vibration analysis are given as follows: (1) Stiffening girder, $L_1 = L_3 = 250$ m, $L_2 = 770$ m, $b = 26$ m, $d = 4{\cdot}5$ m, $w_{s1} = w_{s3} = 1{\cdot}597 \times 10^5$ N/m, $w_{s2} = 1{\cdot}555 \times 10^5$ N/m, $I_{w1} = I_{w3} = 393{\cdot}2$ m^6, $I_{w2} = 324{\cdot}4$ m^6, $J_1 = J_3 = 2{\cdot}901$ m^4, $J_2 = 4{\cdot}169$ m^4, $I_{m1} = I_{m3} = 2{\cdot}170 \times 10^6$ Ns^2, $I_{m2} = 2{\cdot}136 \times 10^6$ Ns^2, $E = 2{\cdot}06 \times 10^{11}$ N/m^2, $G = 7{\cdot}95 \times 10^{10}$ N/m^2, (2) cable, cable sag length in main span $f = 76$ m, $A_c = 0{\cdot}2281$ m^2, $H_w = 1{\cdot}943 \times 10^8$ N, $w_c = 4{\cdot}724 \times 10^4$ N/m, $E_c = 1{\cdot}962 \times 10^{11}$ N/m^2, and (3) tower, $L_{t,1} = L_{t,2} = 138{\cdot}85$ m, $I_{t,1} = I_{t,2} = 3{\cdot}320$ m^4, $E_t = 206 \times 10^{11}$ N/m^2.

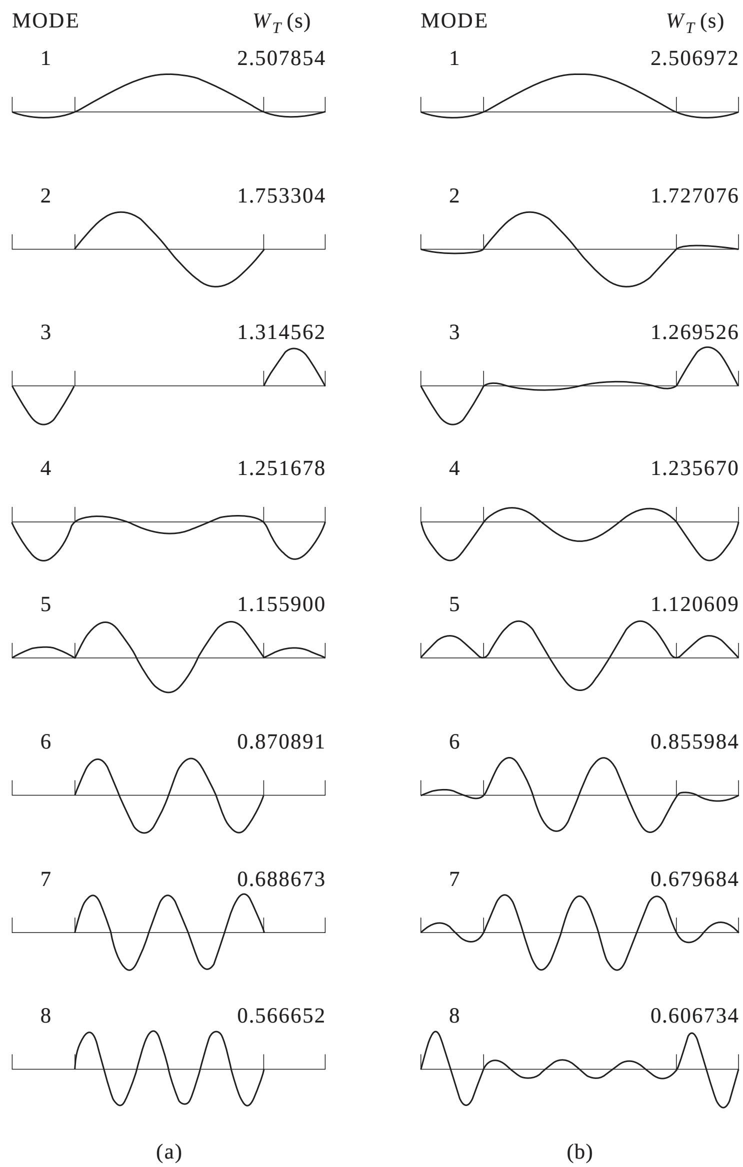

Figure 4. Computed natural mode shapes of torsional vibration. (a) hinged-span type; (b) continuous-span type.

Figure 4 shows the natural mode shapes of torsional vibration in the case of the hinged connection ($N = 0$) as shown in Figure 2(b). In the first few symmetric mode shapes of the hinged-span type bridge, the stiffening girders of the center span and the side spans vibrate together. But in the asymmetric mode shapes, the stiffening girders of the center span and the side spans vibrate separately. Another characteristic of the mode shapes of hinged suspension bridges is that they produce discontinuous forms at the intermediate supports. On the other hand, in both asymmetric and symmetric modes of the continuous-span type bridge, the stiffening girders of the center and side spans always vibrate together from the first mode to the higher modes. The continuous suspension bridges have no discontinuous torsional slope at the intermediate supports.

Some of the computed natural periods of the hinged-span and the continuous-span types are presented for the first five modes of the asymmetric and symmetric vibrations in Tables 2 and 3, respectively. In the lower modes, there is a slight difference between the roller connection and the hinged connection. In the hinged modes, the effect of cable supports at the top of the tower is not recognized.

In the first five natural periods of asymmetric and symmetric modes, there is a considerable difference between the hinged-span type and the continuous-span type bridges. This result can be estimated from the fact that there is a distinct difference in the mode shapes of both types of suspension bridges as shown in Figure 4. In particular, the torsional slope mode shapes of two hinged suspension bridges have a discontinuous form at the intermediate supports of the stiffening girders. The values of the natural periods of hinged suspension bridges are generally large in comparison with those of continuous suspension bridges.

The effect of the gravitational stiffness upon the natural periods is described below. As shown in Tables 2 and 3, the values of the natural periods of suspension bridges with gravitational stiffness are smaller than those of suspension bridges without gravitational stiffness. The natural periods of suspension bridges in either case of the hinged-span type and the continuous-span type have a tendency to decrease in a free torsional vibration analysis which takes into account the gravitational stiffness of the stiffening girders. The effect of the gravitational stiffness become more pronounced for the first natural period

TABLE 2

Effect of gravitational stiffness on natural periods (in seconds) of torsional vibration (hinged-span type)

Mode types of tortional vibration	Mode order	Without gravitational stiffness			With gravitational stiffness		
		Roller connection	Hinged connection ($N = 0$)	Hinged connection ($N \neq 0$)	Roller connection	Hinged connection ($N = 0$)	Hinged connection ($N \neq 0$)
Asymmetric mode	1	1·7761	1·7761	1·7761	1·7533	1·7533	1·7533
	2	1·3251	1·3242	1·3252	1·3154	1·3146	1·3155
	3	0·8736	0·8736	0·8736	0·8709	0·8709	0·8709
	4	0·5674	0·5674	0·5674	0·5667	0·5667	0·5667
	5	0·4112	0·4112	0·4112	0·4109	0·4109	0·4109
Symmetric mode	1	2·5781	2·5760	2·5783	2·5098	2·5079	2·5100
	2	1·2602	1·2600	1·2602	1·2519	1·2517	1·2519
	3	1·1624	1·1624	1·1624	1·1559	1·1559	1·1559
	4	0·6900	0·6900	0·6900	0·6887	0·6887	0·6887
	5	0·4784	0·4784	0·4784	0·4780	0·4780	0·4780

TABLE 3

Effect of gravitational stiffness on natural periods (in seconds) of torsional vibration (continuous-span type)

Mode types of tortional vibration	Mode order	Without gravitational stiffness: Roller connection	Without gravitational stiffness: Hinged connection ($N = 0$)	Without gravitational stiffness: Hinged connection ($N \neq 0$)	With gravitational stiffness: Roller connection	With gravitational stiffness: Hinged connection ($N = 0$)	With gravitational stiffness: Hinged connection ($N \neq 0$)
Asymmetric mode	1	1·7489	1·7489	1·7489	1·7271	1·7271	1·7271
	2	1·2789	1·2782	1·2790	1·2703	1·2695	1·2703
	3	0·8586	0·8586	0·8586	0·8560	0·8560	0·8560
	4	0·6077	0·6077	0·6077	0·6067	0·6067	0·6067
	5	0·5540	0·5540	0·5540	0·5533	0·5533	0·5533
Symmetric mode	1	2·5771	2·5750	2·5774	2·5089	2·5070	2·5091
	2	1·2438	1·2436	1·2439	1·2359	1·2357	1·2359
	3	1·1265	1·1265	1·1265	1·1206	1·1206	1·1206
	4	0·6810	0·6810	0·6810	0·6797	0·6797	0·6797
	5	0·6021	0·6021	0·6021	0·6012	0·6012	0·6012

of the symmetric vibration mode, and the values of a natural period decrease approximately 2·7% in comparison with suspension bridges without the gravitational stiffness. The effect of the gravitational stiffness gradually decreases as the number of mode orders increases. Consequently the effect of the gravitational stiffness upon free torsional

TABLE 4

Effect of δ on natural periods (in seconds) of hinged suspension bridge with roller cable connection

Mode types of tortional vibration	Mode order	δ = 1·31	δ = 2·0	δ = 3·0	δ = 4·0
Asymmetric mode	1	1·7761	1·7631	1·7355	1·6991
		(1·000)†	(0·993)	(0·977)	(0·957)
	2	1·3251	1·2897	1·2215	1·1420
		(1·000)	(0·973)	(0·922)	(0·862)
	3	0·8736	0·8495	0·8032	0·7495
		(1·000)	(0·972)	(0·919)	(0·858)
	4	0·5674	0·5355	0·4811	0·4269
		(1·000)	(0·944)	(0·848)	(0·752)
	5	0·4112	0·3747	0·3202	0·2728
		(1·000)	(0·911)	(0·779)	(0·663)
Symmetric mode	1	2·5781	2·5699	2·5545	2·5367
		(1·000)	(0·997)	(0·991)	(0·984)
	2	1·2602	1·2306	1·1726	1·1041
		(1·000)	(0·977)	(0·930)	(0·876)
	3	1·1624	1·1438	1·1060	1·0584
		(1·000)	(0·984)	(0·951)	(0·911)
	4	0·6900	0·6617	0·6102	0·5549
		(1·000)	(0·959)	(0·884)	(0·804)
	5	0·4784	0·4438	0·3887	0·3375
		(1·000)	(0·928)	(0·813)	(0·705)

†Numbers in brackets indicate ratio of natural frequencies to the parameter $\delta = 1{\cdot}31$.

vibrations of suspension bridges is comparatively small and is limited only to the first few modes. It is necessary to consider the effect of gravitational stiffness in the case of suspension bridges with higher deck such as truss-type stiffening girder, and the decrement of the first natural period of symmetric torsional mode is favorable for the aerodynamic stability. However, in the case of suspension bridges with a large width/depth ratio ($b/2d$) of deck such as flat box girder, the effect of gravitational stiffness is insignificant and may be negligibly small.

The following non-dimensional parameter δ is defined for the study of torsional vibration of suspension bridges:

$$\delta = \sqrt{EI_w/GJb^2}\,, \qquad (24)$$

where EI_w and GJ are the average warping and torsional stiffness of the cross-section of bridge deck, respectively. Tables 4 and 5 show the effect of the torsional parameter δ on the natural periods of the suspension bridge for the hinged and continuous stiffening girders, respectively. The quantities shown in the brackets indicate the ratio of natural frequencies to the parameter $\delta = 1{\cdot}31$. For both tables, the roller cable connection is considered, and the effect of the gravitational stiffness is neglected in order to simplify numerical calculations. The increase of the parameter δ indicates the increase of warping stiffness of the bridge. The natural periods in the case of $\delta = 1{\cdot}31$ shown in Tables 4 and 5 correspond to those of the roller connections in Tables 2 and 3. It is observed that the values of natural periods decrease with the increase of the torsional parameter δ. The

TABLE 5

Effect of δ on natural periods (in seconds) of continuous suspension bridge with roller cable connection

Mode types of tortional vibration	Mode order	δ			
		1·31	2·0	3·0	4·0
Asymmetric mode	1	1·7489	1·7225	1·6764	1·6218
		(1·000)†	(0·985)	(0·959)	(0·927)
	2	1·2789	1·2215	1·1283	1·0331
		(1·000)	(0·955)	(0·882)	(0·808)
	3	0·8586	0·8259	0·7673	0·7029
		(1·000)	(0·962)	(0·894)	(0·819)
	4	0·6077	0·5548	0·4846	0·4238
		(1·000)	(0·913)	(0·797)	(0·679)
	5	0·5540	0·5100	0·4366	0·3705
		(1·000)	(0·921)	(0·788)	(0·669)
Symmetric mode	1	2·5771	2·5674	2·5473	2·5211
		(1·000)	(0·996)	(0·988)	(0·978)
	2	1·2438	1·2114	1·1560	1·0944
		(1·000)	(0·974)	(0·929)	(0·880)
	3	1·1265	1·0833	1·0074	1·9250
		(1·000)	(0·962)	(0·894)	(0·821)
	4	0·6810	0·6476	0·5895	0·5295
		(1·000)	(0·951)	(0·866)	(0·778)
	5	0·6021	0·5419	0·4586	0·3895
		(1·000)	(0·900)	(0·762)	(0·647)

† Numbers in brackets indicate ratio of natural frequencies to the parameter $\delta = 1{\cdot}31$.

calculated results indicate that the natural periods are significantly reduced for the higher modes. From a comparison of natural periods in Tables 4 and 5, the natural periods of the continuous-span type bridge show a more considerable variation than those of the hinged-span type bridge. In other words, it is considered that the continuous suspension bridge is sensitive to the effect of warping stiffness of the deck in comparison with the two-hinged suspension bridge.

6. CONCLUSIONS

An analytical method for free torsional vibration of suspension bridges, including the effect of gravitational stiffness due to the dead load of the stiffening structure, has been developed in this study. The formulation of the dynamic problem is based on the linealized deflection theory and the frequency equation is derived effectively by using the general solutions of the differential equation of motion.

It is concluded from the numerical results that the effect of cable support conditions at the tower top upon the natural periods of suspension bridges is small. However, the effect of the boundary conditions of stiffening girders is considerable. The values of the natural periods of hinges suspension bridges are generally larger than those of continuous suspension bridges. The computed natural periods have a tendency to decrease as the gravitational stiffness of the stiffening girders is considered in a torsional vibration analysis of suspension bridges. The effect of the gravitational stiffness is comparatively small and is limited to ony the first few modes. Also, the natural periods of torsional vibration are significantly reduced with the increase of the warping stiffness of the deck. The continuous suspension bridges are very susceptible to effect of warping stiffness in comparison with the hinged suspension bridges.

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