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LETTERS TO THE EDITOR

ON NON-LINEAR NORMAL VIBRATION MODES THAT EXIST ONLY IN AN INTERMEDIATE AMPLITUDE RANGE

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Non-linear normal vibration modes are a generalization of normal (principal) vibrations of linear systems [1–3]. Posed here is a problem of normal vibrations existing only in an intermediate amplitude range and vanishing by merging in some limiting points as the amplitude tends to zero or infinity.

Consider a conservative system with two degrees of freedom whose kinetic energy is a quadratic function of the velocities that is reduced to a canonical form while the potential energy involves terms of the second, fourth and sixth power. Here it is found that normal vibrations of the above type do occur as the multipliers of the fourth power terms are increased.

The equations of motion of the system are written as

$$\ddot{x} + \partial\Pi(x, y)/\partial x = 0, \qquad \ddot{y} + \partial\Pi(x, y)/\partial y = 0, \tag{1}$$

where

$$\Pi = \sum_{j=0}^{a} \alpha_j x^{2-j} y^j + \rho\left[\sum_{j=0}^{4} b_j x^{4-j} y^j\right] + \sum_{j=0}^{6} d_j x^{6-j} y^j.$$

Substituting $x \to cx$, $y \to cy$, where $c = x(0)$, one obtains a multiplier c^2 for the third power terms in equations (1) and a multiplier c^4 for the fifth power terms. Clearly, this substitution leads to $x(0) = 1$. It is assumed without sacrifice of generality that $\dot{x}(0) = \dot{y}(0) = 0$. At infinitely small amplitudes, $c \to 0$, a linear system is realized, while at infinitely large amplitudes, $c \to \infty$, an essentially non-linear homogeneous system is approached containing terms of the fifth power in equations (1). Both limiting systems allow normal vibrations with rectilinear trajectories in a configuration space of the form $y = kx$.

In what follows, an equation for the trajectories of normal vibrations in a configurational space will be used of the form derived in references [1–3], i.e.,

$$2y''(h - \Pi)/(1 + (y')^2) + y'(-\Pi_x) = -\Pi_y, \tag{2}$$

and a boundary value condition of the form

$$y'(X)(-\Pi_x(X, y(X)) = -\Pi_y(X, y(X)), \tag{3}$$

where X are the values of the variable x on the surface $h - \Pi = 0$, h being the system energy. In the case at hand, $X = 1$.

One can approximate the normal vibration trajectories by straight lines $y = kx$. In order to find k, one first obtains, from equation (2) (with due account of the conditions (3)), the relationship

$$k[-\Pi_x(X, kX)] + \Pi_y(X, kX) = 0. \tag{4}$$

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With respect to set (1), equation (4) is an algebraic equation of the sixth power in k.

For the sake of definiteness, assume that

$$\Pi = x^2/2 + y^2/2 + (x-y)^2 + \rho(x^4 + 6x^2y^2 + 0{\cdot}8xy^3 + 2y^4) + x^6/6 + y^6/6 + (x-y)^6/3;$$

that is, $a_0 = 1{\cdot}5$, $a_i = -2$, $a_2 = 1{\cdot}5$; $b_0 = 1$, $b_1 = 0$, $b_2 = 6$, $b_3 = 0{\cdot}8$, $b_4 = 2$, $d_0 = 0{\cdot}5$, $d_1 = -2$, $d_2 = 5$, $d_3 = -20/3$, $d_4 = 5$, $d_5 = -2$ and $d_6 = 0{\cdot}5$.

Here, both the linear system and the essentially non-linear homogeneous system determined by the sixth power terms in the expression of potential energy allow for normal vibrations of the form $y = kx$ where $k = 1$.

Here k is the amplitude value of the variable y. Further examination reveals that, at not too large values of the parameter ρ, equation (4) has only two real solutions. As ρ is increased in the intermediate range of c, another pair of real solutions appears for $k > 0$, and then yet another pair exists for $k < 0$; the number of normal vibration modes may thus be as large as six. The appearance of these modes with increasing ρ is depicted in Figure 1 as plots of c versus $\beta = \text{arctg}\, k$. Here curve 1) corresponds to $\rho = 1$, 2) to $\rho = 2$, 3) to $\rho = 5$, 4) to $\rho = 10$, 5) to $\rho = 25$, 6) to $\rho = 100$ and 7) to $\rho = 1000$. The arrows indicate further evolution of the normal vibrations. As $\rho \to \infty$, all normal vibrations tend to normal modes with rectilinear trajectories existing in a homogeneous system which is determined by the fourth power terms in the expression for the potential energy. These limiting modes are governed by the equalities $y = kx$ where $k = \{1{\cdot}496, 0, -1{\cdot}279, -5\}$; the latter straight lines are indicated by the cipher 8 in Figure 1. The broken branches merge as $\rho \to \infty$. No solutions of this type exist in the limiting homogeneous system.

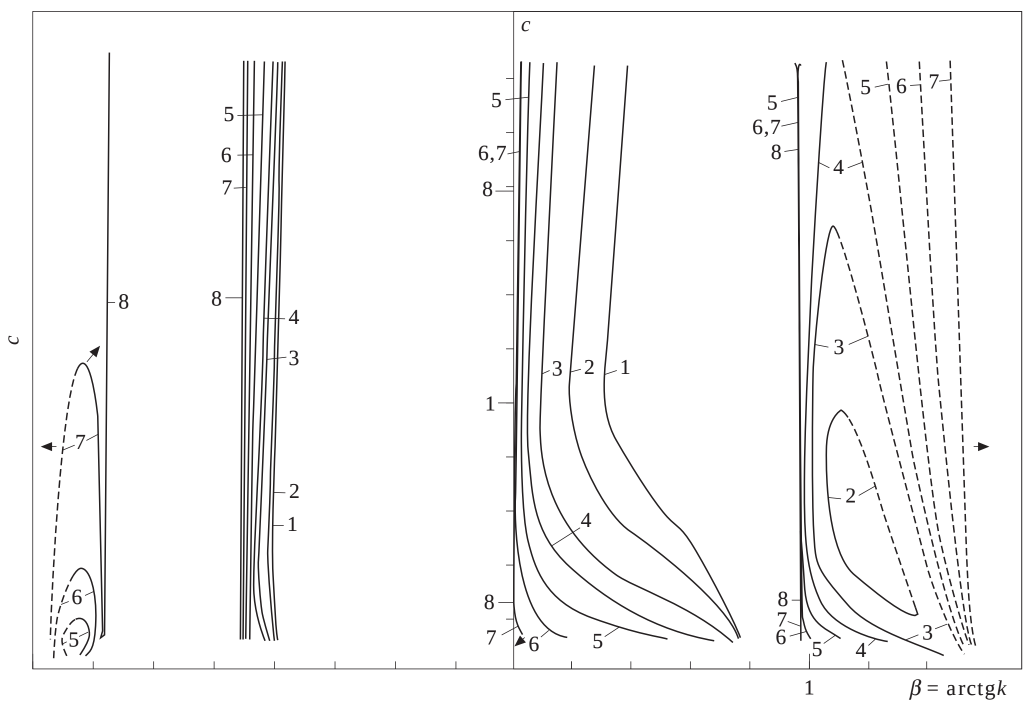

Figure 1. The dependence of non-linear normal modes on the amplitude of oscillations (by using the parameters c and $\beta = \text{arctg}\, k$. The curve 1) corresponds to $\rho = 1$, 2) to $\rho = 2$, 3) to $\rho = 5$, 4) to $\rho = 10$, 5) to $\rho = 25$, 6) to $\rho = 100$ and 7) to $\rho = 1000$. The arrows indicate further evolution of the normal vibrations. The limiting lines (as $\rho \to \infty$) are indicated by the cipher 8. The solutions corresponding to the broken curves are unstable.

The approximate analytical results are fairly consistent with computer simulations. As one would expect, the solutions corresponding to the broken curves in Figure 1 are unstable. Numerical calculations were carried out for the initial conditions corresponding to a rectilinear approximation. The resulting trajectories generated by the computer are nearly rectilinear.

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