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VIBRATION CONTROL OF AN ELASTIC MANIPULATOR LINK

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1. Introduction

In robotics and other applications the members are designed to be rigid to satisfy the requirements, especially in robotics, to minimize the tracking errors. This design consideration results in heavy and slow manipulators which means high energy and time consumptions. Designing lightweight manipulators that can carry heavy loads has been a popular research area recently. The usage of lightweight and fast robots makes the control of the elastic manipulators important and a very difficult problem to overcome. The most important problem is the elimination of vibrations to enable the manipulator to follow the desired trajectory precisely with rather simple control methods.

Researchers have tried various methods for modelling elastic manipulators. Low and Dubey [1] studied a multilink manipulator. They obtained the equations of motion by using Hamilton's principle and solved the equations by perturbation and modal analysis techniques. Chang [2] investigated the dynamics and control of a single-flexible-link arm based on an Equivalent Rigid Link System dynamic model. Mitchell and Bruch [3] studied the problem of finding the total displacement of a rotating beam subject to an arbitrary driving torque. They also used Hamilton's principle to obtain the equation of motion and studied the forced response of the system. Chait *et al*. [4] derived the equivalent but self-adjoint form of a manipulator system. This self-adjoint form led to a natural expansion, where the equations are decoupled. They also showed the reason for the coupling. Transient vibration of a beam mass system fixed to a rotating body is analyzed for a trapezoidal input function in [5].

In this study a single elastic link is modelled by the use of the Euler–Bernoulli beam theory and mode summation techniques are used to solve the transient vibration problem. Cycloidal rise motion is used to drive the link. The spectrum of residual vibration amplitudes to the ratio of the natural frequency to rise motion frequency is obtained. It is shown that at certain values of the rise motion frequency, residual vibration is eliminated which can make the precise position control or precise tracking control of an elastic link possible.

2. Problem formulation and solution method

The manipulator arm is modelled as a fixed-free elastic beam whose position is defined in rotating cartesian co-ordinates xy. The model is shown in Figure 1. If the link is assumed to be an Euler–Bernoulli beam, the equation of motion is,

$$[EIy''(x, t)]'' + m(x)[x\ddot{\theta}(t) + \ddot{y}(x, t)] = 0, \tag{1}$$

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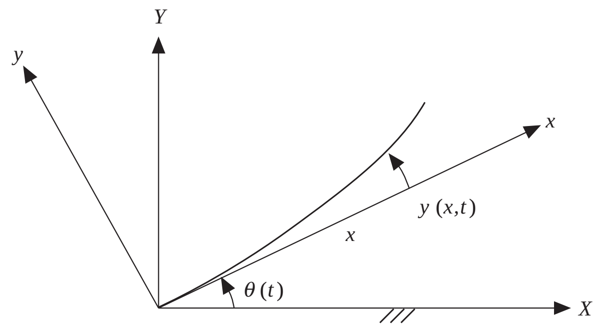

Figure 1. The model of the manipulator arm.

where E is the modulus of elasticity, I is the area moment of inertia, m is the mass per unit length, θ is the angle of rotation. If I and m do not change along the length of the manipulator, Equation (1) becomes,

$$EIy^{IV}(x, t) + m\ddot{y}(x, t)] = -mx\ddot{\theta}(t) \qquad 0 \leqslant x \leqslant L. \tag{2}$$

If the mode summation method is used for the solution of equation (2), the displacement can be assumed as,

$$y(x, t) = \sum q_i(t)\varphi_i(x), \tag{3}$$

in which $\varphi_i(x)$ is the ith normal mode and $q_i(t)$ is the ith generalized co-ordinate. The generalized co-ordinates can be determined from Lagrange's equations by first establishing the kinetic and potential energies of the link [6]. The kinetic energy is,

$$T = \frac{1}{2}\sum M_i\dot{q}_1^2, \tag{4}$$

where the generalized mass M_i is defined as,

$$M_i = \int_0^I \varphi_i(x)^2 m(x)\,\mathrm{d}x. \tag{5}$$

Similarly the potential energy is,

$$U = \frac{1}{2}\sum K_i q_i^2, \tag{6}$$

where the generalized stiffness K_i is

$$K_i = \int_0^I EI[\varphi_i''(x)]^2\,\mathrm{d}x. \tag{7}$$

In addition to T and U, one needs the generalized force Q_i, which is determined from the work done by the applied force for the virtual displacement δq_i.

$$\delta W = \sum_i \delta q_i \int_0^I -mx\ddot{\theta}(t)\varphi(x)\,\mathrm{d}x, \tag{8}$$

where the generalized force is defined as,

$$Q_i = -m\ddot{\theta}(t) \int_0^1 x\varphi(x)\,\mathrm{d}x. \tag{9}$$

Substituting T, U and Q into Lagrange's equation

$$\mathrm{d}/\mathrm{d}t\,(\partial T/\partial \dot{q}_i) - \partial T/\partial q_i + \partial U/\partial q_i = Q_i, \tag{10}$$

the differential equation for $q_i(t)$ is found as

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = \frac{1}{M_i} m\ddot{\theta}(t) \int_0^1 - x\varphi_i(x)\,\mathrm{d}x. \tag{11}$$

Equation (11) can be written as

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = (\Gamma_i/M_i)\,\ddot{\theta}(t), \tag{12}$$

where

$$\Gamma = -m \int_0^1 x\varphi_i(x)\,\mathrm{d}x, \tag{13}$$

which can be defined as the mode participation factor for mode i [6]. For finding the normal modes, the manipulator arm can be assumed as a fixed-free beam. For the assumed boundary conditions the characteristic equation of the system is,

$$\cos\beta L \cosh\beta L = -1. \tag{14}$$

By solving equation (14) numerically, $(\beta L)_i$ values can be found. β_i's are defined as

$$\beta_i^4 = \omega_i^2 m/EI \tag{15}$$

The eigenvalues for the first three modes are $\beta_1 L = 1{\cdot}875$, $\beta_2 L = 4{\cdot}694$ and $\beta_3 L = 7{\cdot}855$. The corresponding frequencies can be found from the relation

$$\omega_i = (\beta_i L)^2 \sqrt{EI/mL^4}. \tag{16}$$

If equation (3) is accepted as the solution for the equation (2), the normal modes are given as

$$\varphi_i(x) = C_1 \sin\beta_i x + C_2 \cos\beta_i x + C_3 \sinh\beta_i x + C_4 \cosh\beta_i x. \tag{17}$$

The constants C_i can be found by using the boundary conditions. For the given boundary conditions of the fixed-free beam the orthogonal modes are [7]

$$\varphi_i(x) = A_i[(\sin\beta_i L - \sinh\beta_i L)(\sin\beta_i x - \sinh\beta_i x) + (\cos\beta_i L - \cosh\beta_i L)(\cos\beta_i x - \cosh\beta_i x)] \tag{18}$$

When these modes are normalized by using the equation below

$$\int_0^1 m\varphi_i^2(x)\,\mathrm{d}x = 1 \tag{19}$$

TABLE 1

The values used in simulations

	Label	Value
Manipulator link length	L (m)	0·7
Width of the cross section	w (m)	0·002
Length of the cross section	h (m)	0·0255
Modulus of elasticity	E (Gpa)	71
Mass per unit length	m (kg/m)	0·1382
First natural frequency	ω_1 (rad/s)	21
Second natural frequency	ω_2 (rad/s)	132
Third natural frequency	ω_3 (rad/s)	372

the unknown coefficients, A_i, in equation (18) can be found. The solution of equation (11) is given as [6],

$$q_i(t) = q_i(0)\cos\omega_i t + \frac{1}{\omega_i}\dot{q}_i(0)\sin\omega_i t + \left(\frac{\Gamma_i}{M_i\omega_i^2}\right)\omega_i\int^{0} f(\xi)\sin\omega_i(t-\xi)\,\mathrm{d}\xi. \tag{20}$$

The initial conditions for the problem on hand are $q_i(0) = 0$ and $\dot{q}_i(0) = 0$, so only the third term remains as the solution. In this term $f(\xi)$ is the time dependent part of forcing function. That is,

$$f(\xi) = f(t)|_{t=\xi} = \ddot{\theta}(t)|_{t=\xi}, \tag{21}$$

where $\ddot{\theta}(t)$ is given by equation (23).

3. EXAMPLE

Cycloidal motion is selected as the trajectory function for the link rotation which is [8],

$$\theta(t) = \Delta\theta[t/t_p - (1/2\pi)\sin\omega_p t], \tag{22}$$

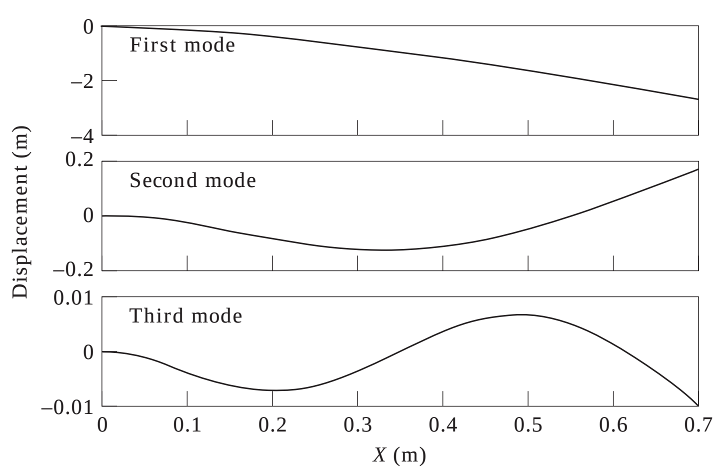

Figure 2. The first, second and third modes of the link.

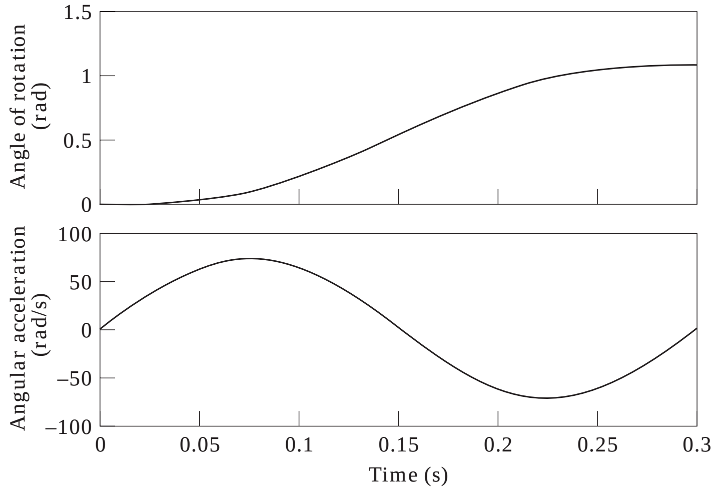

Figure 3. Cycloidal trajectory and angular acceleration for $t_p = 0{\cdot}3$ s.

where $\omega_p = 2\pi/t_p$ is the rise motion frequency and t_p is the rise time. The acceleration for the rise motion is the second derivative of the cycloidal function which is,

$$\ddot{\theta}(t) = \Delta\theta \, \frac{\omega_p^2}{2\pi} \sin \omega_p t = \theta_0 \sin \omega_p t, \tag{23}$$

in which $\Delta\theta$ is the angular path travelled. The third part in equation (20),

$$D_i(t) = \omega_i \int_0^t f(\xi) \sin \omega_i (t - \xi) \, \mathrm{d}\xi = \frac{\Delta\theta \omega_p^2}{2\pi} \omega_i \int_0^t \sin \omega_p \xi \sin \omega_i (t - \xi) \, \mathrm{d}\xi, \tag{24}$$

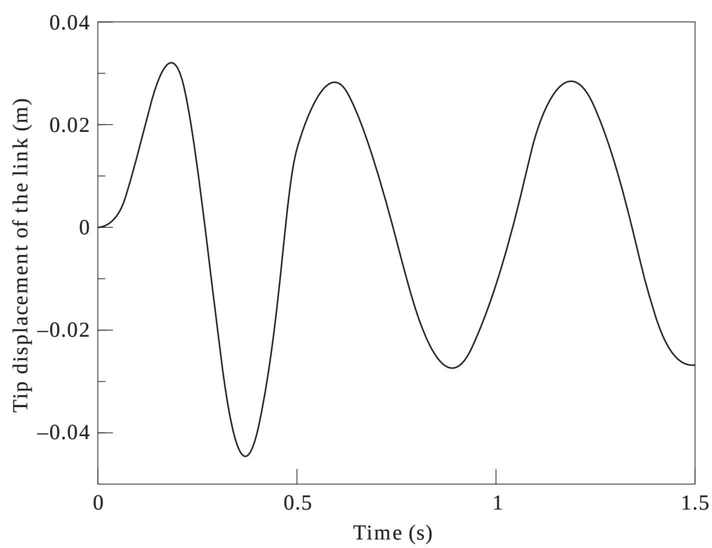

Figure 4. The vibration amplitude of the manipulator tip for $t_p = 0{\cdot}5$ s.

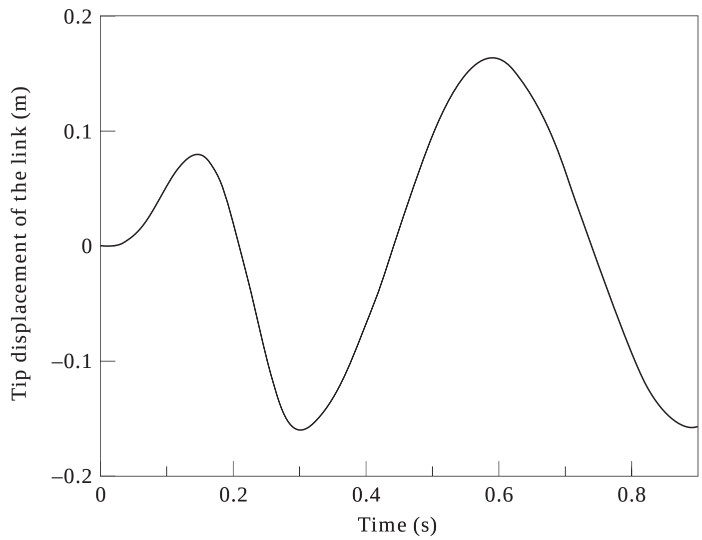

Figure 5. The vibration amplitude of the manipulator tip for $t_p = 0{\cdot}3$ s.

is defined as the dynamic load factor for the ith mode [6]. When the necessary calculations are carried out, $D_i(t)$ is found for $t < t_p$ as,

$$D_i(t) = U\{\sin \omega_i t - (\omega_i/\omega_p) \sin \omega_p t\}. \quad (25)$$

For $t > t_p$ equation (25) can be rewritten in the form

$$D_i(t) = U\{[a(1 - \cos 2\pi b) + b(1 - \cos 2\pi a)] \sin \omega_i t - [a \sin 2\pi b - b \sin 2\pi a] \cos \omega_i t\}, \quad (26)$$

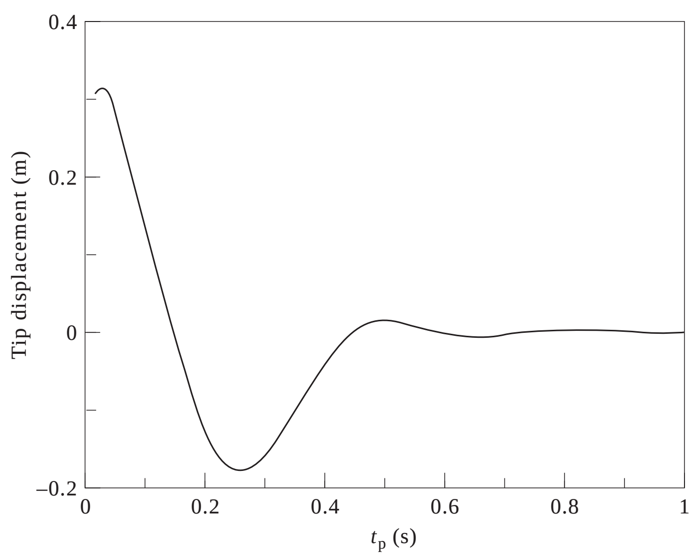

Figure 6. The tip displacement versus rise time at $t = t_{tp}$.

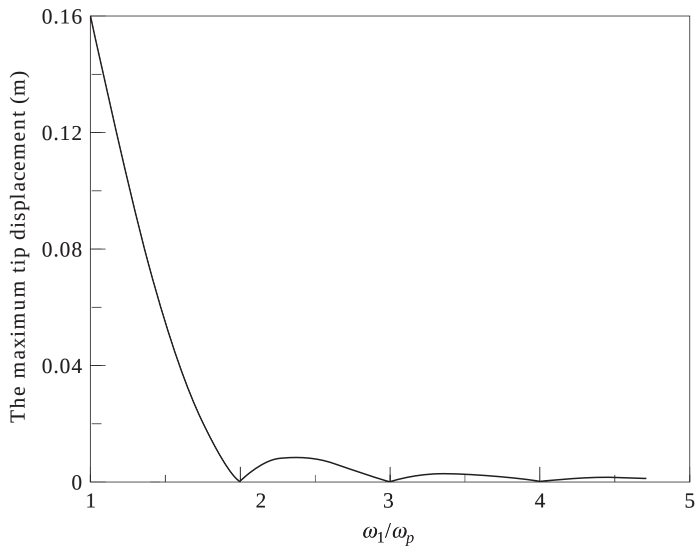

Figure 7. The maximum residual tip displacement versus ω_1/ω_p.

where

$$U = \ddot{\theta}_0(\omega_i/\omega_p)/[1 - (\omega_i/\omega_p)^2], \qquad a = 1 + (\omega_i/\omega_p) \qquad \text{and} \qquad b = 1 + (\omega_i/\omega_p)$$

If equations (25) and (26) are put into equation (20) separately, the solution for the generalized co-ordinates q_i's are obtained for the link before and after the rise time. As a result the solution $y(x, t)$ given in equation (3) is obtained.

3.1. *Results*

For simulation purposes the link length L, cross-section A, and the material are selected. The values are given in Table 1. The coefficients A_i's for the first three modes are 0·4251,

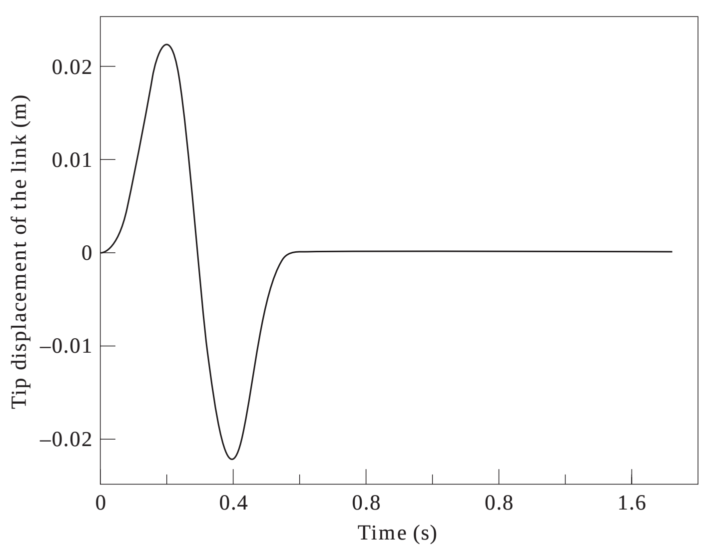

Figure 8. The vibration amplitude of the manipulator tip for $t_p = 0{\cdot}593$ s.

TABLE 2

Effects of the modes on total displacement

Modes	Mode I	Mode II	Mode III
Amplitude at $x = L$ (m)	$-2{\cdot}5$	$0{\cdot}2$	$-0{\cdot}01$

0·0016 and $3{\cdot}8449 \times 10^{-6}$, respectively. The mode participation factors, respectively, are $-0{\cdot}1202$, $-0{\cdot}1119$ and $-0{\cdot}0934$. The first three modes of the beam are shown in Figure 2. The graph of the cycloidal rise motion and the acceleration are given in Figure 3. A plot of the tip amplitude of the elastic link before and after $t_p = 0{\cdot}5$ s is shown in Figure 4. Since damping is not considered, residual vibration with constant amplitude appears at the end of the rise motion. In Figure 5 the tip vibrations occurring before t_p and after t_p are plotted for $t_p = 0{\cdot}3$ s. In Figure 6 the vibration amplitudes of the end point at $t = t_p$, for t_p values, are shown. From this figure it is seen that at certain t_p values the end point amplitudes become zero. Thus, it is possible to drive the link without causing residual vibration just by choosing an appropriate t_p. In Figure 7 the spectrum of the steady state vibration amplitudes for ω_1/ω_p is given. Here ω_1 is the first natural frequency of the link and ω_p is the frequency of the rise function. The t_p values at which zero amplitudes are obtained correspond to ω_1/ω_p ratios equal to 2, 3, 4. As an example, Figure 8 is drawn for $t_p = 0{\cdot}593$ s which corresponds to the ratio of 2. The steady state vibration amplitude becomes zero. During the calculations only the effect of first three modes are taken into account. The effects of the second and third modes with respect to the first ones are negligible as can be seen from Table 2.

4. CONCLUSION

In this study the vibrational behaviour of an elastic manipulator link which is driven by a servomotor is analyzed. The assumed trajectory function is cycloidal. Because of the sinusoidal acceleration caused by the cycloid the manipulator link is only inertially loaded which causes vibrations during the motion and also causes residual vibration. Since the inertial load acts only during the rise time t_p, the vibration problem is transient. The elastic link is assumed to be an Euler beam and the boundary conditions are taken as fixed–free. Mode summation techniques are used to find the solution. Since the second derivative of the cycloid is a sinusoidal function, an analytical solution is possible. A spectrum of the residual vibration amplitude of the link tip for the ratios of the rise motion frequency to the link natural frequency is obtained. It is shown that for certain values of the rise motion frequency, zero residual vibration is possible. Then precise position control or precise trajectory tracking of the elastic link is possible.

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