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FREE VIBRATION CHARACTERISTICS OF ISOTROPIC AND LAMINATED ORTHOTROPIC SPHERICAL CAPS

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This paper deals with the free vibration characteristics of isotropic and laminated orthotropic spherical caps. A first order shear deformable semi-analytical shell finite element has been used to obtain axisymmetric and asymmetric modes of vibration. Two geometric configurations, namely deep and shallow shells, have been considered in the analysis. Both thick and thin ranges of thickness ratio are studied. The effects of thickness ratio, shallowness, material orthotropy and lay-up have been studied. An attempt has been made to explain some of the vibrational characteristics by observing the energies in various modes of vibration.

1. INTRODUCTION

Shell structures are widely encountered in many engineering applications. The most commonly encountered types of shell structures are shells of revolution. With the advent of fibre reinforced composite materials, these structures are being made of composites, especially so in the aero-space industry. A great number of works can be found dealing with the analysis of laminated shells of revolution. Though the free vibration of circular cylindrical and conical shells has been given considerable attention in the literature, the analysis of shells of revolution with curved meridian such as spherical shells or paraboloidal shells are not dealt with extensively. Spherical shells are widely encountered in many structural applications. Spherical pressure vessels are a common phenomenon in the chemical and process industries. Cylindrical and conical pressure vessels with hemispherical or torispherical ends are also found in common practice. The use of composite materials for this type of structure is gaining popularity nowadays. Mirza [1] has reviewed the literature on vibration of layered shells. From this study, it can be seen that, though considerable attention has been given to the study of free vibration of cylindrical and conical shells, the amount of work on laminated spherical shell vibration is limited. Most of the studies available on the spherical shells are limited to isotropic shells. Hence the present study has been concerned with the free vibration characteristics of isotropic and laminated spherical caps.

Considerable attention has been given in the past to the free vibration of thin isotropic spherical shells. Naghdi [2] has derived equations for axisymmetric deformation of a thin isotropic shell of revolution including the effects of transverse shear deformation. Johnson and Reissner [3] have analyzed the transverse vibration of shallow spherical shells for asymmetric modes. They observed that there is an essential difference between the first meridional mode and higher modes for the case of two nodal diameters. For higher

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circumferential modes, transfer of energy takes place from bending to stretching as the curvature of the shell changes from the case of a flat plate to a higher value. Naghdi and Kalnins [4] have investigated axisymmetric and asymmetric vibration of thin elastic spherical shells. De Souza and Croll [5] have investigated the free vibration of isotropic spherical caps using the classical shell theory. They have studied the behaviour of shallow and deep spherical caps by observing the distribution of energy in various modes of vibration. Approximate solutions for the natural frequencies and mode shapes for the axisymmetric vibration of spherical domes have been given by Kunieda [6]. Singh and Mirza [7] have presented extensive results on the free vibration of spherical caps in asymmetric modes. A finite element including the effects of transverse shear deformation and rotary inertia was used in the study.

In contrast to the case of isotopic spherical shells, the literature available on the vibration of laminated spherical shells is limited. Rath and Das [8] have analyzed the axisymmetric vibration of closed layered spherical shells including the effects of transverse shear and rotational inertia. They obtained frequencies corresponding to torsional and axixymmetric flexural motions. Chao *et al*. [9] have studied the axisymmetric free fibration of moderately thick polar–orthotropic hemispherical shells under different boundary conditions such as sliding, guided-pin, clamped and hinged edges. They have studied the effects of thickness, fibre direction and orthotropy on the free vibration of hemispherical shells. Similar studies on the asymmetric vibration of layered spherical shells are not found in the literature. As the understanding of the free vibrational behaviour is necessary for achieving better designs, a comprehensive study on the vibration of single layered and multilayered composite spherical caps has been carried out and is reported in this paper.

2. THEORY AND FORMULATION

2.1. FSDT SHELL ELEMENT

The geometry of the shell considered is a surface of revolution with an arbitrarily curved meridian. The notation for the co-ordinates is shown in Figure 1. The distance of any point from the axis of revolution is r and φ is the angle between the normal to the shell surface and the axis of revolution. The co-ordinate measured along the meridian is s. The distance of any point from the shell mid-surface along the normal is z. The co-ordinate along the circumferential direction is θ. The displacement of any point in the s, θ and z directions are U, V and W respectively and the corresponding reference surface displacements are u, v and w respectively. The rotations of the normal in meridional and circumferential directions are represented by α and β. R_φ and R_θ are the radii of curvature in the meridional and circumferential directions. For an axisymmetric surface R_θ, r and φ are related as follows:

$$R_\theta = r/\sin\varphi, \qquad \partial r/\partial\varphi = r\cos\varphi. \tag{1}$$

For a spherical surface, R_φ and R_θ are constant and equal to the radius of the shell R. The total thickness of the shell is represented by h.

In the first order shear deformation theory an independent rotation of the normal is allowed to accommodate transverse shear. The displacements at a point z away from the shell mid surface are expressed in terms of mid-surface displacements and rotations as

$$U(s,\theta,z,t) = u(s,\theta,t) + z\alpha(s,\theta,t), \qquad V(s,\theta,z,t) = v(s,\theta,t) + z\beta(s,\theta,t),$$

$$W(s,\theta,z,t) = w(s,\theta,t). \tag{2–4}$$

The displacements u, v, w, α and β can be described by Fourier series expansions in the circumferential direction as

$$u(s,\theta,t)=\sum_{n=0}^{\infty}u^N(s,t)\cos(N\theta),\qquad v(s,\theta,t)=v_0(s,t)+\sum_{n=1}^{\infty}v^N(s,t)\sin(N\theta),$$

$$w(s,\theta,t)=\sum_{n=0}^{\infty}W^N(s,t)\cos(N\theta),\qquad \alpha(s,\theta,t)=\sum_{n=0}^{\infty}\alpha^N(s,t)\cos(N\theta),$$

$$\beta(s,\theta,t)=\beta_0(s,t)+\sum_{n=1}^{\infty}\beta^N(s,t)\sin(N\theta). \tag{5}$$

By substituting these expansions in the strain-displacement relations, the strain-displacement relations corresponding to the Nth harmonic can be obtained. They are given by

$$\varepsilon_s^N=\left\{\frac{\partial u^N}{\partial s}+z\frac{\partial\alpha^N}{\partial s}+\frac{w^N}{R_\varphi}\right\},$$

$$\varepsilon_\theta^N=\left\{\frac{\cos\varphi}{r}u^N+\frac{N}{r}v^N+\frac{\sin\varphi}{r}w^N+\frac{z\cos\varphi}{r}\alpha^N+\frac{Nz}{r}\beta^N\right\}, \tag{6, 7}$$

$$\gamma_{s\theta}^N=\left\{\frac{\partial v^N}{\partial s}+z\frac{\partial\beta^N}{\partial s}-\frac{N}{r}u^N+\frac{v^N}{r}\frac{\partial r}{\partial s}-\frac{Nz}{r}\alpha^N-\frac{z}{r}\frac{\partial r}{\partial s}\beta^N\right\}, \tag{8}$$

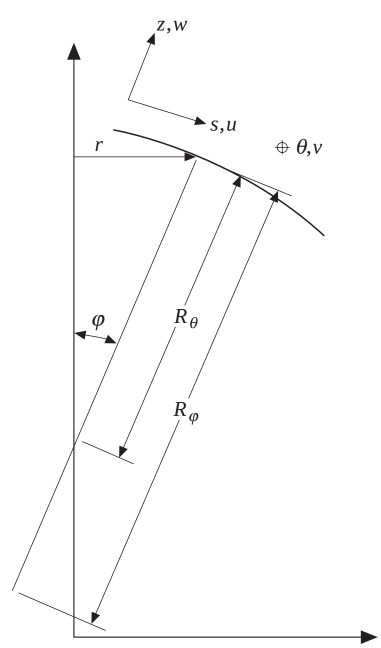

Figure 1. Notation of co-ordinates and displacements.

$$\gamma_{sz}^N = \left\{ -\frac{u^N}{R_\varphi} + \frac{\partial w^N}{\partial s} + \alpha^N \right\}, \qquad \gamma_{\theta z}^N = \left\{ -\frac{v^N}{R_\theta} - \frac{N w^N}{r} + \beta^N \right\}. \tag{9, 10}$$

A three noded finite element can be developed by using these strain-displacement relations. Standard isoparametric shape functions are used for the formulation of element matrices. The shape functions are

$$N_1 = (\xi^2 - \xi)/2, \qquad N_2 = 1 - \xi^2, \qquad N_3 = (\xi^2 + \xi)/2. \tag{11}$$

By using these expressions, the stiffness and mass matrices can be expressed as

$$[K^N]^e = C^N \pi^* \int_s \int_z [B^N]^{\mathrm{T}}[D][B^N] r \, \mathrm{d}s \, \mathrm{d}z, \qquad [M^N]^e = C^N \pi^* \int_s \int_z [N]^{\mathrm{T}}[N] \rho r \, \mathrm{d}s \, \mathrm{d}z \quad (12, 13)$$

The constitutive matrix $[D]$ is given by

$$[D] = \begin{bmatrix} \bar{Q} & 0 & 0 \\ 0 & kG_{13} & 0 \\ 0 & 0 & kG_{23} \end{bmatrix}, \tag{14}$$

where $[\bar{Q}]$ is the standard plane stress constitutive matrix and G_{13} and G_{23} are the transverse shear moduli. k is a shear correction factor. In the present study k is taken as 5/6. The $[D]$ matrix varies across the thickness for a laminated shell.

2.2. EVALUATION OF FREQUENCIES

The governing equation of free vibration for the Nth circumferential mode is

$$[[K^N] - (\omega^N)^2 [M^N]]\{U\} = \{0\}, \tag{15}$$

where $[K^N]$ and $[M^N]$ are global stiffness and mass matrices corresponding to the circumferential mode number N. By solving equation (15) the free vibration frequencies ω^N can be obtained. In the present analysis the frequencies are evaluated by using the simultaneous iteration technique.

2.3. EVALUATION OF ENERGY

Evaluation of potential energy provides insight towards understanding the vibrational behaviour and identification of the modes of vibration. The shell possesses a finite energy during the vibratory motion due to an applied load. In the case of free vibration, the displacements of the shell obtained for different modes are only the relative amplitudes. Hence the evaluation of energy with these amplitudes gives an indication of the energy distribution between various modes and not the exact values. The energy can be primarily divided into three parts consisting of membrane, bending and transverse shear energies. The energies in membrane and bending are obtained by separating the in-plane strains due to extension and bending and evaluating the relative energy. Similarly transverse shear energy, hereafter referred to as shear energy, is obtained from the terms corresponding to the transverse shear. The ratios of membrane, bending and shear energies to the total energy ($\bar{U}$) are denoted by U_M, U_B and U_S respectively. Further, the energy ratios of the five strains are defined as:

$$U_s = \int \tfrac{1}{2}\sigma_s\varepsilon_s \, \mathrm{d}V/\bar{U}, \qquad U_\theta = \int \tfrac{1}{2}\sigma_\theta\varepsilon_\theta \, \mathrm{d}V/\bar{U}, \qquad U_{s\theta} = \int \tfrac{1}{2}\sigma_{s\theta}\varepsilon_{s\theta} \, \mathrm{d}V/\bar{U},$$

$$U_{sz} = \int \tfrac{1}{2}\sigma sz\varepsilon_{sz} \, \mathrm{d}V/\bar{U}, \qquad U_{\theta z} = \int \tfrac{1}{2}\sigma_{\theta z}\varepsilon_{\theta z} \, \mathrm{d}V/\bar{U}. \tag{16}$$

3. RESULTS AND DISCUSSION

Free vibration characteristics of spherical caps with various geometric and material parameters are investigated in this section. First, a study of isotropic spherical caps has been carried out. Energy contributions from different modes have been evaluated for carrying out such a study. Subsequently, the free vibration characteristics of single layered composite spherical caps have been studied. Later, studies on lay-up, shallowness and orthotropy are presented.

In the following discussion a *deep shell* refers toa a hemispherical shell ($S/a = 1$) and a *shallow shell* refers to a shell with $S/a = 0{\cdot}1$, where S is the height of the apex above the base and a is the base radius as shown in Figure 2. These two configurations are chosen such that the two near extreme cases of curvature effects are covered. The half-angle subtended by the shell at the centre of the arc ϕ is 90° for the shell with $S/a = 1$ and for the shell with $S/a = 0{\cdot}1$ it is 11·42°.

In the present study, mild steel is considered for the example of an isotropic shell. Its properties are given by $E = 2{\cdot}1 \times 10^7$ N/cm^2 and $v = 0{\cdot}3$. The properties of the composite material considered are as follows: material, graphite/epoxy [10], $E_L = 13{\cdot}8 \times 10^6$ N/cm^2, $E_T = 1{\cdot}06 \times 10^6$ N/cm^2, $G_{LT} = 0{\cdot}6 \times 10^6$ N/cm^2, $G_{TT} = 0{\cdot}39 \times 10^2$ N/cm^2, $v_{LT} = 0{\cdot}28$, $v_{TT} = 0.34$, $\rho = 1{\cdot}5$ kg/m^3.

3.1. VALIDATION OF RESULTS

The FSDT element discussed in section 2 was used for obtaining the frequencies for axisymmetric and asymmetric modes of vibration. A preliminary convergence study was carried out for a few typical cases to estimate the number of elements needed for the study. The frequencies obtained by using the present finite element are validated in this section by comparisons with reported results.

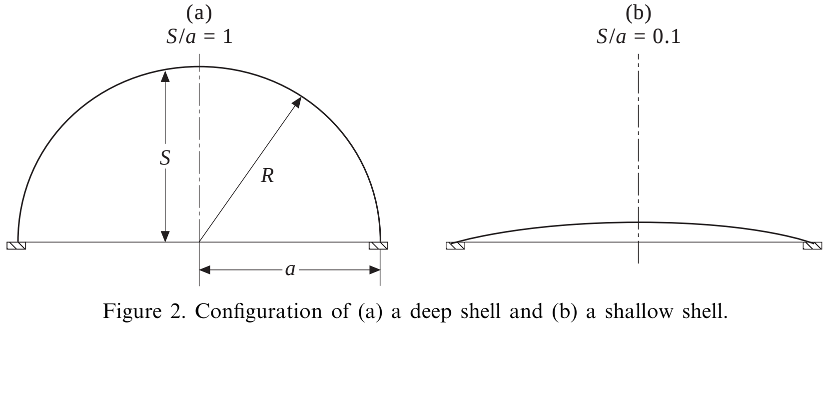

Figure 2. Configuration of (a) a deep shell and (b) a shallow shell.

TABLE 1

Comparison of non-dimensional frequencies of an isotropic clamped hemispherical cap with those of reference [11]; $\Omega = \omega R(\rho(1-\nu^2)/E)^{1/2}$

Theory	Mode 1	2	3	4
FSDT	0·8050	1·175	1·508	1·838
Ref. [11]	0·809	1·176	1·517	1·854

3.1.1. *Problem* 1

The axisymmetric frequencies of a clamped isotropic hemispherical shell with $h/R = 0{\cdot}1$ are compared with the reported results of Tessler and Spirichigliozzi [11]. These values are shown in Table 1. The frequencies predicted by the present element are in good agreement with the reported results.

3.1.2. *Problem* 2

The axisymmetric free vibration frequencies of a three layered cross ply [0/90/0] complete spherical shell with $h/R = 0{\cdot}1$ given by Rath and Das [8] are compared with the results obtained by the present study and are presented in Table 2. The three layers are of equal thickness. The material properties of the shell are: layers I and III; $E_L = 9 \times 10^6$ lb/in^2, $E_T = 2 \times 10^6$ lb/in^2, $\nu_{LT} = 0{\cdot}3$, $G_{LT} = 0{\cdot}75 \times 10^6$ lb/in^2, $G_{TZ} = 0{\cdot}7 \times 10^6$ lb/in^2, $G_{LZ} = 0{\cdot}8 \times 10^6$ lb/in^2, $\rho = 0{\cdot}193 \times 10^{-3}$ lb s^2/in^4; layer II, $E_L = 20 \times 10^6$ lb/in^2, with all other properties the same as those of layer I. It can be seen from Table 2 that the frequencies are in good agreement.

3.1.3. *Problem* 3

The asymmetric frequencies of a clamped 60° isotropic shell are compared with the results reported by Singh and Mirza [7] for a shell with $h/R = 0{\cdot}05$. The frequencies are tabulated in Table 3 and can be seen to be in good agreement.

3.2. BOUNDARY CONDITIONS AND NON-DIMENSIONALIZATION

In the present study only clamped spherical caps are considered. The outer edge of the spherical cap is assumed to be clamped. At the apex, the boundary conditions are given from the symmetry considerations. The displacement boundary conditions are given $u = v = \beta = 0$ at the apex, and $u = v = w = \alpha = \beta = 0$ at the clamped edge.

TABLE 2

Comparison of non-dimensionl frequencies of orthotropic three-layered complete spherical shell with those of reference [8]; $\Omega = \omega R(\rho_A/A_{22})^{1/2}$

Theory	Mode 2	3	4	5
FSDT	0·8034	0·9398	1·048	1·197
Ref. [8]	0·8181	0·9837	1·141	1·355
Classical [8]	0·8204	1·0047	1·203	1·502

TABLE 3

Comparison of non-dimensional asymmetric frequencies of isotropic clamped 60° cap with those of reference [7]; $\Omega = \omega R(\rho(1-\nu^2)/E)^{1/2}$

Theory	Mode (N, m) 1, 1	1, 2	2, 1	2, 2	4, 1	4, 2
FSDT	0·8529	1·145	1·025	1·397	1·325	1·994
Ref. [7]	0·8576	1·1535	1·0293	1·4076	1·3289	2·0059

The frequencies are non-dimensionalized as

$$\Omega = \omega R\sqrt{\rho/E}, \quad (17)$$

where ω is the frequency in rad/s, ρ is the density, E is the Young's modulus, R is the radius of the shell and Ω is the non-dimensional frequency. In the case of a laminated composite shell E is replaced by E_T, the modulus in the matrix direction of the bottom-most lamina: then

$$\Omega = \omega R\sqrt{\rho/E_T}. \quad (18)$$

3.3. ISOTROPIC SHELLS

This section deals with the free vibration characteristics of isotropic spherical caps. Three thickness ratios of $a/h = 100$, 10 and 5 are considered for the analysis. Both shallow and deep shells are studied for their vibrational behaviour. First thirteen circumferential modes are obtained for each case. The frequencies corresponding to the first two meridional modes and thirteen circumferential modes are presented in tables. To study the vibrational characteristics and identify the modes of vibration, the energies of membrane, bending and shear are evaluated and their magnitudes relative to the total energy are plotted against the circumferential wave number for the first meridional mode. Also, the energies from the strains in different directions as given in equations (16) are evaluated and their variation with N is plotted.

3.3.1. *Deep shell*

Table 4 shows the non-dimensional frequencies corresponding to the first two meridional ($m = 1, 2$) and thirteen circumferential ($N = 0, 1, \ldots, 12$) modes of a clamped deep isotropic spherical shell for three different thickness ratios. It can be seen that the lowest frequency occurs at $N = 1$ for all the three thickness values considered here. The mode of vibration corresponding to $N = 1$ along the axis of symmetry will be similar to that of a beam in bending. At $N = 0$, the circumference will expand about the axis of revolution. In the case of a deep shell, the distance between the height of the apex above the clamped edge is considerable, and hence the beam-like mode is expected to be more flexible. This will lead to a lower frequency at $N = 1$.

Figure 3 shows the variation with N of energy contributions from membrane, bending and shear deformations and also the variation of energy from strain in different directions for the first meridional mode. It can be observed that for deep shells, the lower circumferential modes have predominant membrane effects. As a higher N is reached, the influence of bending increases. In the case of thicker shells, the bending energy increases at a lower value of N (see Figure 3). The value of N at which the bending energy ratio crosses the membrane energy ratio is higher for thin shells. But beyond a certain N the fraction of bending energy comes down with an increase in shear energy contribution. It

TABLE 4

Non-dimensional frequencies of clamped deep isotropic spherical cap for different thickness ratios

	a/h					
	100		10		5	
N	1	2	1	2	1	2
0	0·767	0·956	0·843	1·221	0·944	1·218
1	0·716	0·917	0·785	1·068	0·835	1·285
2	0·900	0·992	0·966	1·301	1·104	1·750
3	0·948	1·020	1·127	1·571	1·427	2·234
4	0·970	1·044	1·315	1·881	1·806	2·734
5	0·986	1·069	1·559	2·234	2·246	3·259
6	1·001	1·106	1·858	2·626	2·734	3·806
7	1·016	1·116	2·206	3·051	3·255	4·369
8	1·033	1·117	2·596	3·505	3·802	4·945
9	1·053	1·143	3·020	3·983	4·366	5·530
10	1·078	1·187	3·474	4·482	4·942	6·124
11	1·109	1·235	3·953	4·999	5·527	6·725
12	1·146	1·268	4·452	5·531	6·118	7·334

can also be seen from Figure 3 that the energy due to circumferential strain (U_θ) is higher for thin shells at low values of N, i.e., at $N = 0$ and $N = 1$, whereas U_s is higher for thicker shells. However, the influence of U_θ increases as N increases. This is due to the increased bending effects in the circumferential direction arising from the higher circumferential modes.

It has been observed that $U_{s\theta}$ has a peak value at $N = 1$ for all the thickness ratios considered. $U_{s\theta}$ will be zero for $N = 0$ as the in-plane shear strain corresponding to the axisymmetric mode will be zero. However, the value of $U_{s\theta}$ will be non-zero at $N = 1$. As N increases, the contribution of $U_{s\theta}$ decreases. This is due to the increase in the stiffness from the formation of lobes in the circumferential direction as N increases. The lowest meridional frequency at $N = 12$ for $a/h = 100$ is approximately 1·6 times the lowest frequency ($N = 1$). The second meridional frequency at $N = 1$ is 1·3 times and the fifth meridional frequency is 1·6 times the lowest frequency. It is observed that the lowest meridional frequency at a very high circumferential wave number of $N = 30$ gives a frequency of 4·4 times the lowest frequency. This relatively small change in frequency can be attributed to the predominant membrane action governing the vibration. Increase in the value of N introduces additional bending and shear in the circumferential direction. However, at lower values of thickness, the additional bending and shear energies will be small and hence the increase in the net energy is also small. This leads to the frequencies being closely spaced. This implies that the higher circumferential modes also play a vital role in the dynamic characteristics of thin deep spherical shells. At a thickness of $a/h = 10$, the ratio of the lowest meridional frequency corresponding to $N = 12$ to the lowest frequency is approximately 4·5 which can still play an important role in the transient dynamic characteristics. As the bending energy varies as the third power of the thickness, the influence of bending increases at higher values of thickness.

The ratio of lowest frequencies corresponding to $a/h = 5$ and $a/h = 100$ is approximately 1·2. This is a rather small increase for an increase of thickness by 20 times. It is due to the predominant membrane action present in the shell vibration as can be seen from

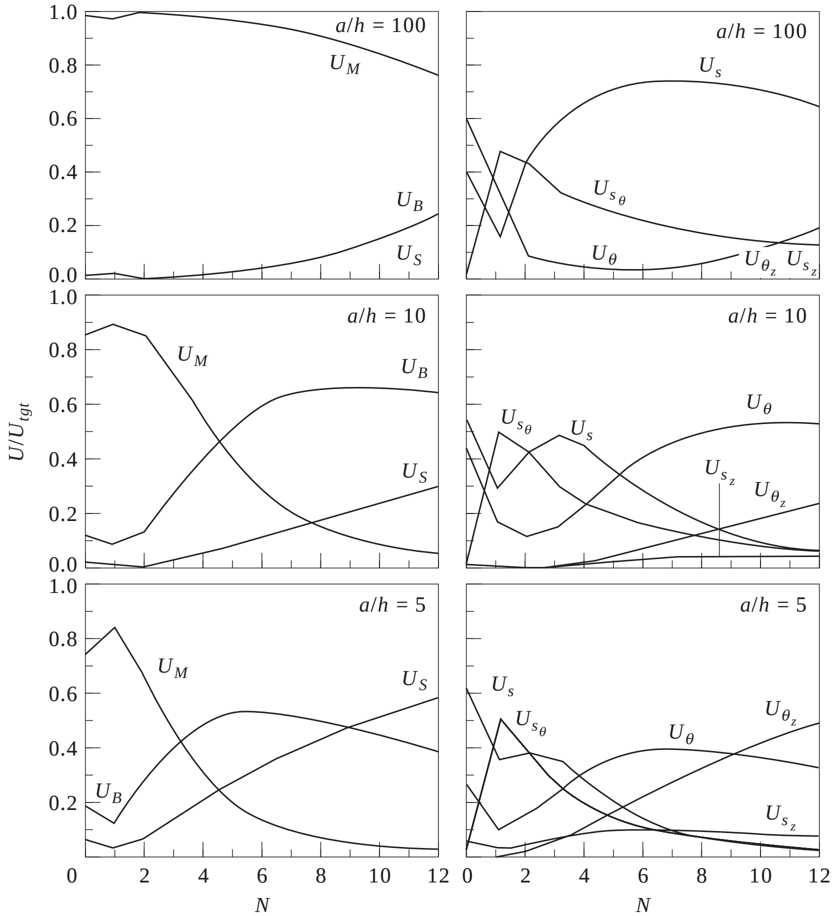

Figure 3. Energy contributions from various terms for deep isotropic shells.

Figure 3. The stiffness as well as the mass in membrane mode are proportional to the thickness of the shell and hence the membrane frequency is independent of the thickness. Only at higher circumferential modes, a considerable difference of frequencies at these two thickness ratios is observed (5·3 at $N = 12$).

The first meridional mode shapes corresponding to various values of N are plotted in Figure 4. The displacements are normalized to have a peak amplitude of magnitude one. While studying the mode shapes of the shell, it has to be kept in mind that, unlike in the case of plates, for shells the transverse normal displacement w cannot be considered to be representative of bending. A transverse normal displacement, in the case of a curved surface causes both bending and membrane actions. An extreme case is that of a full spherical shell loaded by an uniform internal pressure which will have only the transverse normal displacement at all the points on the surface though the load is supported by only the membrane action.

It can be seen that the mode shapes corresponding to thick and thin shells are similar with a noticeable difference only at $N = 1$. It can be noted that $N = 1$ corresponds to the lowest frequency. Also it can be seen from Figure 3 that at $N = 1$ the energy corresponding to U_s has a minimum. It has been seen earlier that various theories differ considerably at $N = 1$. It is observed from the mode shapes that the amplitudes corresponding to in-plane displacements u and v are small at higher values of N though they have considerable influence at lower circumferential modes. The decrease in the amplitudes of u and v with increase in N can be attributed to the increase in bending action with increase in circumferential mode number. Another feature to be noted from these figures is that the location of peak amplitude shifts towards the clamped edge with increase in the circumferential wave number. At these modes, the displacements near the apex remain almost zero. The strain-displacement relations contain terms corresponding to N/r. As the apex is approached, r tends to zero and hence the corresponding displacements also have to tend to zero for a finite strain. This can be seen in Figure 4 where all the displacements except at $N = 0$ are zero at the apex. As the value of N increases, the zone of influence of $1/r$ increases and hence the displacements over a longer meridional length from the apex tend to be low. The high amplitude near the clamped edge is an indication that, for increasing the frequencies, increasing the stiffness near the clamped edge will be effective. This can possibly be achieved through an increase in the thickness at the clamped edge or by introducing stiffeners in this zone.

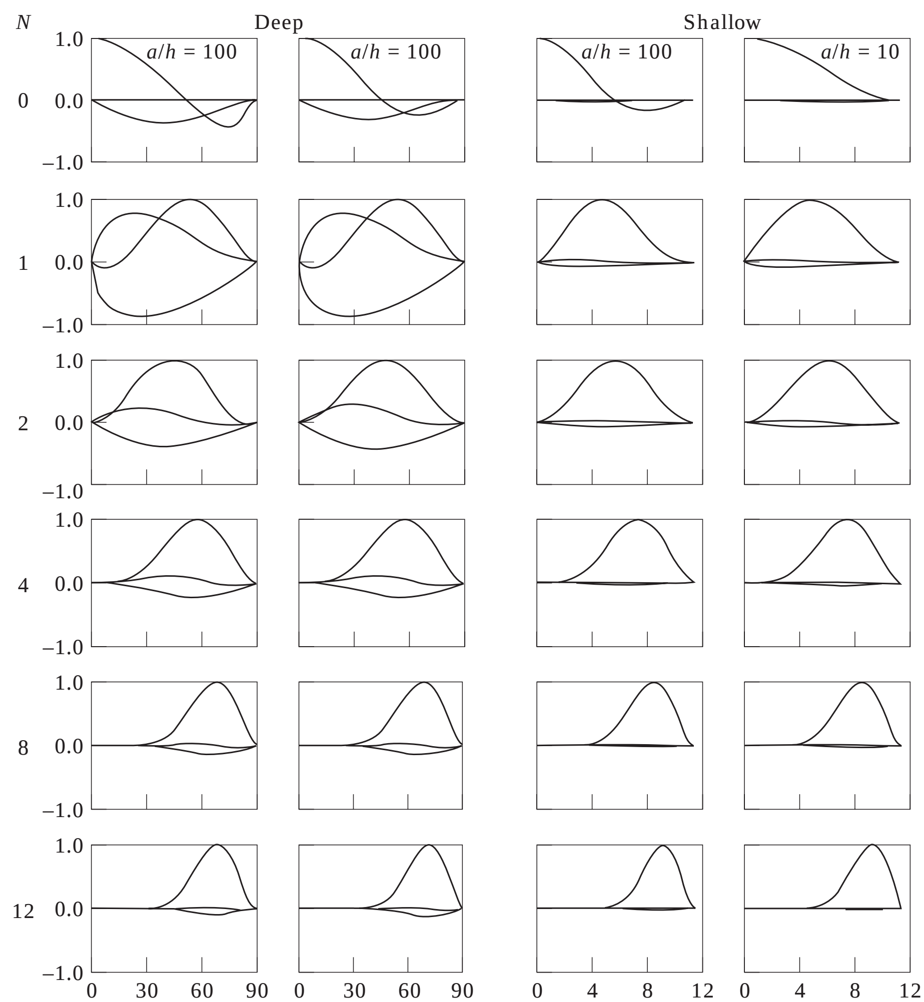

Figure 4. Mode shapes for isotropic shells ($m = 1$).

3.3.2. *Shallow shell*

The non-dimensional frequencies of a shallow spherical shell of $S/a = 0{\cdot}1$ for the first two meridonal modes and thirteen circumferential modes are given in Table 5. The energy plots are given in Figure 5 and the corresponding mode shapes are given in Figure 4.

It can be seen from the table that the lowest frequency occurs at $N = 0$ for all three thickness ratios considered. It has been noted earlier that the lowest frequency occurs at $N = 1$ for deep shells. In the case of shallow shells, the height of the apex over the clamped edge is very low and hence the resistance for bending in $N = 1$ will be higher. This causes the lowest frequency to be at $N = 0$. It can be observed from the energy plots that the shallow shell vibration is predominantly governed by the bending and shear energies. The contribution of membrane is felt only at lower circumferential modes of thin shells. At $a/h = 100$ ($R/h = 505$) the contribution of meridional energy is higher than that of bending energy for $N < 5$ only. However, the contribution from the shear is very low at this thickness. In the case of thick shells, The membrane energy drops rapidly from $N = 0$ to $N = 2$ and approaches almost zero beyond this value. It can be seen from the strain-displacement relations that the contribution of w to the membrane strain is proportional to $1/R$. As the shallowness of the shell increases the corresponding radius of curvature increases and this in turn decreases the contribution of transverse displacement to the in-plane strain and hence the membrane effects are low.

The contribution of bending increases initially as N increases and beyond some value of N (e.g., $N = 2$ for $a/h = 10$ and $N = 1$ for $a/h = 5$) the bending energy ratio decreases. This drop is due to the increase in the contribution of transverse shear energy to the total energy. At very high circumferential modes, the contribution of shear dominates the other two. The major contribution to this shear energy comes from the shear in the circumferential direction. It can be seen from the figure that the energy contribution due to the strain in the meridional direction is predominant at lower circumferential modes. However, as N increases, the values of U_θ and U_{θ_z} increase. The predominant contribution to the transverse shear energy comes from U_{θ_z}.

TABLE 5

Non-dimensional frequencies of clamped shallow isotropic spherical cap for different thickness ratios

	a/h					
	100		10		5	
N	1	2	1	2	1	2
0	1·119	1·473	2·043	5·611	3·103	9·186
1	1·216	1·424	3·342	8·310	5·584	11·53
2	1·179	1·650	4·960	10·95	8·207	16·19
3	1·301	1·977	6·985	13·85	11·16	20·14
4	1·481	2·366	9·224	16·87	14·25	23·83
5	1·721	2·809	11·64	19·98	17·42	27·50
6	2·016	3·301	14·20	23·16	20·66	31·15
7	2·361	3·839	16·88	26·39	23·93	34·79
8	2·753	4·420	19·66	29·67	27·23	38·41
9	3·189	5·044	22·52	32·99	30·55	42·00
10	3·665	5·707	25·46	36·33	33·87	45·57
11	4·181	6·410	28·45	39·70	37·18	49·13
12	4·735	7·151	31·49	43·09	40·50	52·67

Mode shapes of a shallow shell also follow a pattern similar to that of a deep shell except at $N = 1$. A feature that can be noticed is that the peak amplitudes in the u and v directions are much less than those of a deep shell.

The variation of frequency with N is significant in the case of shallow shells even at low values of thickness ratio. For $a/h = 100$, the ratio of frequencies at $N = 12$ to the lowest frequency is approximately 4·2 as against 1·6 for the case of a deep shell. The ratio for a thick shell with $a/h = 5$ is approximately 12·9. This is due to the predominant effect of bending. Also, the frequency increases considerably with the thickness. The ratio of lowest frequencies for $a/h = 5$ and $a/h = 100$ is 2·8 as against 1·2 in the case of deep shells. Also, for a particular N, frequencies vary considerably with the meridional mode.

3.4. ORTHOTROPIC SINGLE LAYERED SHELLS

In the previous section, the free vibrational behaviour of deep and shallow spherical shells has been discussed. This section deals with the vibrational characteristics of a single layered composite (graphite epoxy) shell with fibre along the meridian or circumference. The material properties are as above. Both deep and shallow shells are discussed together.

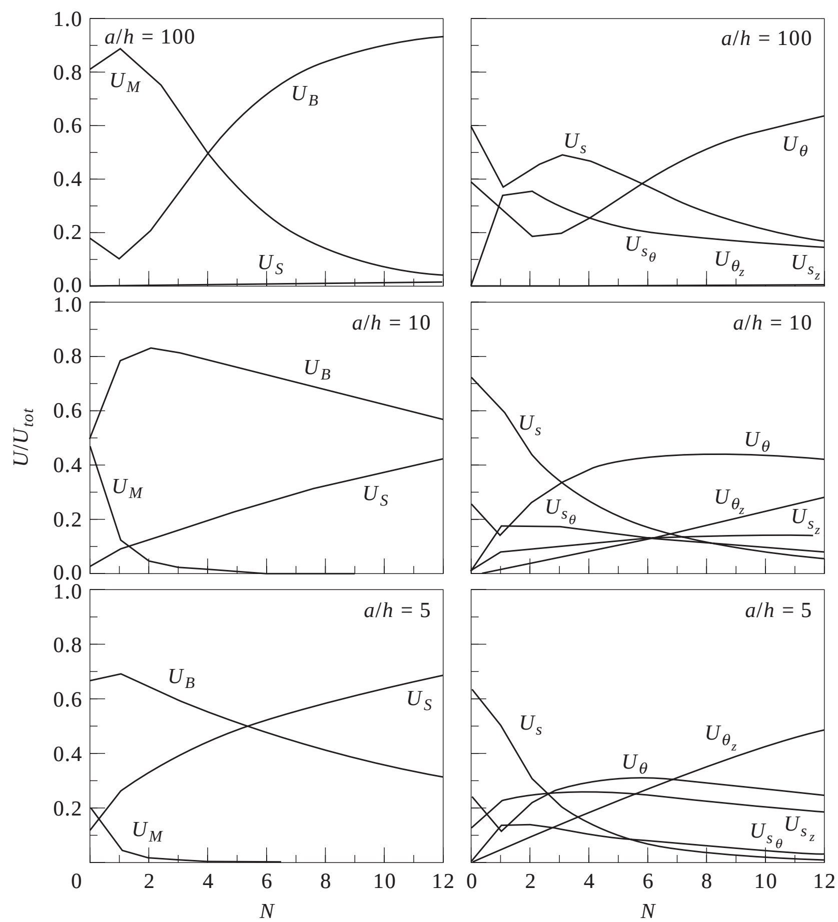

Figure 5. Energy contributions from various terms for shallow isotropic shells.

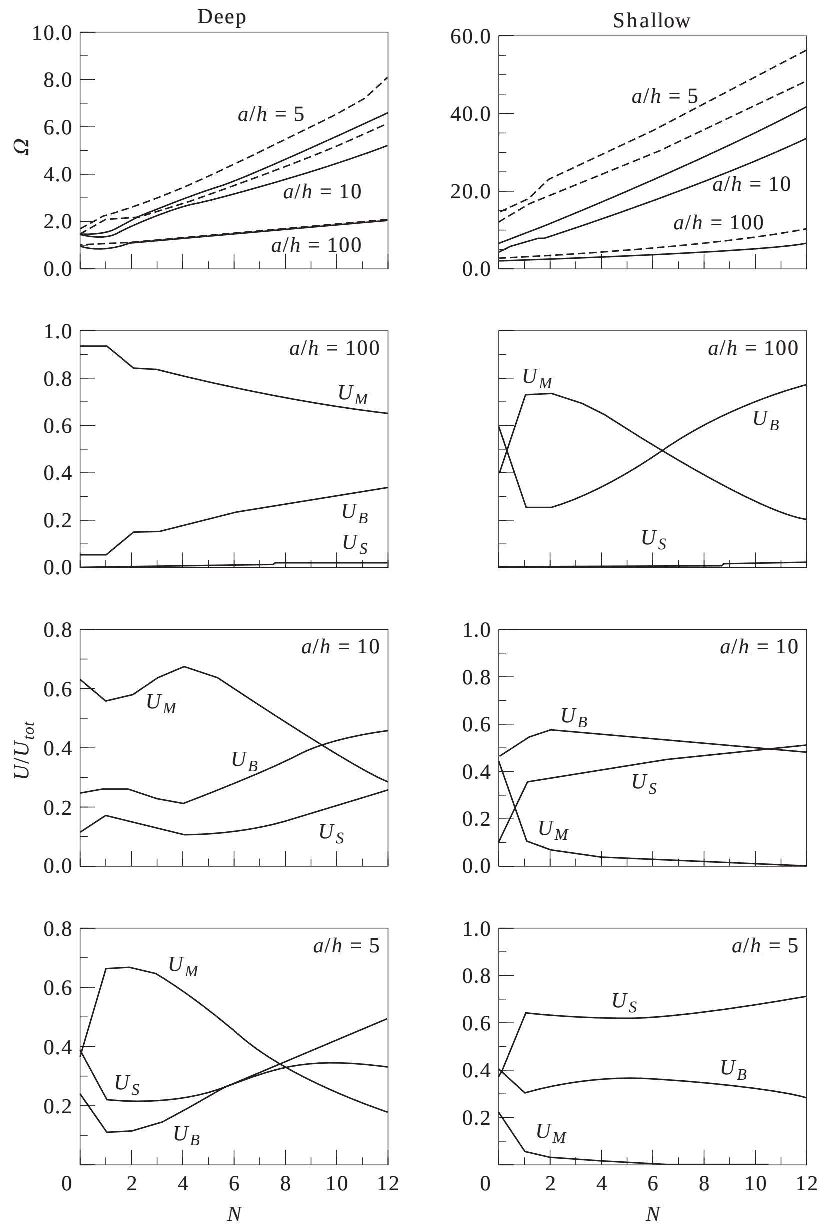

Figure 6. Variations of frequencies and energy ratios for shells with fibre along the meridian: —, $m = 1$; – – –, $m = 2$.

3.4.1. *Meridional fibre orientation*

The variation of the first two non-dimensional meridional frequencies with N for both deep and shallow shells at three a/h ratios of 100, 10 and 5 are plotted in Figure 6. The variation with N of the contributions from meridional, bending and transverse shear to the total energy of the shell for the first meridional mode are also shown in Figure 6. As the fibre is oriented along the meridian, the modulus in the meridional direction is much higher than the circumferential direction.

It is observed from the frequency variation that the lowest free vibration frequency occurs at $N = 1$ for deep shells with thickness ratios $a/h = 100$ and 10. At $a/h = 5$, the deep shell has its lowest frequency at $N = 0$. It has been observed that in the case of a deep isotropic shell the lowest frequency occurs at $N = 1$ at all three thickness ratios considered. As the fibre is in the meridional direction, the stiffness corresponding to $N = 1$, the beam-like mode, increases while the expansion of the circumferential direction decreases due to the low modulus. Hence, as the thickness increases, the lowest frequency shifts from $N = 1$ to $N = 0$. In the case of a shallow shell, the lowest frequency occurs at $N = 0$ for all three values of thickness considered here. It can be observed from the frequency variation that the frequencies corresponding to higher thickness ratios increase steeply with N. This is due to the increased effect of bending and shear.

It can be observed by comparing the energy plots of deep isotropic and meridional fibre shells that the membrane energy contribution, though the predominant one at lower circumferential modes even for fibre-reinforced deep shells, is a smaller fraction of the total energy than that of an isotropic shell. In the case of a shallow shell with meridional fibre, the energy plots follow a pattern similar to that of an isotropic shell. The variation of membrane energy is very similar to that of an isotropic shell. The contribution of shear energy increases significantly for fibre reinforced shells. Even at lower circumferential modes, the influence is much higher for the composite shell, unlike in the case of an isotropic shell.

The ratio of the first meridional frequency corresponding to $N = 12$ to the lowest frequency for a deep shell with $a/h = 100$ is approximately 2·3 and for $a/h = 10$ it is 3·8. The corresponding values for a shallow shell are 3·04 and 6·04. The ratio of lowest frequencies corresponding to $a/h = 5$ and 100 are 1·57 and 3·1 for deep and shallow shells respectively. These values are higher than those for an istropic shell, but still are significantly low.

3.4.2. *Circumferential fibre orientation*

In the previous section, the vibrational characteristics of a shell with fibre along the meridian are discussed. The next configuration considered is that of a shell with fibre oriented along the circumference (perpendicular to the meridian). Figure 7 shows the variation of the first two meridional frequencies and the corresponding variation of energy ratios for the first meridional mode with N.

It is observed from the frequency plots that the lowest frequency occurs at $N = 1$ for a deep shell and at $N = 0$ for a shallow shell. This behaviour is similar to that observed in the case of an isotropic shell. It can be seen from the energy variation that the membrane energy drops more rapidly with the increase in N than that of a shell with meridional fibre. In the case of deep shells, the membrane action is predominant at lower values of N. Even at a thickness of $a/h = 5$, the membrane action at $N = 0$ and $N = 1$ is very high. This is clearly reflected in the frequency variation. The ratio of lowest frequencies for a deep shell with thickness ratios $a/h = 5$ and 100 is 1·05 while it is 1·57 for a shell with meridional fibre. The bending energy ratio, except in the case of a thin deep shell, shows an increase initially and then decreases. As explained earlier, with the increase in the circumferential mode number the bending action in the circumferential direction increases and this in turn increases the bending energy. However, as the value of N increases further, the contribution of shear increases and this causes a fall in the relative magnitude of bending energy though the absolute value increases. At very high values of N, the shear energy becomes predominant for thick shells ($a/h = 10$ and 5).

For shallow shells, except for $a/h = 100$, the shear energy becomes predominant at a considerably low value of N. Even for the axisymmetric mode the shear contribution is significant for thick shallow shells. The shear energy is about 10% of the total energy for a shell with $a/h = 10$ and for $a/h = 5$, it is approximately 30%. The ratio of lowest frequencies corresponding to shallow shells with $a/h = 5$ and 100 is approximately 3.

The ratios of the first meridional frequency at $N = 12$ to the lowest frequency for deep and shallow shells with circumferential fibre are 1·95 and 7·8 for a shell with $a/h = 100$ and 6·9 and 16·8 for $a/h = 10$. The corresponding values for a shell with meridional fibre are 2·3, 3·0 and 3·8, 7·1. The higher ratios for circumferential fibre shells are expected. As

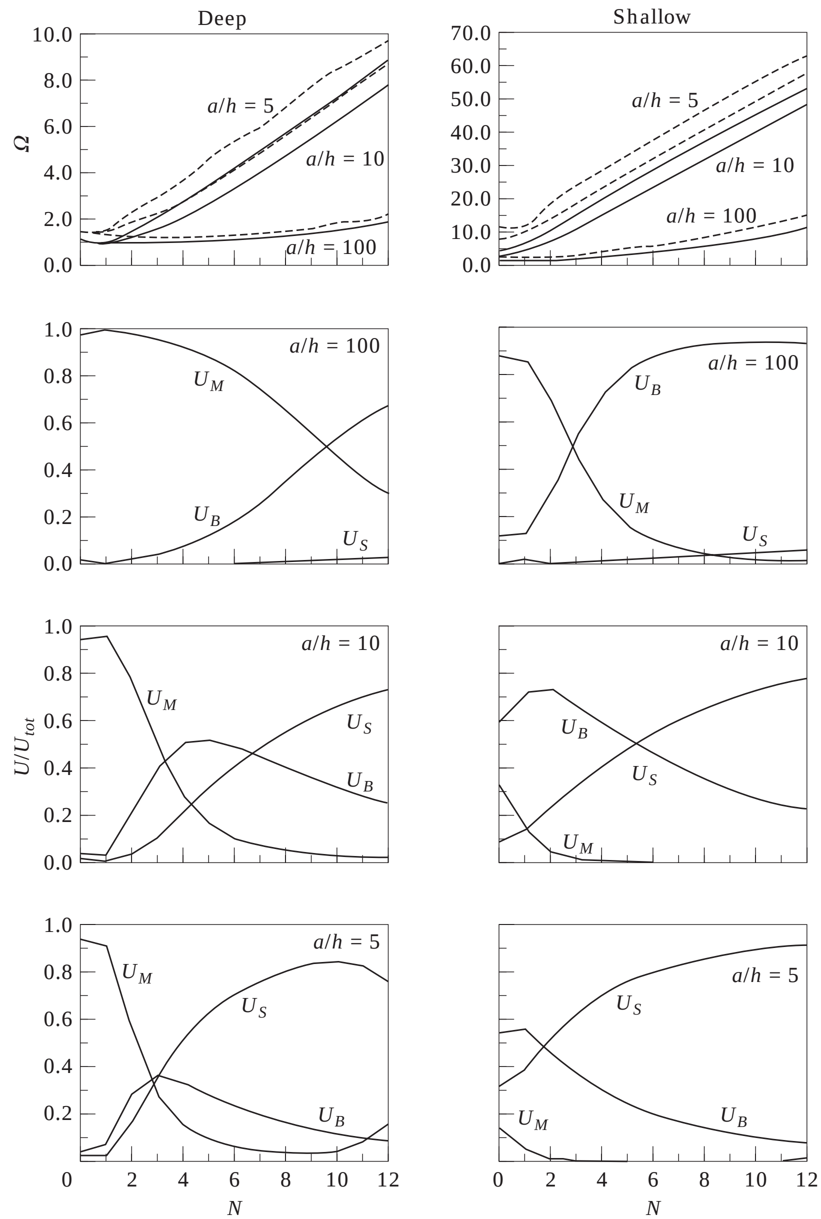

Figure 7. As Figure 6 but with fibre along the circumference. Key as Figure 6.

the fibre is in the circumferential direction, the resistance to the deformation in the circumferential direction will be more and hence the frequencies increase rapidly with N.

3.5. EFFECT OF FIBRE ORIENTATION AND LAYUP

In the previous section, the vibration of single layered shells with isotropic or orthotropic properties has been discussed. The vibrational characteristics may change significantly when the shell is layered. Hence this section deals with the vibration of layered shells.

In the following discussion, 0° fibre orientation refers to fibre in the meridional direction and 90° fibre refers to a circumferential fibre. Four types of layered configurations namely two-layered [0/90], three-layered [0/90/0] and [90/0/90] and eight-layered $[0/90/0/90]_s$ configurations are considered. All layers are assumed to be of equal thickness. For the convenience of discussion, these configurations are labelled as follows: L1A, single layer, meridional fibre; L1B, single layer, circumferential fibre; L2, two-layered [0/90]; L3A, three-layered, [0/90/0]; L3B, three-layered [90/0/90]; L8, eight-layered $[0/90/0/90]_s$. The first two configurations are single layered with higher modulus in the meridional (L1A) and circumferential (L1B) directions respectively. The L2 configuration is a two-layered shell where the effective modulus in both the directions is the same. The L3A shell has a higher modulus in the meridional direction but the circumferential direction is also stiffened by fibre in the middle layer. The L3B configuration is similar to L1B with additional reinforcement in the meridional direction. The eight-layered shell L8 is similar to L2 with the effective modulus in both directions being the same.

Figure 8 shows the variation of the first meridional frequencies with circumferential mode number for different lay-up and thickness ratios. The curves are labeled by the material configuration as discussed above.

The lowest frequencies at different thickness values and lay-up for deep and shallow shells are given in Tables 6 and 7 respectively. The numbers in the brackets indicate the circumferential mode number at which the lowest frequency occurs. The last column indicates the ratio of the highest and lowest frequencies corresponding to a particular thickness.

It can be observed from the figure and tables that at lower circumferential modes, the L1B configuration gives the lowest frequency for all lay-ups except for the range of a deep shell. At higher circumferential modes, the frequencies corresponding to L1B are above the frequencies of other configurations except L3B. The increase in the frequencies corresponding to L1B and L3B at higher circumferential wave numbers can be attributed to the higher modulus in the circumferential direction arising from the fibre in that direction. Although L1B and L3B give higher frequencies at higher N, the lowest frequencies of these two are lower than those of other configurations except, as mentioned earlier, for the thin deep shell. The curves representing L3A and L1A are similar in most of the cases. In both these configurations, the modulus in meridional direction is higher as the fibre is predominantly oriented along the meridian.

For thin deep shells, L1B gives higher frequencies than L1A. This can be explained by examining the energy plots for isotropic shells. For deep shells in the thin range at $N = 0$ and $N = 1$, U_s is observed to be less than U_θ. Hence, the circumferential fibre gives higher frequencies over the meridional fibre as additional stiffness is added in the direction that carries the maximum energy. This can be observed from the frequencies obtained for orthotropic shells. For example, at $a/h = 100$ the frequencies of a deep shell with meridional and circumferential fibres are 0·884(1) and 0·976(1) respectively. As the lowest frequency occurs at $N = 1$ for these shells, it can be said that, in single-layered shells, a higher frequency can be obtained by circumferential fibre for thin deep shells while the meridional fibre is preferable for thick deep as well as thick and thin shallow shells.

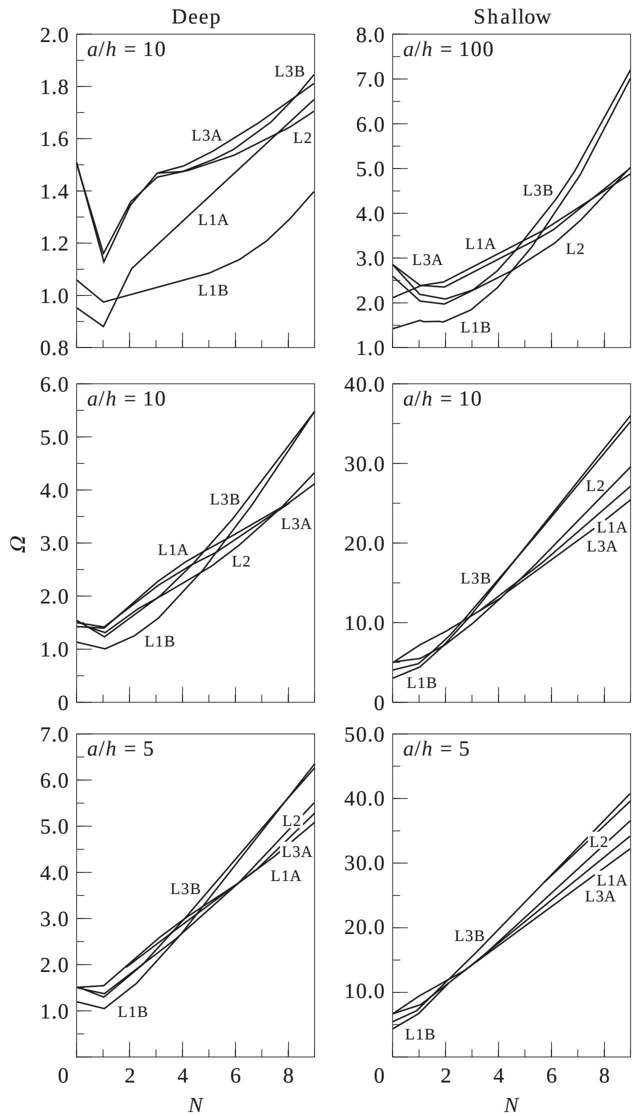

Figure 8. Variations of frequencies with N for deep and shallow shells with different lay-ups.

In the case of deep shells, at low thickness values, the two-layered shell gives the highest frequency. It is observed previously that the circumferential fibre is preferable over the meridional fibre in case of single-layered shells. However, as the predominant modes of vibration are extensional with considerable influence of extension in both the directions, stiffening the shell in both the directions by L2 lay-up gives higher frequencies. As the thickness increases, the maximum lowest frequency occurs at L3A configuration. This is due to the increase in energy from the strain in the meridional direction as the thickness increases. An increase in the frequency of 31% is obtained for a shell with $a/h = 100$ by choosing a suitable lay-up. The highest increase in the frequency is about 50% at $a/h = 5$ among the cases considered.

For a shallow shell, the highest frequency is observed for the L3A configuration for lower thickness values and for L2 for higher thickness values. The highest gain in the

TABLE 6

Lowest frequencies and corresponding circumferential wave numbers of a clamped hemispherical cap for different lay-ups

a/h	0	90	0/90	0/90/0	90/0/90	$[0/90/0/90]_s$	Ratio
100	0·884(1)	0·976(1)	1·157(1)	1·133(1)	1·125(1)	1·142(1)	1·31
50	0·932(1)	0·978(1)	1·183(1)	1·180(1)	1·145(1)	1·181(1)	1·27
20	1·130(1)	0·983(1)	1·237(1)	1·284(1)	1·185(1)	1·260(1)	1·30
14·2	1·253(1)	0·988(1)	1·263(1)	1·340(1)	1·202(1)	1·300(1)	1·36
10	1·386(1)	0·995(1)	1·292(1)	1·404(1)	1·222(1)	1·346(1)	1·41
6·67	1·498(0)	1·009(1)	1·326(1)	1·470(1)	1·247(1)	1·396(1)	1·46
5	1·491(0)	1·002(1)	1·348(1)	1·491(0)	1·268(1)	1·429(1)	1·50

frequency, which is almost double the lowest, is observed at a thickness of $a/h = 20$. However, the frequencies of the shells with predominantly meridional fibre (L1A, L3A) and two-layered or eight-layered do not show much difference while for L1B and L3B, where the circumferential fibre is influential, the frequency is seen to be much lower.

From the above study it can be seen that a significant increase in the lowest natural frequency can be obtained by choosing a suitable lay-up. The gain is more for a shallow shell. Finally, the designer has to choose a configuration based on other requirements such as static stresses and manufacturing considerations.

3.6. EFFECT OF S/a

The dynamic characteristics of a spherical cap change greatly from deep to shallow ranges. The shell is considered shallow when S/a is low; that is when $\sin\phi$ is small where ϕ is the subtended angle of the shell. A value of $S/a = 1$ represents a hemispherical cap which is a deep shell. In this section, the frequencies of a spherical cap are investigated for their variation from deep to shallow ranges of S/a for two thickness ratios of $a/h = 10$ and 100, representing thick and thin shells. Three material cases, namely isotropic, composite meridional fibre and circumferential fibre, are considered. The variation of the first meridional frequencies with N a different values of A/a is shown in Figure 9. It can be observed from the figure that the increase of shallowness increases the frequency monotonically for all the cases considered. As the shallowness of the shell increases, the length of the meridian decreases. This reduces the wave length of the meridional modes. Moreover, as seen earlier, as the shallowness increases the radius of curvature increases and this decreases the influence of transverse normal displacement w on the meridional energy as the inplane strain from w varies as $1/R$. With the reduction in the meridional

TABLE 7

Lowest frequencies and corresponding circumferential wave numbers of a clamped shallow spherical cap for different lay-ups

a/h	0	90	0/90	0/90/0	90/0/90	$[0/90/0/90]_s$	Ratio
100	2·127(0)	1·453(0)	2·070(2)	2·374(2)	1·986(2)	2·255(2)	1·63
50	2·907(1)	1·603(0)	2·464(1)	2·855(1)	2·239(1)	2·676(1)	1·78
20	3·721(0)	2·040(1)	3·522(1)	3·908(1)	3·906(1)	3·748(0)	1·92
14·2	4·142(0)	2·369(0)	4·307(1)	4·314(0)	3·528(0)	4·155(0)	1·82
10	4·815(0)	2·886(0)	5·107(0)	4·955(0)	3·954(0)	4·789(0)	1·77
6·67	4·828(0)	3·689(0)	5·978(0)	5·902(0)	4·670(0)	5·711(0)	1·62
5	6·600(0)	4·350(0)	6·623(0)	6·603(0)	5·298(0)	6·387(0)	1·47

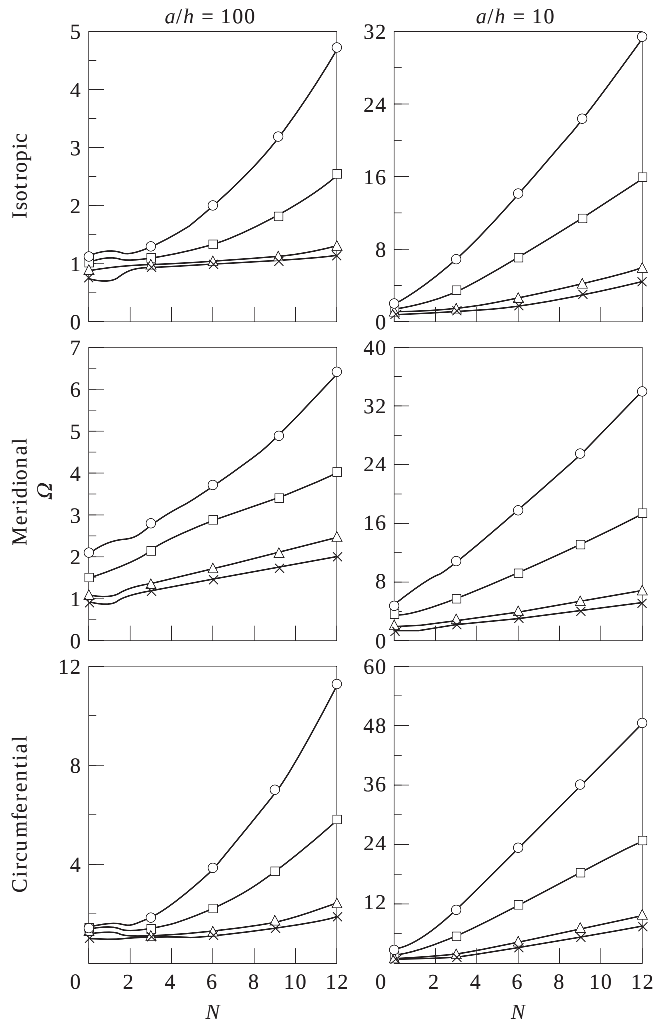

Figure 9. Variations of frequencies with N of thin and thick shells, for different S/a: ○, 0·1; □, 0·2; △, 0·6; ×, 1.

effects, the bending and shear effects become predominant. Also, it can be noted that the frequency increases sharply with N at shallow ranges. This is also due to the higher influence of bending and shear. The variation of frequencies with S/a is higher for thicker shells. The lowest frequencies corresponding to different cases are presented in Table 8.

The ratio of the lowest frequencies corresponding to $S/a = 0{\cdot}1$ and $S/a = 1$ for an isotropic shell at $a/h = 100$ is 1·6 and at $a/h = 10$, it is 2·6. The corresponding values for a meridional fibre shell are 2·4 and 3·5 and those for a circumferential fibre shell are 1·5 and 2·9. From these values it can be seen that the meridional fibre shell has a higher influence of S/a on the lowest frequency. The ratios corresponding to the isotropic shell and meridional shell are very close. It has been observed earlier that, at the lowest frequency, the energy from the strain in the meridional direction contributes predominantly to the total energy except at thin ranges of deep shell. Hence, it can be said that the stiffness in the meridional direction will have higher influence on the lowest frequencies of the shell which in turn leads to higher influence of shallowness in the case of meridional fibre shells.

3.7. EFFECT OF ORTHOTROPY

Fibre reinforced composites in general have orthotropic properties with the preferred directions oriented along the fibre and its perpendicular directions for an uni-directional fibre reinforced composite. The properties of these materials depend not only on the properties of individual matrix and fibre but also on the volume fraction of the fibre. Wide ranging materials with varying properties from a high level of orthotropy to near isotropy can be found in use. The two basic factors by which the orthortropy can be judged are the ratio of in-plane moduli (E_L/E_T) and the ratio of the transverse shear modulus to that of the in-plane modulus (G_{LZ}/E_T). Any variation in these ratios effects the final behaviour of the shell. In this context, here, a study has been carried out to observe the influence of E_L/E_T and G_{LZ}/E_T. In carrying out such a study, the values of all other material constants are kept constant, with only E_L or G_{LZ} varied. Although in a real situation all these properties vary when the material constitution is changed, this kind of a study is expected to give some insight into the effect of orthortropy on the free vibration. The study is carried out on clamped deep and shallow shells with $a/h = 100$ and 10.

3.7.1. *Effect of E_L/E_T*

The variation of non-dimensional frequencies corresponding to the first meridional modes of the first six circumferential modes ($N = 0$ to 5) with E_L/E_T for three material configurations, namely single-layered meridional fibre, single-layered circumferential fibre and eight-layered $[0/90/90/0]_s$ symmetric lamination, is shown in Figure 10. A range of E_L/E_T from moderately orthotropic 5 to highly orthotropic 40 is considered for the study. This is achieved by keeping all other material constants except E_L the same as that of glass/epoxy.

For thin deep shells, the lowest frequency occurs at $N = 1$ for all three material configurations considered. The frequencies corresponding to $N = 0$ are seen to be above $N = 2$ and $N = 3$ for both circumferential fibre and eight-layered shells. The variations of frequencies of both these shells are seen to be similar. In the discussion for thin isotropic deep shell, it is seen that the energy corresponding to the strain in the circumferential direction is the predominant contributor to the total energy. Hence, this similarity is

TABLE 8

Lowest frequencies and corresponding circumferential wave numbers of a clamped spherical cap for different S/a

Material	S/a					
	0·1	0·2	0·4	0·6	0·8	1·0
			$a/h = 100$			
Isotropic	1·119(0)	1·021(0)	0·962(0)	0·919(0)	0·824(1)	0·713(1)
Glass epoxy (meridional)	2·126(0)	1·532(0)	1·216(1)	1·063(1)	0·965(1)	0·882(0)
Glass epoxy (circumf.)	1·453(0)	1·336(2)	1·189(2)	1·110(2)	1·052(2)	0·971(1)
			$a/h = 10$			
Isotropic	2·043(0)	1·555(0)	1·294(0)	1·079(1)	0·917(1)	0·782(1)
Glass epoxy (meridional)	4·807(0)	3·612(0)	2·615(1)	2·095(1)	1·681(1)	1·382(1)
Glass epoxy (circumf.)	2·885(0)	1·949(0)	1·530(0)	1·352(0)	1·174(1)	0·990(1)

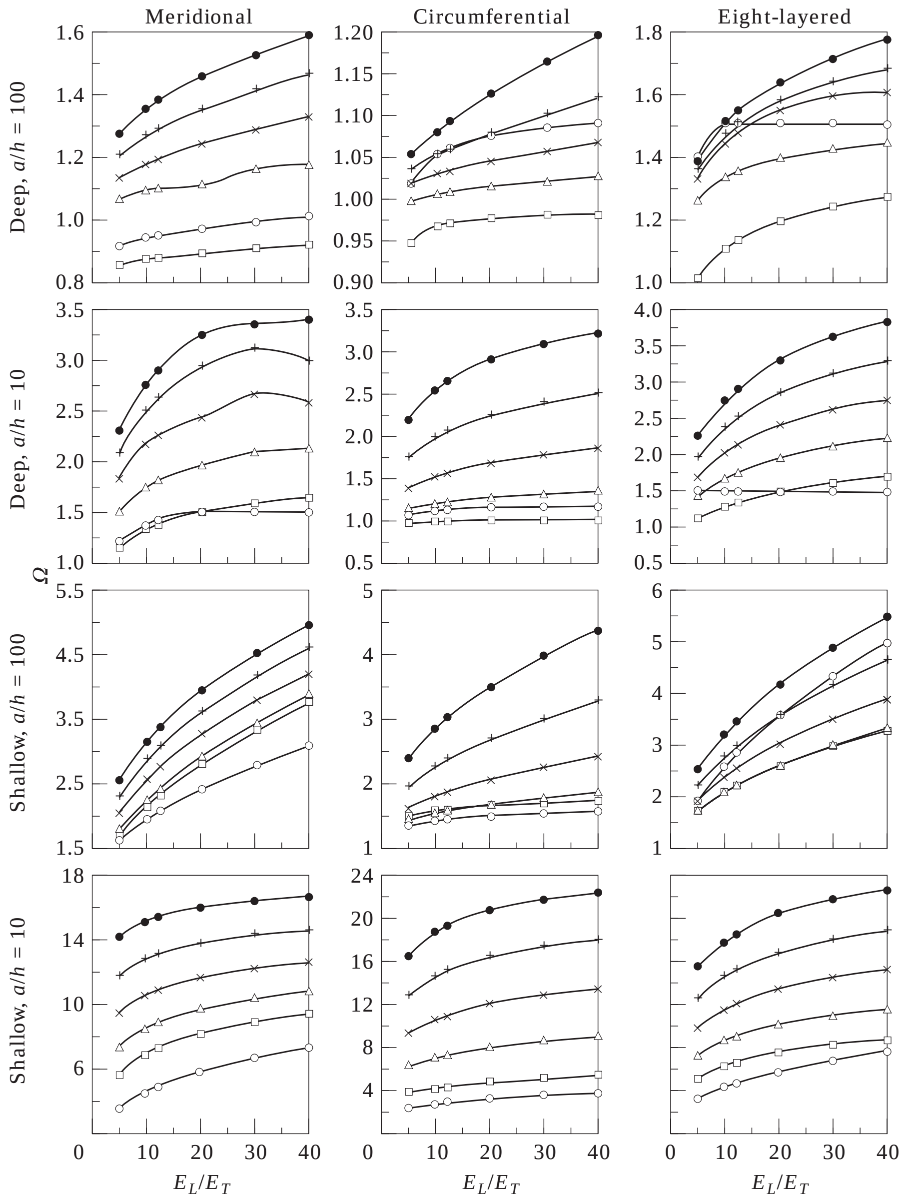

Figure 10. Effect of E_L/E_T for deep and shallow shells. ○, $N = 0$; □, $N = 1$; △, $N = 2$; ×, $N = 3$; +, $N = 4$; ⊗, $N=5$.

expected. In the case of thick deep shells, the lowest frequency corresponding to the shell with circumferential fibre occurs at $N = 1$ whereas in the case of meridional fibre and eight-layered shells, the lowest frequency shifts from $N = 1$ to $N = 0$ as E_L increases. As observed earlier, the $N = 1$ mode corresponds to a beam-like mode. Increasing the modulus in the meridional direction by increasing E_L leads to higher resistance to $N = 1$ and thus the corresponding frequency increases. Hence, beyond a certain value of E_L the frequency corresponding to $N = 0$ will be lower. The frequency corresponding to $N = 0$ is almost unaffected by E_L/E_T for deep eight-layered shells except for very low range of

E_L/E_T for a thin shell. This can be attributed to the predominant membrane effects and also the near isotropic properties of an eight-layered shell.

The ratios of the lowest frequencies corresponding to $E_L/E_T = 40$ and $E_L/E_T = 5$ for thin deep shells with $a/h = 100$ are 1·08, 1·04 and 1·28 respectively for the meridional fibre, circumferential fibre and eight-layered shells. The corresponding values for the thicker shell with $a/h = 10$ are 1·31, 1·04 and 1·34. From these values, it can be said that E_L has very little effect on the shells with circumferential fibre orientation. This is in contrast to a significant gain observed for eight-layered shells. The increased stiffness in both the meridional and circumferential directions can be attributed to such a high gain in the frequencies for eight layered shells. In the case of meridional fibre orientation, the thin shell is not much influenced by E_L but a significant gain is observed for the thicker shell. This is due to the higher influence of bending in the meridional direction in thick shells.

It can easily be seen from the figures that the E_L/E_T ratio has a more pronounced effect on shallow shells than on deep shells. This is as expected. In contrast to deep shells, in the case of shallow shells, the lowest frequency occurs at $N = 0$ except for thin eight-layered shells. This phenomenon can be seen in Table 7 also. The frequencies corresponding to various circumferential modes are well separated in case of thick shallow shells. This is as expected. The ratios of the lowest frequencies of thin shallow shells corresponding to the extreme cases of E_L/E_T considered in this study are 1·87, 1·17 and 1·87 for meridional fibre, circumferential fibre and eight-layered shells respectively, and the corresponding values for thick shells are 2·1, 1·52 and 2·3. Here also, as in the case of a deep shell, the circumferential fibre orientation experiences the lowest influence of E_L/E_T of the three cases considered, though the gain in the frequency is much higher than for deep shells. Very high gain in the frequency is observed for both meridional fibre and eight-layered shells. The gain in the case of thick shells is seen to be close to the square-root of the ratio of extreme values of E_L, i.e., $\sqrt{40/5} = 2{\cdot}83$.

From this study, it can be said that the use of high orthotropy materials for deep single-layered shells will not increase the frequencies significantly but a considerable gain can be obtained for a layered shell within the lower range. In the case of shallow shells, the frequency can be increased significantly by using a high orthotropy material.

3.7.2. *Effect of* G_{LZ}/E_T

Figure 11 shows the variation of non-dimensional frequencies corresponding to the first meridional modes of the first six circumferential modes ($N = 0$ to 5) with G_{LZ}/E_T for three material configurations, namely single-layered meridional fibre, single-layered circumferential fibre and two-layered [0/90] lay-up. A range of G_{LT}/E_T from 0·2 to 0·7 is considered. The material properties considered in this study are as follows: $E_L/E_T = 25$; $E_Z/E_T = 1$; $G_{TZ}/E_T = 0{\cdot}2$; $G_{LZ}/E_T = G_{LT}/E_T = 0{\cdot}2$–$0{\cdot}7$. Here, G_{LZ} and G_{LT} are assumed to be the same as this is the case in many real materials. The ratios of the lowest frequencies corresponding to $G_{LZ}/E_T = 0{\cdot}7$ and 0·2 are 1·32, 1·60 and 1·70 respectively for a thin deep shell of meridional fibre, circumferential fibre and two layers. The corresponding values for a thick deep shell are 1·82, 1·56 and 1·74. From these, it can be seen that the shear modulus has a significant influence on deep shells even in the thin range. Here, unlike in the case of E_L/E_T, even the shell with circumferential fibre orientation is greatly influenced. Low shear modulus of at $G_{LZ}/E_T = 0{\cdot}2$ ($G_{LZ}/E_T = 0{\cdot}008$) leads to very high shear flexibility. Hence, even though the membrane effects are predominant for deep isotropic shells, the influence of shear is also felt considerably. The lowest frequency ratios for thin shallow shells are 1·01, 1·18 and 1·24 and for a thick shallow shell they are 1·14, 1·0 and 1·35 respectively for the three material configurations considered. From this, it can be seen that the influence of G_{LZ}/E_T is low for shallow shells. This in contrast to what is expected. The ratio of 1 for the circumferential fibre is due to the lowest frequency occurring at the axisymmetric mode ($N = 0$). For the axisymmetric mode of shallow shells with circumferential fibre, the frequency is unaffected by G_{LZ}/E_T. Here, G_{LZ} is in the θ-z plane and as there is no shear strain in this plane for the axisymmetric mode, this can be expected. In the case of a deep shell, $N = 0$ curve is almost flat except at very low shear moduli. The variation at low shear moduli can be attributed to the mode of vibration being torsional. As the shear modulus increases, the mode of vibration changes to that of flexure–extension and remains flat thereafter.

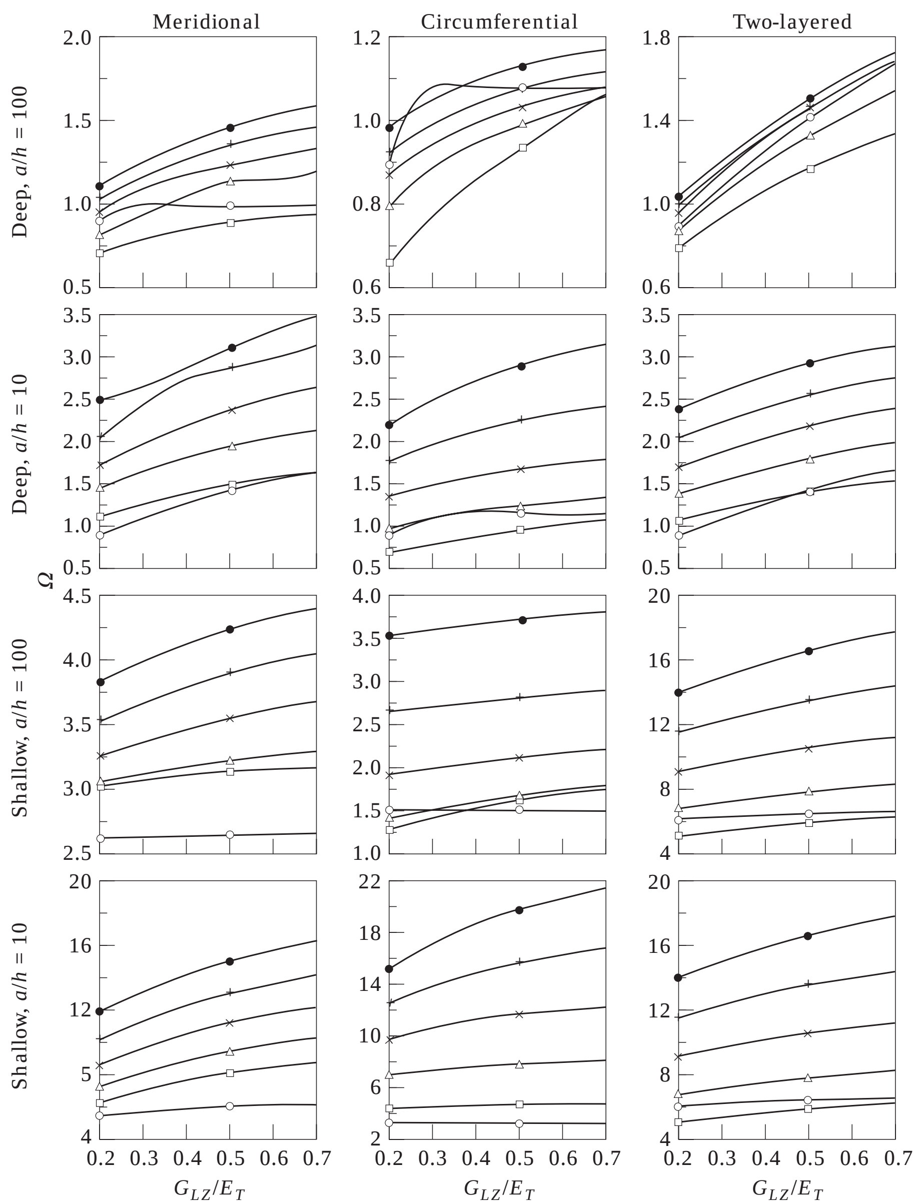

Figure 11. As Figure 10 but effect of G_{LZ}/E_T.

4. CONCLUSIONS

In this paper, a study of free vibration characteristics of isotropic and laminated spherical caps has been carried out by using a first order shear deformable semi-analytical shell finite element. Clamped spherical caps in deep and shallow ranges and covering thick and thin ranges are considered in the study. The effects of material orthotropy, fibre angle and lay-up of laminated shells are studied in detail. From the above study, the following observations are made.

1. The vibration of a deep shell is primarily governed by the membrane effects whereas that of a shallow shell is governed by bending and shear.
2. As the circumferential wave number increases, the influence of bending and shear increases.
3. Deep shells have very close frequencies and the frequency increases very slowly with circumferential mode number; hence higher circumferential modes play a vital role in the dynamics of deep shells.
4. A significant gain only in the lowest frequencies can be achieved by choosing a suitable lamination scheme.
5. In most cases, except for thin deep shells meridional fibre gives a higher fundamental frequency. For thin deep shells, circumferential fibre is preferable.
6. Although lamination increases the frequencies in many cases, increasing the number of layers beyond three does not increase the frequency considerably. Hence, an appropriate two- or three-layered shell is advisable.
7. The effect of curvature is higher in the case of shells with fibre oriented along the meridian.
8. Using materials with high modulus in the fibre direction does not yield any considerable increase in the frequencies of deep single layered shells but leads to a significant increase in the frequencies of shallow shells.
9. For a multilayered shell, the frequencies can be significantly increased by choosing a material with high orthotropy ratio.

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