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FORCED OSCILLATIONS AND RESONANCE OF INFINITE PERIODIC STRINGS

P. M. BELOTSERKOVSKIY

Department of Higher Mathematics, Moscow Institute for Railway Transport Engineers, Obraztsov St. 15, Moscow 101475, Russia

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An infinite string, supported by the equidistantly spaced identical suspensions, is considered. Each suspension consists of a spring and a dashpot with viscous damping, in parallel. The small transverse oscillations of the string are affected by the viscous drag of an external medium. The concentrated harmonic force, moving steadily along the string, causes steady-state oscillations. This means that the displacement along the string across the suspension spacing brings the time delay and the phase lead in the string's transverse deflection. The former is equal to the time of the exciting force motion over the spacing. The latter is equal to the change of the exciting force phase over this time. The steady-state nature of the oscillations and the linear property of the infinite periodic structure allows one to consider just one segment of the string between the neighbouring suspensions.

The string deflection is governed within the segment by a partial differential equation and two boundary conditions that include the time delay and the phase lead. Both approach infinity as the speed of exciting force approaches zero and so stationary excitation cannot be directly included in the present consideration. It is supposed that the string segment had been at rest before the force approached and returned to rest after the force had moved away. Fourier transformation is used to solve the boundary problem in the infinite strip. The solution is represented in the form of a single integral that is as good for calculation as for qualitative analysis. These show that the periodic string resonance takes place, if any viscous resistance is absent and the integrand has a real pole of the second order.

The string oscillations' dependence on the suspension stiffness is studied. If the stiffness is small enough, then a Doppler effect takes place. Two limit cases which correspond to the stiffness that approaches zero or infinity are considered. In the first case, the string is free. In the second one, the rigid suspensions divide the string into an infinite sequence of isolated segments. Any isolated segment is exposed to the moving exciting force over a limited time. Therefore, string resonance is impossible. The integrand has no second order real pole as well.

In order to include stationary excitation into consideration, a suitable limit procedure is used. Resonance in response to such an excitation is studied and resonant frequencies are found. If the excitation point coincides with the suspension location or with the string mid-span, then the exciting force produces symmetric oscillations in the string. In these particular cases of excitation, resonance at some resonant frequencies disappears, but anti-resonance, that seems to be impossible in response to moving excitation, appears instead. In some cases, the exciting force produces a standing wave in the string, each suspension coincides with the wave node and so any suspension is strictly fixed as well as the excitation point. Such anti-resonance is not affected by suspension viscous damping. In other cases, if viscous damping is small, the excitation point and suspensions experience small oscillations.

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1. INTRODUCTION

An infinite periodic structure which is an infinite string, supported by equidistantly spaced identical suspensions, is considered. Each suspension consists of a spring and a dashpot with viscous damping, in parallel. Small transverse oscillations of the string in viscous medium are considered. If any resistance to the string motion is absent, free undamped oscillations are possible. If the string is finite, then only the specific values of the free oscillation frequencies can occur [1]. Such frequencies are the discrete spectrum of free oscillations. The finite string experiences resonance as soon as the frequency of the string forced oscillations coincides with any frequency of its free oscillations. As will be shown, the frequencies of the infinite periodic string free oscillations belong to the infinite sequence of the segments which are separated by another sequence of segments. Such frequencies fill the segments of the first sequence continuously. This means that the frequency spectrum of the free oscillations is not discrete, but continuous.

The infinite string oscillations, caused by a concentrated harmonic force, moving steadily along the string, are studied. The study shows that the resonance of the infinite periodic structure, excited by the moving harmonic force, occurs, if the structure and the force parameters comply with some condition that is, obviously, quite different from the above-mentioned condition, corresponding to the finite string resonance. The string deflection, obtained by means of the Fourier transformation in the form of a single integral, is as good for calculation as for qualitative analysis. This shows that resonance takes place, if the integrand has a real pole of the second order. The resonance of the infinite structure with the continuous spectrum of frequencies was displayed, when oscillations of infinite uniform beams, resting on uniform elastic foundation and excited by the moving force, had been studied [2–5]. It turns out that the condition of the infinite periodic structure resonance is the same.

2. BOUNDARY PROBLEM

An infinite string is supported by periodic visco-elastic suspensions with spacing l, identical stiffness k and viscous damping k_1 (see Figure 1). The string complex transverse deflection $y(x, t)$ is caused by the concentrated harmonic force $a_0 \exp(\mathrm{i}\omega_0 t)$ of the amplitude a_0 and the angular velocity ω_0, moving steadily along the string with non-zero speed v_0. The variables t and x denote the time and the longitudinal co-ordinate along the string. The point $x = 0$ corresponds to one of the suspension points. Small steady-state oscillation of the string are considered. Therefore, the string trannsverse deflection is in proportion to a_0 and any change in the value of a_0 causes no interest. The dimensionless value $\Phi_0 = \omega_0 l/v_0$ is the change of the exciting force phase over the time l/v_0 as the force moves over the distance l. This depends on both interesting parameters of excitation ω_0 and v_0. The value of exciting force gains the factor $\exp(\mathrm{i}\Phi_0)$ over this distance. The value

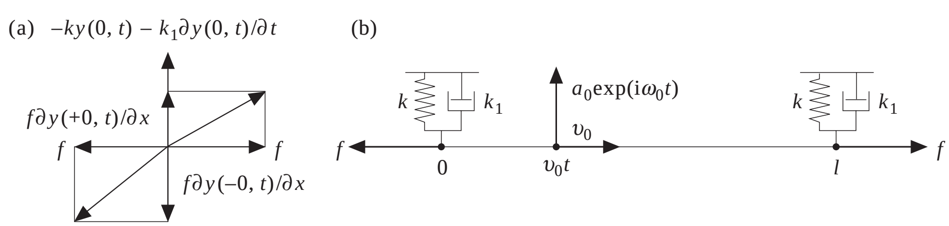

Figure 1. Periodic structure; (a) Suspension; (b) Span of a string.

of the string deflection gains this factor too, in accordance with the linear property of the periodic structure. Therefore, one supposes that

$$y(x + l, t + l/v_0) = \exp(\mathrm{i}\Phi_0) y(x, t). \tag{1}$$

This is similar to Mead's approach (see [6, 7], for example). The condition (1) has been used to describe the steady-state oscillations of periodic beams, excited by a moving harmonic force [8]. If $\omega_0 = 0$, then the exciting force is constant, $\Phi_0 = 0$ and $\exp(\mathrm{i}\Phi_0) = 1$. In this particular case, the condition (1) easily reduces to

$$y(x + l, t + l/v_0) = y(x, t) \tag{2}$$

and shows that the displacement along the string of the space l causes the time delay l/v_0 in the string deflection and that the exciting force speed v_0 cannot equal zero in such consideration. This means that the particular case of the stationary exciting force can not be considered as easily as the case of the constant exciting force, but can be obtained by means of a suitable limit procedure, as will be shown. If n is an integer and ω_0 is related to v_0 by the following equality

$$\omega_0 = 2\pi n v_0, \tag{3}$$

then the exciting force is in phase again after its movement over l. In this case, $\Phi_0 = 2\pi n$, $\exp(\mathrm{i}\Phi_0) = 1$ and one obtains the condition (2) again. The condition (3) has been used to describe the oscillations of rails by a moving loaded wheel or harmonic force [9, 10].

The string transverse deflection is governed by the equation

$$\rho \frac{\partial^2 y(x, t)}{\partial t^2} + r \frac{\partial y(x, t)}{\partial t} - f \frac{\partial^2 y(x, t)}{\partial x^2} = a_0 \exp(\mathrm{i}\omega_0 t)\delta(x - x_0), \tag{4}$$

where f and ρ are the string tension force and linear density, r is viscous drag per the string unit length, δ denotes the Dirac function. A string has no bending stiffness, contrary to a beam. As it will be shown, some peculiarities in the periodic string oscillations follow from this. A string tangent is discontinuous at any point where a concentrated force is applied. The co-ordinate $x_0 = v_0 t$ locates the moving point of excitation. The function $y(x, t)$ is continuous at this point. But the function $f\partial y(x, t)/\partial x$ experiences a sudden change which is equal to the exciting force. Similarly, the function $y(x, t)$ is continuous at the point $x = 0$, but the function $f\partial y(x, t)/\partial x$ experiences the sudden change $ky(0, t) + k_1 \partial y(0, t)/\partial t$, which is equal to the force, acting upon the suspension at this location. Here k and k_1 are the suspension stiffness and viscous damping. Taking this and the condition (1) into account, one obtains

$$y(l, t + l/v_0) = \exp(\mathrm{i}\Phi_0) y(0, t), \tag{5}$$

$$f\partial y(l, t + l/v_0)/\partial x = \exp(\mathrm{i}\Phi_0)(f\partial y(0, t)/\partial x - ky(0, t) - k_1 \partial y(0, t)/\partial t). \tag{6}$$

It is further supposed that the string segment was at rest long before the exciting force approached and returns to rest due to resistance in the periodic structure long after the force moves away. This means that $y(x, t)$ and its derivatives vanish as $t \to \pm\infty$. The last is not necessary and can be replaced by other suitable suppositions.

The deflection $y(x, t)$ is determined within the string segment $0 \leqslant x \leqslant l$ from the solution of the partial differential equation (4), together with the two boundary conditions (5), (6) and the last supposition. Over the remainder of the string it is determined by the condition (1). The boundary conditions (5) and (6) include the time delay l/v_0. So one has a boundary problem with a time lag [11, 12] in the strip $(0, l) \times (-\infty, +\infty)$, or, in the terminology of

[13], one has a differential-difference boundary problem. Such a problem arises, for example, if one studies the interaction between an electric railway feeding wire and a moving locomotive pantograph [14].

Denote dimensionless values

$$X = x/l, \qquad X_0 = x_0/l = v_0 t/l, \qquad T = v_0 t/l, \qquad Y(X, T) = y(x, t)/l, \qquad K = kl/f,$$

$$K_1 = k_1/(\rho f)^{1/2}, \qquad R = rl/(\rho f)^{1/2}, \qquad A_0 = a_0/f, \qquad V_0^2 = \rho v_0^2/f = (v_0/v_*)^2,$$

where $v_* = (f/\rho)^{1/2}$ is the speed of waves, propagating in the free string. Except V_0, these dimensionless values do not depend on v_0. The dimensionless value V_0 is similar to the Mach number in aerodynamics. Variables X and T denote the dimensionless longitudinal co-ordinate along the string and the dimensionless time. The dimensionless co-ordinate X_0 of the moving excitation point coincides with T. Introducing the dimensionless values into equations (4–6) and taking into account that $\delta(x - x_0) = \delta(X - T)/l$, one obtains

$$V_0^2 \frac{\partial^2 Y(X, T)}{\partial T^2} + RV_0 \frac{\partial Y(X, T)}{\partial T} - \frac{\partial^2 Y(X, T)}{\partial X^2} = A_0 \exp(\mathrm{i}\Phi T)\delta(X - T), \qquad (7)$$

$$Y(1, T + 1) = \exp(\mathrm{i}\Phi_0) Y(0, T), \qquad (8)$$

$$\partial Y(1, T + 1)/\partial X = \exp(\mathrm{i}\Phi_0)(\partial Y(0, T)/\partial X - KY(0, T) - K_1 V_0 \partial Y(0, T)/\partial T). \qquad (9)$$

3. FREQUENCY SPECTRUM OF FREE PROPAGATING WAVES IN THE PERIODIC STRING

If any resistance to the string motion and the exciting force are absent, then R, K_1 and A_0 are equal to zero, but V_0 should be considered now as the unknown dimensionless value of the free undamped wave propagation speed. Therefore, the partial differential equation (7) and the boundary conditions (8) and (9) reduce to

$$V_0^2 \partial^2 Y(X, Y)/\partial T^2 - \partial^2 Y(X, T)/\partial X^2 = 0, \qquad Y(1, T + 1) = Y(0, T), \qquad (10, 11)$$

$$\partial Y(1, T + 1)/\partial X = \partial Y(0, T)/\partial X - KY(0, T). \qquad (12)$$

It is supposed in this section that any point of the string experiences harmonic oscillations and so the solution of the equation (10) can be written as

$$Y(X, T) = A(X) \exp(\mathrm{i}\Omega_0 T/V_0) = A(X) \exp(\mathrm{i}\omega_0 t), \qquad (13)$$

where $A(X)$, $0 \leqslant X \leqslant 1$, is the dimensionless complex amplitude of oscillations of the string point X. The value $\Omega_0 = V_0 \Phi_0 = \omega_0 l(\rho/f)^{1/2}$ depends on ω_0 not v_0, and so can be understood as a dimensionless frequency of oscillations. By substituting the expression (13) into equations (10–12), one obtains

$$\mathrm{d}^2 A(X)/\mathrm{d}X^2 + \Omega_0^2 A(X) = 0, \qquad \exp(\mathrm{i}\Omega_0/V_0) A(1) = A(0), \qquad (14, 15)$$

$$\exp(\mathrm{i}\Phi_0/V_0)\, \mathrm{d}A(1)/\mathrm{d}X = \mathrm{d}A(0)/\mathrm{d}X - KA(0). \qquad (16)$$

The fundamental solution of the ordinary differential equation (14) can be written as

$$A(X) = C_1 \sin(\Omega_0(1 - X)) + C_2 \sin(\Omega_0 X),$$

where C_1 and C_2 are arbitrary constants. By substituting this solution into the boundary conditions (15) and (16), one obtains the following equality

$$\cos(\Omega_0/V_0) = \cos\Omega_0 + K \sin\Omega_0/(2\Omega_0), \qquad (17)$$

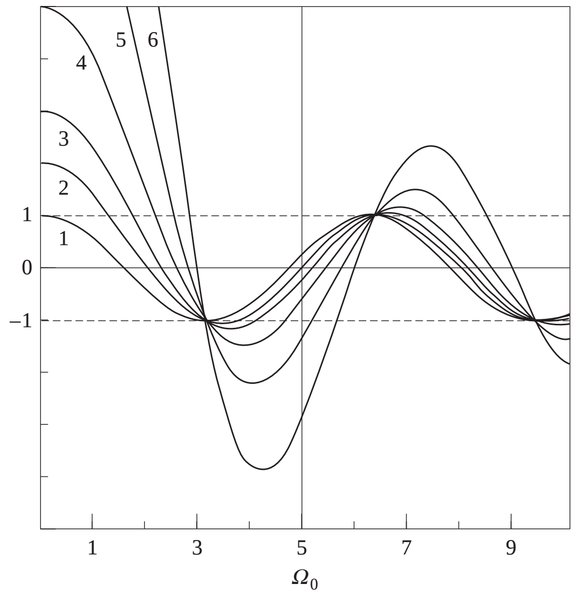

Figure 2. Dependence on Ω_0 of the right side of the equation (17); 1, $K = 0$; 2, $K = 1$; 3, $K = 2$; 4, $K = 4$; 5, $K = 8$; 6, $K = 16$.

that links Ω_0 and V_0. This shows that the free wave propagation speed v_0 depends on the frequency ω_0 of the string's free oscillations. The finite string can freely oscillate only with frequencies which belong to the discrete spectrum that consists of the infinite set of isolated points. The infinite free string can freely oscillate with any frequency. The spectrum is entire in this case. The equality (17) shows that a periodically suspended infinite string presents an intermediate kind of spectrum. The value of the equality (17) left side is bounded by -1 and 1. If n is an integer and $\Omega_0 = \pi n$, then the value of the right side of equality (17) turns into $(-1)^n$. The curves in Figure 2 show how the values of the right side of equality (17) calculated with different values of the suspension dimensionless stiffness K, depend on Ω_0. The two dotted lines correspond to -1 and 1. The equality (17) with a given Ω_0 can take place, only if

$$-1 \leqslant \cos \Omega_0 + K \sin \Omega_0/(2\Omega_0) \leqslant 1. \tag{18}$$

In this case, the value of V_0 can be found by means of the equality (17). As one sees from Figure 2, the value of Ω_0, which satisfy the inequalities (18), belong to the infinite sequence of the segments, separated by another sequence of segments. Such values of Ω_0 fill any segment of the first sequence continuously. This means that the frequency spectrum of the free propagating waves in the periodically suspended string is continuous, but not entire like that in the free string. As K tends to zero, the length of the segments increase. In the limiting case $K = 0$, the string becomes free and the segments merge. Therefore, the spectrum becomes entire and includes all the angular velocities. The right borders of the segments are the points πn. As K increases, the left borders of the segments tend to these points. In the limiting case $K = \infty$, the segments tighten into the points πn. This means that the spectrum of the angular velocities ω_0 becomes discrete and consists of the points $(\pi n/l)(f/\rho)^{1/2}$. If $K = \infty$, the suspensions are rigid and divide the string into the infinite sequence of the isolated segments of the length l. It is known that these points are the discrete spectrum of the finite string free oscillations [1].

4. FORCED OSCILLATIONS

Let

$$\int_{-\infty}^{+\infty} Y(X, T) \exp(-\mathrm{i}\Phi T)\, \mathrm{d}T = \hat{Y}(X),$$

where Φ is a dimensionless parameter. Then

$$\int_{-\infty}^{+\infty} Y(X+1, T+1) \exp(-\mathrm{i}\Phi T)\, \mathrm{d}T = \exp(\mathrm{i}\Phi)\, \hat{Y}(X+1).$$

By performing a Fourier transformation on equation (7), applying the boundary conditions (8) and (9) and integrating by parts, one obtains

$$-d^2\hat{Y}(X)/\mathrm{d}X^2 - S^2\hat{Y}(X) = A_0 \exp(\mathrm{i}(\Phi_0 - \Phi)X), \tag{19}$$

where the complex value S is determined by the following

$$S^2 = V_0^2\Phi^2 - \mathrm{i}RV_0\Phi, \tag{20}$$

and

$$\exp(\mathrm{i}(\Phi - \Phi_0))\,\hat{Y}(1) = \hat{Y}(0), \qquad \exp(\mathrm{i}(\Phi - \Phi_0))\,\mathrm{d}\hat{Y}(1)/\mathrm{d}X = \mathrm{d}\hat{Y}(0)/\mathrm{d}X - K(\Phi)\,\hat{Y}(0), \tag{21, 22}$$

where

$$K(\Phi) = K + \mathrm{i}K_1V_0\Phi. \tag{23}$$

The solution of the ordinary differential equation (19) may be written as

$$\hat{Y}(X) = \frac{C_1 \sin(S(1-X)) + C_2 \sin(SX) + A_0 \exp(\mathrm{i}(\Phi_0 - \Phi)X)}{(\Phi - \Phi_0)^2 - S^2}.$$

By substituting this solution into boundary conditions (21) and (22), one obtains two equations to determine two unknown values of C_1 and C_2. By evaluating them and substituting into the solution of equation (19), one finally obtains

$$\hat{Y}(X) = A_0(1 + K(\Phi)N(X, \Phi)/D(\Phi)) \exp(\mathrm{i}(\Phi_0 - \Phi)X)/((\Phi - \Phi_0)^2 - S^2),$$

where

$$N(X, \Phi) = \exp(\mathrm{i}(\Phi - \Phi_0)X) \sin(S(1-X)) + \exp(\mathrm{i}(\Phi - \Phi_0)(X-1)) \sin(SX),$$

$$D(\Phi) = 2S(\cos(\Phi - \Phi_0) - \cos S) - K(\Phi) \sin S.$$

These equations show that Φ is similar to Φ_0. Both $N(X, \Phi)$ and $D(\Phi)$ depend on S. Two complex values $\pm S$ are determined by the equality (20). If $-S$ is substituted for S, then both $N(X, \Phi)$ and $D(\Phi)$ change their signs, but the fraction $N(X, \Phi)/D(\Phi)$ does not and so $\hat{Y}(X)$ does not change its value. If S is zero, then both $N(X, \Phi)$ and $D(\Phi)$ are zero too and so the fraction stays limited as well as $\hat{Y}(X)$.

By performing an inverse Fourier transformation

$$Y(X, T) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \hat{Y}(X) \exp(\mathrm{i}\Phi T)\, \mathrm{d}\Phi$$

one obtains the string dimensionless deflection

$$Y(X, T) = \frac{A_0 \exp(\mathrm{i}\Phi_0 X)}{2\pi} \int_{-\infty}^{+\infty} \left[1 + \frac{K(\Phi)N(X, \Phi)}{D(\Phi)}\right] \frac{\exp(\mathrm{i}\Phi(T - X))\,\mathrm{d}\Phi}{(\Phi - \Phi_0)^2 - S^2} \qquad (24)$$

within the segment $0 \leqslant X \leqslant 1$. Behaviour of integral (24) is determined by the poles of its integrand. If $(\Phi - \Phi_0)^2 - S^2 = 0$ and $K(\Phi) \neq 0$, then $N(X, \Phi) = \sin S$ and $D(\Phi) = -K(\Phi) \sin S$. Therefore, the value in the square brackets of the integral (24) is equal to zero too and the integrand value is limited in this case. These show that only the roots of the equation $D(\Phi)/S = 0$ may be the poles of the integrand (24) and should be taken into account for calculations and qualitative analysis.

5. FREE STRING

If $K(\Phi) = 0$, then the integral (24) reduces to the integral

$$Y(X, T) = \frac{A_0 \exp(\mathrm{i}\Phi_0 X)}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(\mathrm{i}\Phi(T - X))\,\mathrm{d}\Phi}{(\Phi - \Phi_0)^2 - S^2}, \qquad (25)$$

which describes forced oscillations of the free string and can be calculated by means of residues. In order to study the poles of the integrand (25) with $V_0 < 1$, one may consider the following equation

$$(\Phi - \Phi_0)^2 - V_0^2\Phi^2 + \mathrm{i}RV_0\Phi \equiv ((1 + V_0)\Phi - \Phi_0)((1 - V_0)\Phi - \Phi_0) + \mathrm{i}RV_0\Phi = 0. \qquad (26)$$

The equality (20) has been taken into account. If $R = 0$, then the values $\Phi_{1,2}^0 = \Phi_0/(1 \pm V_0)$ are the roots of this equation. If $R \neq 0$, then the roots of the equation (26) may be represented as

$$\Phi_{1/2} = \Phi_{1,2}^0 + \Delta\Phi_{1,2}. \qquad (27)$$

If R and $\Delta\Phi_{1,2}$ are small, then, in accordance with (26), these small values are connected by the equality

$$2(\Phi_{1,2}^0 - \Phi_0)\Delta\Phi_{1,2} - 2V_0^2\Phi_{1,2}^0\Delta\Phi_{1,2} + \mathrm{i}RV_0\Phi_{1,2}^0 = 0$$

By evaluating $\Delta\Phi_{1,2}$ and substituting them into (27), one can represent the roots of the equation (26) as

$$\Phi_{1,2} = (\Phi_0 \pm \mathrm{i}R/2)/(1 \pm V_0).$$

This last relationship shows there are two poles $\Phi_{1,2}$ of the first order in the integrand of equation (25). The imaginary part of one of these poles is positive, while the other one is negative. If $T \geqslant X$, then the integral is equal to the residue at the pole Φ_1, multiplied by the factor $2\pi\mathrm{i}$. If $T \leqslant X$, then the integral is equal to the residue at the pole Φ_2, multiplied by the factor $-2\pi\mathrm{i}$. By evaluating the residue and taking after this $R = 0$, one can obtain

$$Y(X, T) = \exp(\mathrm{i}\Phi_0(T \mp V_0 X)/(1 \mp V_0))/(2\mathrm{i}V_0\Phi_0).$$

By using the initial variables, this can be rewritten as

$$y(x, t) = \frac{a_0 v_*}{2\mathrm{i}f\omega_0} \exp\left(\frac{\mathrm{i}\omega_0(v_* t \mp x)}{v_* \mp v_0}\right).$$

The top sign corresponds to the excitation point $x_0 \leqslant x$ and the lower one corresponds to $x_0 \geqslant x$. The last expression represents the direct and inverse waves, which propagate with the same velocity v_*. Every point x of the string oscillates with the same amplitude $a_0 v_*/(2f\omega_0)$. The angular velocity of the point x oscillations changes from the greater value $\omega_0 v_*/(v_* - v_0)$ to the smaller one $\omega_0 v_*/(v_* + v_0)$, when the excitation point x_0 reaches the point x. This is the Doppler effect, that is well known in optics and acoustics. To show this effect, one should separate the real and imaginary parts of $y(x, t)$ or $Y(X, T)$. The real part of the latter with $X = 0$, $\Phi_0 = \pi$ and $V_0 = 0{\cdot}5$ is shown in Figure 3. The amplitude of the point x oscillations unlimitedly grows as $\omega_0 \to 0$. It is similar to the string resonance. In this case, Φ_0 tends to zero and two poles $\Phi_{1,2}$ of the first order merge and form the single pole of the second order at the point $\Phi = 0$. The same takes place as resonance in an infinite beam, resting on uniform elastic foundation, appears [2–5].

6. RIGID SUSPENSIONS

If the suspension dimensionless stiffness K approaches infinity, then the suspensions become rigid. In this case, $K(\Phi) = \infty$ and the integral (24) reduces to

$$Y(X, T) = \frac{A_0 \exp(\mathrm{i}\Phi_0 X)}{2\pi} \int_{-\infty}^{+\infty} \left[1 - \frac{N(X,\Phi)}{\sin S}\right] \frac{\exp(\mathrm{i}\Phi(T - X))\,\mathrm{d}\Phi}{(\Phi - \Phi_0)^2 - S^2}. \tag{28}$$

This represents the dimensionless deflection of the string within the isolated segment $0 \leqslant X \leqslant 1$ between two rigid suspensions. If X is equal to 0 or 1, then $N(X, \Phi) = \sin S$ and $Y(0, T) = (1, T) = 0$. The integral (28) can be directly obtained by solving the equation (19), together with two boundary conditions $\hat{Y}(1) = \hat{Y}(0) = 0$.

7. RESONANCE

If any resistance to the periodic structure motion is absent, then $R = 0$, and $K_1 = 0$. Therefore, $K(\Phi) = K$ and $S = V_0\Phi$. The periodic structure properties are determined by the poles of the integrand (24). The roots of the equation $D(\Phi)/(V_0\Phi) = 0$ can provide such poles. Taking into account the free string propetries, one assumes that the real poles of

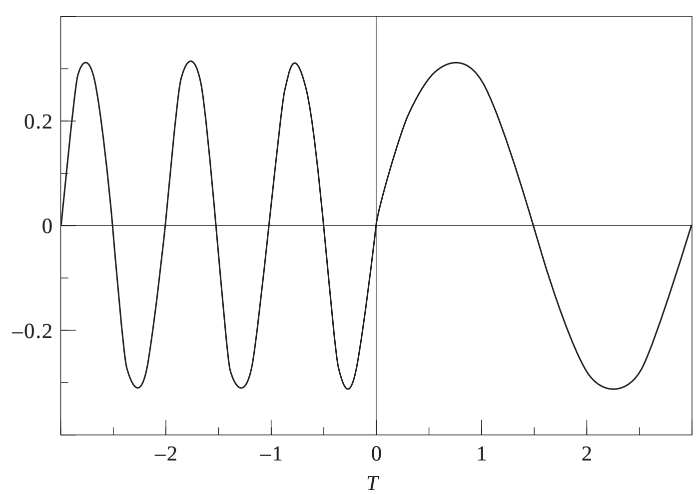

Figure 3. Forced oscillations of the free string at point $X = 0$; $\Phi_0 = \pi$, $V_0 = 0{\cdot}5$.

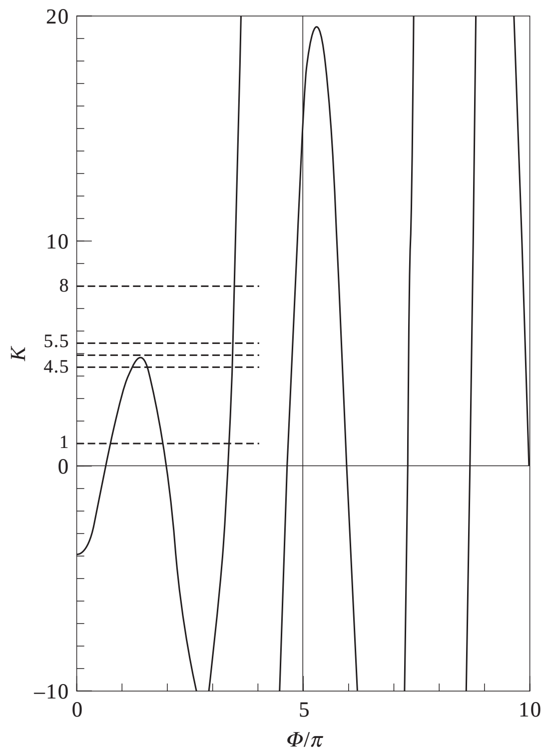

Figure 4. Dependence on Φ of the right side of the equation (29).

the first order give rise to undamped propagating waves. If the parameters of the excitation force and/or the periodic structure change, then the poles change too. One assumes that the periodic structure resonance takes place as soon as a pair of the real poles of the first order merge and form the single pole of the second order. This corresponds to the studies [2–5] on infinite beams, resting on uniform elastic foundation.

Let one consider how the suspension dimensionless stiffness K affects the string oscillations. If $K = \infty$, then the above-mentioned equation should be replaced by the equation $\sin(V_0\Phi)/(V_0\Phi) = 0$, as one can see from the integral (28). All the roots of the last equation are real and simple. According to the assumption, the string resonance is impossible in this case. But the segment transient response to excitation can be large enough and remain such after excitation. There is another explanation of this. Any segment of the string is isolated from the neighbouring ones by the rigid suspensions and exposed to excitation over limited time. The last is not true, if a periodically supported beam is considered. All the neighbouring segments of the beam cohere due to bending stiffness. Returning to the equation $D(\Phi)/(V_0\Phi) = 0$. Under consideration, one presents this equation as

$$K = 2V_0\Phi(\cos(\Phi - \Phi_0) - \cos(V_0\Phi))/\sin(V_0\Phi).$$

Let $\Phi_0 = \pi$ and $V_0 = 0{\cdot}5$ again. In this case, the above equation reduces to

$$K = -\Phi\cos(3\Phi/4)/\sin(\Phi/4). \quad (29)$$

The dependence on Φ of the even function in the right side of the equation (29) is presented in Figure 4 as a continuous line. If $K = 0$, then $\cos(3\Phi/4) = 0$ and all the roots

$\Phi_n = \pi(4n - 2)/3$, $n = 1, 2, 3, \ldots$ of the equation (29) are real and simple, the roots $\Phi_1 = 2\pi/3$ and $\Phi_2 = 2\pi$ correspond to the free string. As K increases, all the roots change. Their values can be found graphically as the intersections of the curve in Figure 4 and the line, which corresponds to the given value of K. Such lines are shown as the dotted ones and correspond to different values of K. If the value of K is sufficiently small, then all the roots are real and simple and give rise to undamped propagating waves. The root Φ_1 and Φ_2 merge and form a single root of the second order as K reaches the specific value of 4·85. The unmarked dotted line, which corresponds to this specific value, touches the top of the curve in this case. According to the assumption, this value of K corresponds to resonance. As K becomes slightly more than 4·85, the second order root branches into the pair of the first order complex-conjugated roots, that give rise to decaying waves. These waves vanish quickly as soon as the exciting force moves away. If $K = \infty$, then $\sin(\Phi/4) = 0$ and all the roots of the equation (29) are real and simple again. The roots Φ_{3n}, $n = 2, 2, 3, \ldots$ stay real and simple as K changes from zero to ∞. These roots change from $\pi(4\pi - 2/3)$ to $4\pi n$. All other ones merge and give rise to the following resonances.

To prove the assumption on the string oscillations and resonance, the calculations of the integral (24) with the external medium viscous drag $R = 0$ and the small value $K_1 = 0{\cdot}1$ of the suspension viscous resistance were made. In this case, there is no real pole in the integral and its direct calculation is possible. The most interesting values $Y(T, T)$ and $Y(0, T)$ with the dimensionless parameters $\Phi_0 = \pi$ and $V_0 = 0{\cdot}5$ were obtained. These complex values represent the transverse deflection of the string at the moving excitation point, where $X = T$, and at the suspension $X = 0$. The first one, with accordance to the condition (1), obeys the equality $|Y(T + 1, T + 1)| = |Y(T, T)|$, so $|Y(T, T)|$ is a periodic function. Figure 5 shows a relief map of $Y(T, T)$ over the period $0 \leqslant X = T \leqslant 1$ with the

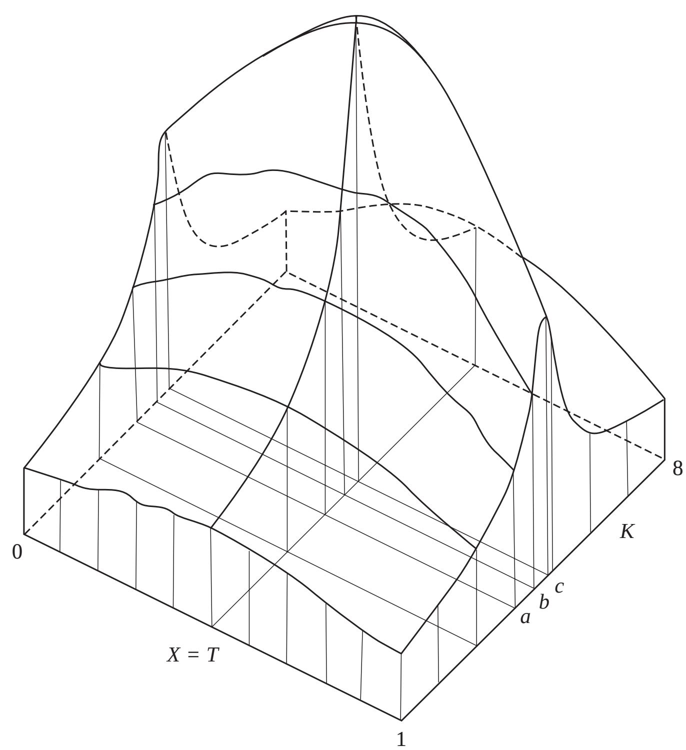

Figure 5. Oscillations of the moving excitation point; $\Phi_0 = \pi$, $V_0 = 0{\cdot}5$. a, $K = 4$; b, $K = 4{\cdot}5$; c, $K = 4{\cdot}85$.

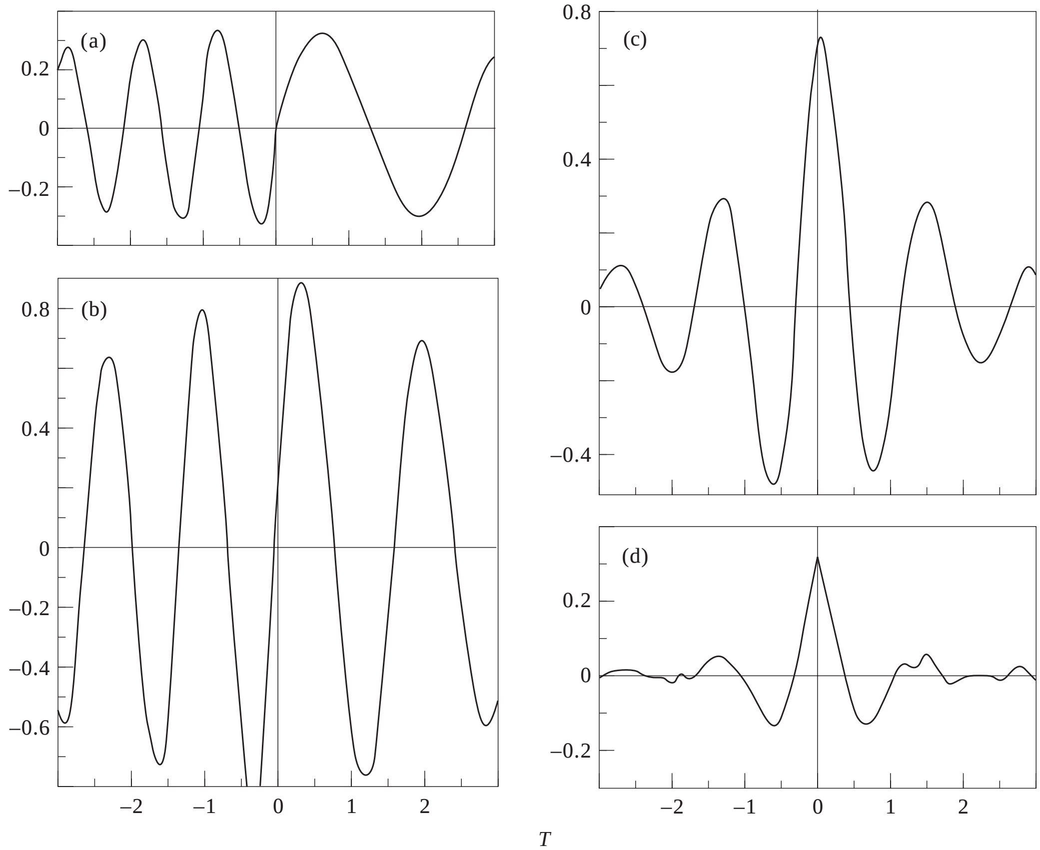

Figure 6. Forced oscillations of the suspension; $\Phi_0 = \pi$, $V_0 = 0{\cdot}5$. (a) $K = 1$; (b) $K = 4{\cdot}5$; (c) $K = 5{\cdot}5$; (d) $K = 8$.

suspension dimensionless stiffness K that ranges from 1 to 8. The resonance at $k = 4{\cdot}85$ is clearly seen.

8. PROPAGATING WAVES AND DOPPLER EFFECT

To show the propagating waves and the Doppler effect in the periodic string, one can separate the real and the imaginary parts of $Y(0, T)$ that corresponds to the suspension location. If $T < 0$, then the moving excitation point approaches the suspension $X = 0$. If $T > 0$, this moves away. The dependence on T of the real part is shown in Figure 6. If $K = 1$, then the oscillations of the suspension $X = 0$ are similar to those of the free string shown in Figure 3. The propagating waves and the Doppler effect are clearly seen. If $K = 4{\cdot}5$, then propagating waves are still seen, but the Doppler effect is absent due to coincidence of the roots Φ_1 and Φ_2. If $K = 5{\cdot}5$ or $K = 8$, there is no propagating wave as well.

9. STATIONARY EXCITING FORCE

If the exciting force speed V_0 tends to zero, then only $V_0 = v_0(\rho/f)^{1/2}$ tends to zero too and only $\Phi_0 = \omega_0 l/v_0$ approaches infinity. Therefore, the string deflection, caused by the standing exciting force cannot be directly obtained from the integral (24), but by means of the following limit procedure. At first, the integration variable Φ is replaced by the variable $\Omega = \Phi - \Phi_0$. After this, the integral can be represented in the form

$$Y(X, T) = \frac{A_0 \exp(\mathrm{i}\Phi_0 T)}{2\pi} \int_{-\infty}^{+\infty} \left[1 + \frac{K(\Omega + \Phi_0) N(X, \Omega + \Phi_0)}{D(\Omega + \Phi_0)} \right] \frac{\exp(\mathrm{i}\Omega(X_0 - X))\, \mathrm{d}\Omega}{\Omega^2 - S^2},$$

where $X_0 = x_0/l = v_0 t/l$ is the dimensionless co-ordinate of the excitation point. This coincides with $T = v_0 t/l$. One sees that $\Phi_0 T = \omega_0 t$. Let $R = 0$ again. Taking into account (2) and (23), one, further, obtains

$$S^2 = (V_0 \Omega + \Omega_0)^2, \qquad K(\Omega + \Phi_0) = K + \mathrm{i}K_1(V_0 \Omega + \Omega_0)$$

where $\Omega_0 = V_0 \Phi_0$ can be considered now as the dimensionless angular velocity of excitation. Let X_0 stay constant, but V_0 tends to zero in the expression obtained. Then, S^2 and $K(\Phi)$ should be replaced by the real constant Ω_0^2 and the complex constant $K_0 = K + \mathrm{i}K_1\Omega_0$. Thus,

$$y(x, t) = lY(X, T) = (a_0 l/f) \exp(\mathrm{i}\omega_0 t) A(X, X_0).$$

The last expression shows that the string point X performs harmonic oscillations with the angular velocity ω_0. The quantity

$$A(X, X_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[1 + K_0 \frac{\exp(\mathrm{i}\Omega X) \sin(\Omega_0(1 - X)) + \exp(\mathrm{i}\Omega(X - 1)) \sin(\Omega_0 X)}{2\Omega_0(\cos\Omega - \cos\Omega_0) - K_0 \sin\Omega_0} \right] \times \frac{\exp(\mathrm{i}\Omega(X_0 - X))\, \mathrm{d}\Omega}{\Omega^2 - \Omega_0^2} \tag{30}$$

is the complex amplitude of oscillations of the string at point X, caused by the stationary harmonic exciting force at the point X_0. In the integral (30), the excitation point co-ordinate X_0, that has replaced T, can be an arbitrary real number, but $0 \leqslant X \leqslant 1$. If $X_0 \to \pm\infty$, then $A(X, X_0) \to 0$. This means that at the string point X oscillations decay, if the excitation point X_0 moves away from the point X. If $X = X_0$, then the integral (30) represents the complex amplitude of the excitation point. If, finally, Ω_0 tends to zero, then K_0 tends to K. Replacing $\cos\Omega_0$, $\sin\Omega_0$, $\sin(\Omega_0 X)$ and $\sin(\Omega_0(1 - X))$ by 1, Ω_0 $\Omega_0 X$, and $\Omega_0(1 - X)$, one obtains the string dimensionless static deflection

$$A_0(X, X_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[1 + K_0 \frac{(1 - X) \exp(\mathrm{i}\Omega X) + X \exp(\mathrm{i}\Omega(X - 1))}{2(\cos\Omega - 1) - K} \right] \times \frac{\exp(\mathrm{i}\Omega(X_0 - X))\, \mathrm{d}\Omega}{\Omega^2}$$

and, if $X = X_0$ again, the value $A_0(X_0, X_0)$ is the string static dimensionless deflection at the excitation point. The latter can be understood as the dimensionless local flexibility of the periodic structure. Figure 7 shows how the values of $A_0(X_0, X_0)$ depend on X_0. These values have been calculated with different K. As K decreases, then the values of $A_0(X_0, X_0)$ increase and the difference between $A_0(1/2, 1/2)$ and $A_0(0, 0)$ lessens at that time. This means that the string discrete suspensions act as continuous uniform suspension, if K is sufficiently small.

10. RESONANCE AND ANTI-RESONANCE IN RESPONSE TO STATIONARY EXCITATION

In order to study the resonance of the periodic structure, caused by the stationary harmonic exciting force, one should take $K_1 = 0$, $K_0 = K$ and search for the second order real poles of integrand (30). By studying the roots of the equation

$$\cos \Omega = \cos \Omega_0 + K \sin \Omega_0/(2\Omega_0), \tag{31}$$

such poles can be found. The right side of the equation depends on two parameters K and Ω_0. The dependence has been shown in Figure 2. If Ω coincides with the second order root of the equation (31), then $\sin \Omega = 0$ and so $\cos \Omega = \pm 1$. Thus, the equation (31) has the second order root, if K and Ω_0 obey the following condition:

$$\cos \Omega_0 + K \sin \Omega_0/(2\Omega_0) = \pm 1 \tag{32}$$

One sees, that the values $\Omega_0 = \pi n$, $n = \pm 1, \pm 2, \pm 3, \ldots$, obey this condition with any real and even complex K. As Figure 2 shows, there is another value Ω_0 between πn and $\pi(n+1)$, obeying the condition (32) with any given real K. This value depends on K.

If the excitation point X_0 coincides with 0 or 0·5, then the exciting force produces the string oscillations, which are symmetric with respect to the excitation point, and so, in these particular cases of excitation, resonance may not take place in despite of fulfilment of the condition (32). Indeed, in this case, the denominator of the fraction in the square brackets in the integrand (30) turns into $2\Omega_0(\cos \Omega \mp 1)$. If $X_0 = 0$, then the fraction reduces to

$$K \sin \Omega_0/[2\Omega_0(\cos \Omega \mp 1)]$$

and turns into zero as soon as $\Omega_0 = \pi n$. Thus, there is no pole of the second order and no resonance as well. Further, if $X_0 = 0{\cdot}5$, then the same reduces to

$$K \sin (\Omega_0/2) \cos (\Omega/2)/[\Omega_0(\cos \Omega \mp 1)]$$

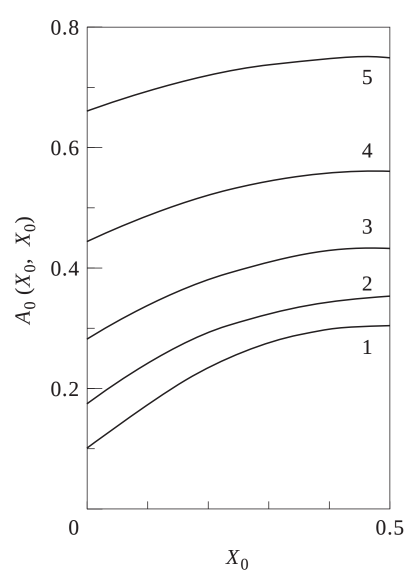

Figure 7. Static flexibility of the periodic structure; 1, $K = 0{\cdot}5$; 2, $K = 1$; 3, $K = 2$; 4, $K = 4$; 5, $K = 8$.

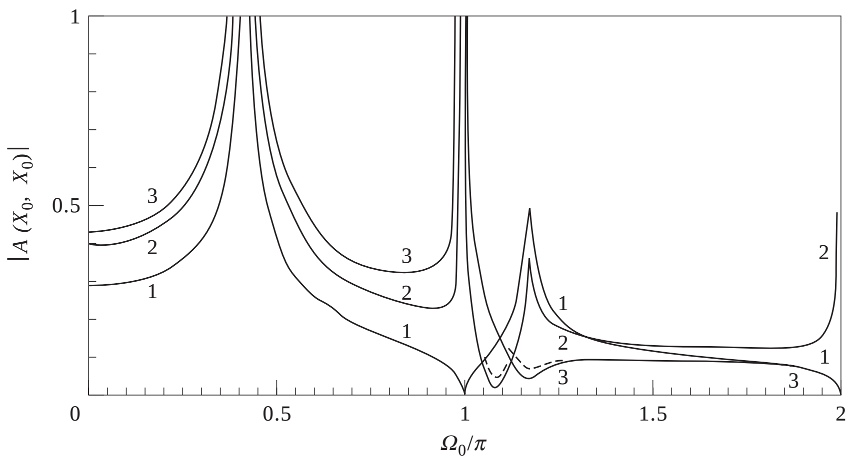

Figure 8. Frequency response to stationary excitation; 1, $X_0 = 0$; 2, $X_0 = 0{\cdot}25$; 3, $X_0 = 0{\cdot}5$.

and turns into zero as soon as $\Omega_0 = 2\pi n$. There is no resonance again. If the right side of (32) is equal to -1, then the last expression is

$$K \sin(\Omega_0/2)\cos(\Omega/2)/[\Omega_0(\cos\Omega + 1)] = K\sin(\Omega_0/2)/[2\Omega_0(\cos(\Omega/2)]$$

and so the integrand reduces to the following

$$[1 + K\sin(\Omega_0/2)/[2\Omega_0(\cos(\Omega/2)]\exp(\mathrm{i}\Omega(X_0 - X))/(\Omega^2 - \Omega_0^2)$$

which shows there is a pole of the second order and resonance, only if $\Omega_0 = \pi(1 + 2n)$.

In order to prove this analysis, the calculations of the integral (3) with the small value 0·04 of the suspension viscous damping K_1 were made. If $\Omega_0 = \pi n$, then the denominator of the fraction in the square brackets in the integrand (30) becomes $2\Omega_0(cos\ \Omega \mp 1)$, as has been mentioned. This means that the integrand has real poles in despite of presence of the positive K_1 and the integral cannot be calculated with such Ω_0. Therefore, values of Ω_0 between zero and π and between π and 2π were chosen for calculations. According to Figure 2, there are four resonant values $0{\cdot}415\pi$, π, $1{\cdot}17\pi$ and 2π of Ω_0 in this band. Three values 0, 0·25 and 0·5 of X_0 were chosen to present the string symmetric and asymmetric excitation. The values of $|A(X_0, X_0)|$, calculated by means of the integral (30) with $K = 2$ and $X = X_0$, are shown in Figure 8 and represent the periodic structure frequency response to excitation by the stationary harmonic force. The curve 1 presents the frequency response to the symmetric excitation at the point $X_0 = 0$, that coincides with the suspension location. The string does not experience resonance if $\Omega_0 = \pi$ and $\Omega_0 = 2\pi$. This is in accordance with the previous analysis. The curve 2 presents the frequency response to the asymmetric excitation at $X_0 = 0{\cdot}25$. In this case, the string experiences resonance, if Ω_0 coincides with any resonant value, listed above. The curve 3 presents the frequency response to the symmetric excitation at the mid-span $X_0 = 0{\cdot}5$. In accordance with the previous analysis, there is no resonance at $\Omega_0 = 1{\cdot}17\pi$ and $\Omega_0 = 2\pi$.

One sees that Figure 8 provides some additional information. There are two types of anti-resonance, that seem not to be possible in response to moving excitation. The first type corresponds to the symmetric excitation and is not affected by the suspension viscous damping. It is seen at $\Omega_0 = \pi$, if $X_0 = 0$, and at $\Omega_0 = 2\pi$, if $X_0 = 0$ or $X_0 = 0{\cdot}5$. It can be easily proved that K_0 disappears from the integrand (30) in all of these cases as well as the suspension damping K_1. The excitation point X_0 is strictly fixed and so there is no energy influx into the periodic structure. If steady-state oscillations take place, then energy outflow must not exist, occuring only if the string experiences oscillations in the form of a standing wave and all suspension locations coincide with the wave nodes. In such case, there is no energy flow along the string and no energy loss in the strictly fixed suspensions. Figure 9 shows the string shapes in the form of the standing wave. Vertical strokes locate the suspensions. Vertical arrows locate the exciting force that is balanced by the tension forces in two string wings. The curve 1 corresponds to $\Omega_0 = \pi$ and $X_0 = 0$, curves 2 and 3 to $\Omega_0 = 2\pi$, $X_0 = 0$ and $X_0 = 0{\cdot}5$. Such string deflection cannot be obtained by means of the integral (30) because there is no decay far from the excitation point. But it can be easily obtained from the equation (10), if one considers oscillations of the string wings which cannot take place in periodically supported beams because of their bending stiffness. Taking into account the physical explanation of anti-resonance, one should expect such anti-resonance to be caused by asymmetric excitation. The second type of anti-resonance can be seen from Figure 8 at the resonant dimensionless angular velocity $\Omega_0 = 1{\cdot}17\pi$, if $X_0 = 0{\cdot}5$, and at the non-resonant $\Omega_0 = 1{\cdot}08\pi$, if $X_0 = 0{\cdot}25$. Such anti-resonance is affected by the suspension viscous damping and so the excitation point experiences small oscillations making possible energy influx in the periodic structure that compensates for energy loss in the oscillating suspensions. The excitation point amplitude depends on K_1, increasing with K_1. This is shown by means of dotted curves which correspond to the greater value 0·16 of K_1. In the limit case $K_1 = 0$, the excitation point is strictly fixed.

The calculations of $|A(X_0, X_0)|$ show there is, sometimes, an infinite response, if $X \neq 0$ and $\Omega_0 = \pi n$, with any non-zero K_1. It can appear only, if the exciting force produces a standing wave, all the suspensions coincide with the wave nodes and so there is no energy loss again. This is similar to the pinned–pinned resonance periodically supported beams [15].

11. CONCLUSIONS

An infinite string, supported by equidistantly spaced identical visco-elastic suspensions, was considered as well as a concentrated harmonic force, that moves steadily along the string and causes steady-state small oscillations in the string. This allows one to consider just one segment of the string, between neighbouring suspension points, and creates a boundary problem that includes the time delay and the phase lead between two points of

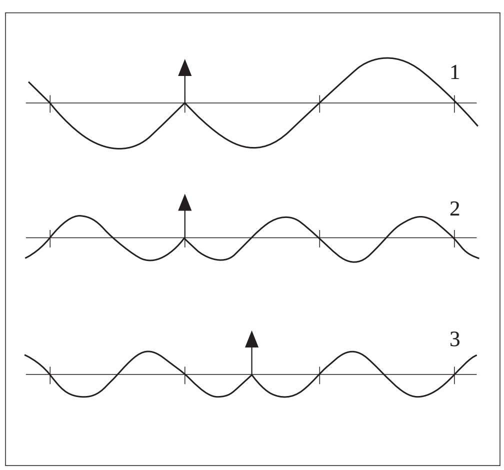

Figure 9. Anti-resonance; 1, $X_0 = 0$, $\Omega_0 = \pi$; 2, $X_0 = 0$, $\Omega_0 = 2\pi$; 3, $X_0 = 0{\cdot}5$, $\Omega_0 = 2\pi$.

the string separated by the suspension space. Both approach infinity as the speed of the exciting force approaches zero and so a stationary exciting force cannot be directly included in the consideration.

Fourier transformation was used to solve the boundary problem and the solution was presented in the form of a single integral that was used for calculation and for qualitative analysis. This showed that the periodic string, having a continuous spectrum of free oscillations, experiences resonance, if the integrand has a real pole of the second order. The string oscillations' dependence on the suspension stiffness was studied. A Doppler effect occurs if the stiffness is small enough or equal to zero. The latter corresponds to the free string. As the study showed, there is no resonance if the suspension stiffness approaches infinity. This can be explained as follows. The rigid suspensions divide the string into an infinite sequence of isolated segments. Any isolated segment is exposed to the moving exciting force over a limited time and, therefore, string resonance is impossible.

In order to consider stationary excitation, a suitable limit procedure was used. Resonance occurred again as well as anti-resonance, that seems to be impossible in response to moving excitation. If the excitation point coincides with the suspension location or with the string mid-span, then the exciting force produces symmetric oscillations in the string. In these particular cases of excitation, resonance at some resonant frequencies disappears, but anti-resonance appears instead. In some cases, the exciting force produces a standing wave in the string, each suspension coincides with the wave node and so any suspension is strictly fixed as well as the excitation point. Such anti-resonance is not affected by suspension viscous damping and cannot appear in periodically supported beams.

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