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ERRORS IN THE MEASUREMENT OF STRUCTURE-BORNE POWER FLOW USING TWO-ACCELEROMETER TECHNIQUES

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This paper presents a theoretical study of three errors in the measurement of structure-borne power flow in one-dimensional structures using two-accelerometer techniques. It is assumed that the physical and material properties of the test structure are known. For the measurement of bending wave power flow, the errors due to the neglect of shear force and rotating inertia and the presence of longitudinal waves are examined individually. It is shown that the omission of shear force and rotating inertia results in a large bias error at high frequencies. The presence of incoherent longitudinal waves results in no bias error for the biaxial accelerometer technique or usually a negligible bias error for the two-accelerometer array technique. However, if longitudinal waves are coherent with bending waves the bias error increases with increasing the coherence and the longitudinal to bending wave energy ratio and becomes large when the bending wave energy is smaller than the highly coherent longitudinal wave energy. For the measurement of longitudinal wave power the effect of bending waves is important. As long as the bending wave power is not much smaller than the longitudinal wave power, the bias error is large even if bending and longitudinal waves are incoherent. Measures should be taken during measurements to eliminate or reduce the contribution from bending waves to a certain extent, otherwise, the longitudinal wave power cannot be measured using the two-accelerometer array technique.

1. INTRODUCTION

The ability of structural intensity techniques to measure both the magnitude and direction of power flow makes them attractive techniques for investigating a wide variety of structures. However, intensity measurement techniques are prone to errors. These errors depend on the properties of the structure to be measured, the type of waves present on the structure and the instrumentation employed in the measurement. Attempting to analyse all the errors at the same time results in a very complicated analysis and makes it difficult to draw clear conclusions. Therefore, it is necessary to examine each of the errors in isolation. Some of the errors have been theoretically analysed [1–4] and some still need to be further investigated.

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These errors can be classified as either bias errors or random errors. Bias errors can be calculated exactly and result in either an over or under estimation of the true power flow. Random errors cannot be calculated exactly and for any given situation they too can either over or under estimate the true power flow. This paper looks at three bias errors that can occur in the measurement of structure-borne power flow using the two accelerometer techniques. For simplicity the analysis is restricted to one dimensional beams. However, in some cases the method of analysis can be extended to two dimensional plates.

Of all structural intensity techniques, both the two-accelerometer array and biaxial accelerometer techniques are convenient to implement and widely used in practical engineering. These two techniques are briefly described in the first part of this paper, then three bias errors associated with them are examined individually. The errors due to the presence of longitudinal waves and the neglect of shear deformation and rotatory inertia are examined for the measurement of bending wave power. It is shown that the error due to the neglect of shear deformation and rotatory inertia increases with increasing frequency and becomes large at high frequencies. The effect of coherent longitudinal waves could be important if the coherence is high and the longitudinal wave energy is greater than the bending wave energy, however, the effect of incoherent longitudinal waves is usually negligible. For the measurement of longitudinal wave power flow, the error due to the presence of bending waves is examined. It is shown that the contribution from either coherent or incoherent bending waves is important and must be eliminated or reduced to a certain extent, otherwise, the measured results will be corrupted.

2. MEASUREMENT OF STRUCTURE-BORNE POWER FLOW

For a one-dimensional beam lying along the x direction of a Cartesian co-ordinate system, as shown in Figure 1, the translational acceleration at any farfield location due to the propagation of travelling bending waves can be expressed as

$$a = (A_+ \, \mathrm{e}^{-\mathrm{j}kx} + A_- \, \mathrm{e}^{\mathrm{j}kx}) \, \mathrm{e}^{\mathrm{j}\omega t} \tag{1}$$

where $\mathrm{j} = \sqrt{-1}$; k is the bending wavenumber; $\omega = 2\pi f$ is the radian frequency; A_+ and A_- represents the amplitudes of the travelling bending waves in the positive and negative directions respectively. The longitudinal wave acceleration is given as

$$a_L = (B_+ \, \mathrm{e}^{-\mathrm{j}k_L x} + B_- \, \mathrm{e}^{\mathrm{j}k_L x}) \, \mathrm{e}^{\mathrm{j}\omega t} \tag{2}$$

where k_L is the longitudinal wavenumber; B_+ and B_- are the amplitudes of the longitudinal waves propagating in the positive and negative directions respectively.

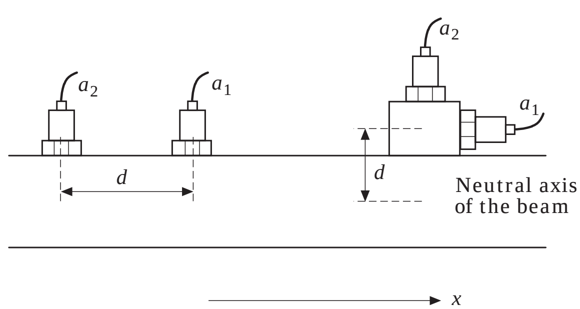

Figure 1. Diagram of the two-transducer technique probes used in the measurement of bending wave power flow on a uniform beam.

The spectral densities of the bending and longitudinal wave power flows, P_B and P_L in a thin beam can be given as [1, 5, 6]

$$P_B = (2m_b c_b^2/\omega^3)\ \mathrm{Im}\,\{G(\partial a/\partial x, a)\} \tag{3}$$

$$P_L = (m_b c_l^2/\omega^3)\ \mathrm{Im}\,\{G(\partial a_L/\partial x, a_L)\} \tag{4}$$

where m_b is the mass per unit length of the beam; c_b and c_l are the bending and longitudinal wavespeeds; Im $\{G(\partial a/\partial x, a)\}$ is the imaginary part of the cross spectrum, $G(\partial a/\partial x, a)$ between the acceleration a and its first spatial derivative. In practice, either the bending or longitudinal wave acceleration can be measured directly using a translational or rotational accelerometer, however, the measurement of the first spatial derivative of an acceleration is not a straightforward procedure. The first spatial derivative can only be approximately obtained from other measured data. Based on the principle of finite difference, the first spatial derivative of either bending or longitudinal wave acceleration can be estimated from the difference of two accelerometer signals so that the spectral densities of the bending and longitudinal wave power flows in a thin beam can be measured using a two-accelerometer array and are given by [6]

$$P_B = (2m_b c_b^2/\omega^3 d)\ \mathrm{Im}\,\{G(a_1, a_2)\} \tag{5}$$

$$P_L = (m_b c_l^2/\omega^3 d)\ \mathrm{Im}\,\{G(a_{L1}, a_{L2})\} \tag{6}$$

where d is the accelerometer separation and a_1 (a_{L1}) and a_2 (a_{L2}) are the bending (longitudinal) wave acceleration signals measured using the accelerometer array. For bending waves, the first spatial derivative of a translational acceleration is the rotational acceleration so that it can also be indirectly measured using a rotational accelerometer. Therefore, the spectral density of the bending wave power flow in a thin beam can be measured alternatively using a biaxial accelerometer [5] as shown in Figure 1, and estimated using equation (5) where d is the distance from the neutral axis of the beam to the axis of the rotational accelerometer.

It can be seen from equations (3) to (6) that the power flow depends on both the imaginary part of the cross spectrum and the material properties of the beam being tested. The errors associated with the physical and material properties are relatively straight forward and are not discussed here. It is assumed in the following analysis that the physical and material properties of the beam are known.

Bias errors are systematic errors that can be computed for any given situation and can be given in normalized form as

$$\varepsilon = (P_{me} - P)/P = P_{me}/P - 1 \tag{7}$$

where ε is the normalized bias error; P_{me} and P are the measured and true values of the power flow. Measured power can sometimes be corrected for bias errors during data processing using the following expression

$$P = P_{me}/(1 + \varepsilon) \tag{8}$$

3. BIAS ERROR DUE TO THE EFFECTS OF SHEAR DEFORMATION AND ROTATORY INERTIA

Equation (3) is derived from classical Bernoulli–Euler bending theory, which deals with pure bending waves, and is valid only for bending wavelengths much larger than the thickness of the structure along which propagation occurs. For short wavelengths, shear deformation and rotatory inertia become important [7, 8], and the estimation of bending

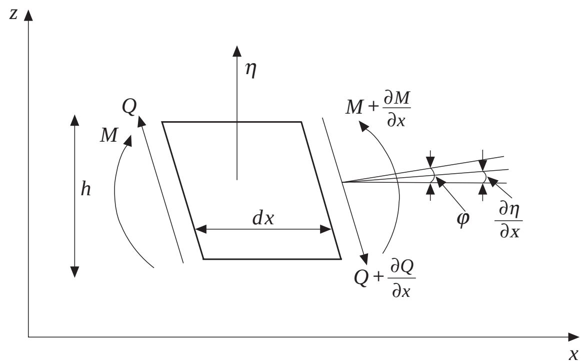

Figure 2. An element of a beam subjected to bending and shear deformation

wave power flow using equation (3) will result in an estimate error. In order to estimate this error, the effects of shear deformation and rotatory inertia were included.

Consider an element of a uniform beam, as shown in Figure 2. At rest the beam is straight and lies along the x-axis. If the angle φ denotes the rotation of the neutral axis of the beam due to bending, η denotes the displacement due to bending, Q and M represent the shear force and bending moment respectively, the following differential equations will hold [9]

$$\frac{\partial\eta}{\partial x}=\varphi-\frac{Q}{g}; \quad \frac{\partial^2\varphi}{\partial x\partial t}=-\frac{1}{B}\frac{\partial M}{\partial t}; \quad -\frac{\partial M}{\partial x}=Q+\rho J\frac{\partial^2\varphi}{\partial t^2}; \quad -\frac{\partial Q}{\partial x}=m_b\frac{\partial^2\eta}{\partial t^2} \qquad (9)$$

where B is the bending stiffness; ρ is the material density; J is the second moment of area of the cross-section of the beam; g is the modified shear stiffness and given by [9]

$$g = Ehb/2(1+\mu)\kappa \qquad (11)$$

where h and b are the thickness and width of the beam; μ is Poisson's ratio and κ is a constant, equal to 1·2 for a rectangular cross-section [9]. The Timoshenko bending wave equation can be derived from equations (9) as [9, 10]

$$\frac{B}{m_b}\frac{\partial^4\eta}{\partial x^4}+\frac{\partial^2\eta}{\partial t^2}-\left[\frac{h^2}{12}+\frac{B}{g}\right]\frac{\partial^4\eta}{\partial x^2\partial t^2}+\frac{\rho J}{g}\frac{\partial^4\eta}{\partial t^4}=0 \qquad (12)$$

which describes two types of bending waves each with a different wavenumber. At low frequencies one wave is a travelling bending wave and the other is a decaying bending wave as in the solution to the wave equation for thin beams. The two wavenumbers, k_1 and k_{2n}, are given by [10]

$$k_1=\sqrt{\frac{m_b\omega^2}{2B}\left(\frac{h^2}{12}+\frac{B}{g}\right)+\frac{m_b}{2B}\sqrt{\left(\frac{h^2}{12}+\frac{B}{g}\right)^2\omega^4+4\frac{B\omega^2}{m_b}\left(1-\frac{\rho J\omega^2}{g}\right)}} \qquad (12)$$

$$k_{2n}=\sqrt{\frac{m_b}{2B}\sqrt{\left(\frac{h^2}{12}+\frac{B}{g}\right)^2\omega^4+4\frac{B\omega^2}{m_b}\left(1-\frac{\rho J\omega^2}{g}\right)}-\frac{m_b\omega^2}{2B}\left(\frac{h^2}{12}+\frac{B}{g}\right)} \qquad (13)$$

where k_1 is the wavenumber of travelling bending wave and k_{2n} is the wavenumber of the decaying bending wave. It can be shown from equation (13) that when the frequency is greater than

$$f_0 = (1/2\pi)\sqrt{g/\rho J} \approx c_l/\pi h \tag{14}$$

the decaying bending wave becomes a travelling bending wave [10]. When the frequency of interest is much smaller than f_0, then

$$4B/m_b \gg \left(\frac{h^2}{12} + \frac{B}{g}\right)^2 \omega^2; \qquad g \gg \rho J \omega^2$$

equations (12) and (13) can be simplified to

$$k_1 \approx k_{2n} \approx k = \sqrt[4]{m_b \omega^2 / B} \tag{15}$$

which is the bending wavenumber for a thin beam.

The bending moment and shear force can be derived from equation (9) as

$$M = -B\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{m_b \omega^2}{g}\eta\right), \qquad Q = \frac{gB}{g - \rho J \omega^2}\frac{\partial^3 \eta}{\partial x^3} + \frac{(\rho J g + m_b B)\omega^2}{g - \rho J \omega^2}\frac{\partial \eta}{\partial x} \tag{16, 17}$$

When the frequency is smaller than f_0, the bending wave power flow in the far field is due to the travelling wave and its spectral density is given by

$$P_B = \mathrm{Im}\left\{G(Q, v) + G\left(M, \frac{\partial v}{\partial x}\right)\right\} = \frac{\tau}{\omega^3}\,\mathrm{Im}\left\{G\left(\frac{\partial a}{\partial x}, a\right)\right\} \tag{18}$$

where $v = \mathrm{j}\omega\eta$ is the translational velocity and τ is given by

$$\tau = \frac{gBk_1^2 - (\rho J g + m_b B)\omega^2}{g - \rho J \omega^2} + B\left(k_1^2 - \frac{m_b \omega^2}{g}\right) \tag{19}$$

When the bending wavelength is much larger than the thickness of the beam, equation (18) can be simplified to equation (3). Equation (3) is therefore an approximation of equation (18) and the use of equation (3) instead of equation (18) results in an estimate error, which leads to a normalized bias error ε_a

$$\varepsilon_a = \frac{2[1 - (f/f_0)^2](k/k_1)^2}{2 - 0{\cdot}716(f/f_0)^2(\lambda/h)^2 - (f/f_0)^2[1 - 0{\cdot}316(f/f_0)^2(\lambda/h)^2]} - 1 \tag{20}$$

At frequencies far below f_0, this equation can be simplifed to

$$\varepsilon_a = 0{\cdot}9 f/f_0 \tag{21}$$

which shows that low frequencies the normalized bias error, ε_a, is linearly proportional to the frequency.

At frequencies greater than f_0, the decaying bending wave becomes a travelling bending wave (the wavenumber k_{2n} in equation (13) becomes an imaginary number) so that there are two travelling bending waves propagating simultaneously along the beam in each direction. If a_{k1} and a_{k2} represent the accelerations of the travelling bending waves with wavenumbers k_1 and k_2 ($k_2 = -\mathrm{j}k_{2n}$ is a real number) respectively, the spectral densities of the force and moment related power components, P_{BQ} and P_{BM}, can be expressed as

$$P_{BQ} = \mathrm{Im}\,\{G(Q, v)\} = \frac{gB}{g - \rho J \omega^2}\sum_{i=1}^{2}\frac{k_i^2}{\omega^3}\,\mathrm{Im}\left\{G\left(\frac{\partial a_{ki}}{\partial x}, a\right)\right\} - \frac{\rho J g + m_b B}{\omega(g - \rho J \omega^2)}\,\mathrm{Im}\left\{G\left(\frac{\partial a}{\partial x}, a\right)\right\} \tag{22}$$

$$P_{BM} = \mathrm{Im}\left\{G\left(M, \frac{\partial v}{\partial x}\right)\right\} = \frac{B}{\omega^3}\sum_{i=1}^{2}\left(k_i^2 - \frac{m_b\omega^2}{g}\right)\mathrm{Im}\left\{G\left(\frac{\partial a}{\partial x}, a_{ki}\right)\right\} \tag{23}$$

where $a = a_{k1} + a_{k2}$ is the total bending wave acceleration. The total bending wave power is the sum of the force and moment related components for both wavetypes. Using only two accelerometers it is not possible to differentiate between a_{k1} and a_{k2} and so meaningful measurements cannot be made.

The error due to the omission of shear and rotatory effects below the frequency f_0 can be seen by considering bending waves propagating on a masonry column with a cross-section of $0{\cdot}44 \times 0{\cdot}44$ m. The frequency f_0 for this column is 1596 Hz which was calculated from equation (14). The normalized bias error, ε_a, is shown in Figure 3 which shows both the exact estimate of the error using equation (20) and the approximate estimate of the error from equation (21). It can be seen that the approximate estimate from equation (21) is accurate at low frequencies where the frequency ratio, f/f_0, is smaller than 0·2 and reasonable even at high frequencies close to the frequency f_0. Equation (3) overestimates the bending wave power and the estimate error increases with increasing frequency. For this type of building structure the error is important in all but the lowest frequencies and therefore equation (18) rather than equation (3) should be used to calculate bending wave power flow.

Figure 3 also shows the negative normalized bias error $(-(k - k_1)/k_1)$ of the bending wavenumber for the masonry column. It can be seen that the error in the power flow is much larger than the error in the wavenumber. A 10% error in the estimate of powerflow occurs at about one octave below the frequency where there is a 10% error in the wavenumber.

4. BIAS ERROR DUE TO THE PRESENCE OF LONGITUDINAL WAVES

In real structures other types of structural waves are usually present in addition to bending waves. These can cause bias errors in the measurement of bending wave power. Longitudinal waves travelling in a uniform beam will produce lateral displacements, as

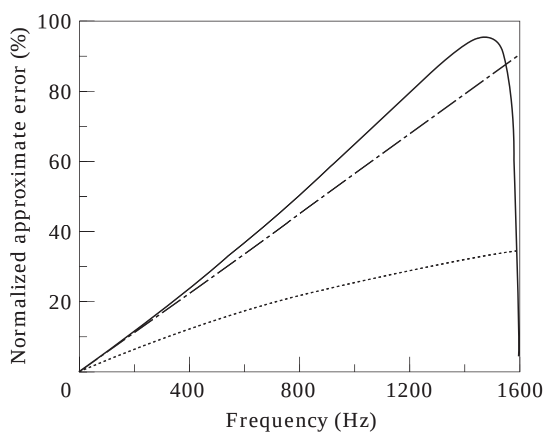

Figure 3. Normalized bias error due to the omission of the effects of shear deformation and rotatory inertia. Error in the estimate of bending wave power: ——, exact error calculated using equation (20); — - — -; approximate error calculated by using equation (21), - - - - -, error in the estimatae of bending wavenumber.

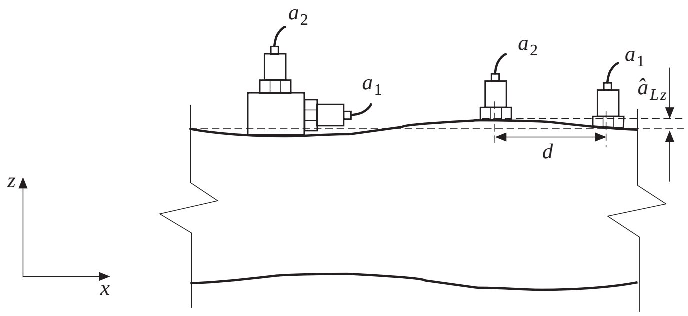

Figure 4. Cross-contraction associated with longitudinal waves in a uniform beam.

shown in Figure 4, because of the cross-contraction phenomenon [9]. For a rectangular beam with a thickness h, the amplitude of the lateral acceleration $\hat{a}_{LZ}$ resulting from the presence of longitudinal waves is given by [9].

$$\hat{a}_{Lz} = \pm(\pi\mu h/\lambda_L)\hat{a}_L \tag{24}$$

where $\hat{a}_L$ is the amplitude of the longitudinal wave acceleration, a_L, and λ_L is the longitudinal wavelength.

Since $\pi\mu$ is approximately unity for most structural materials, the amplitude ratio of the lateral acceleration a_{Lz} to the longitudinal wave acceleration a_L is approximately equal to the ratio of the thickness h to the longitudinal wavelength λ_L. For a constant longitudinal wave acceleration, the higher the frequency the greater the lateral acceleration. When longitudinal wavelengths are of the same order as, or shorter than, the thickness h, equation (24) can no longer be valid as the lateral displacements on opposite surfaces of the beam are no longer in phase over the cross-section and do not vary linearly with distance from the neutral axis of the beam [9].

In practice, translational accelerometers also have a non-zero transverse sensitivity which can also result in the accelerometers generating signals when they are excited by inplane waves. When both bending and longitudinal waves propagate along a uniform beam, the measured bending wave acceleration a_{BM} consists of three components, the bending wave acceleration a and the lateral acceleration a_{Lz} due to the cross-contraction phenomenon and the contribution from the horizontal motions due to the non-zero tranverse sensitivity of the accelerometer, and is given by

$$a_{Bm} = a + a_{Lz} + \alpha(a_L + a^R) = a + \alpha a^R + \alpha_0 a_L \tag{25}$$

where $\alpha_0 = \alpha \pm \pi\mu h/\lambda_L$; α is the transverse sensitivity of the accelerometer; $a^R = d(\partial a/\partial x)$ is the rotational acceleration associated with bending waves in the absence of longitudinal waves. For the measurement performed using the biaxial accelerometer technique, the rotational acceleration that is measured, a_{Rm}, will also include three components: the bending rotation, the longitudinal wave acceleration and the contribution from the vertical motions, and is given by

$$a_{Rm} = a^R + a_L + \alpha(a + a_{Lz}) = a^R + \alpha a + (1 + \alpha\pi\mu h/\lambda_L)a_L$$

In practice, the transverse sensitivity of any accelerometer, α, is usually smaller than 0·05, therefore, the quantity, $\alpha\pi\mu h/\lambda_L$, is much smaller than 1, the above equation can be approximated to

$$a_{Rm} \approx a^R + \alpha a + a_L \tag{26}$$

Theoretically, the effects of longitudinal waves on either translational or rotational acceleration of bending waves can be eliminated if two identical accelerometers (both phases and amplitudes are well matched) are fixed on the same position but opposite sides of the test structure [6]. Therefore, for the intensity measurements, the probe should consist of four identical accelerometers and two signal sum/difference preamplifiers, as shown in Figure 5. The phases among the accelerometers and between the signal sum/difference preamplifiers must be well matched. If not, the effects of longitudinal waves cannot be eliminated or reduced to a given extent, and the use of the four-accelerometer probes shown in Figure 5 could result in a larger bias error than the use of the two-accelerometer probes shown in Figure 1. However, it is difficult, in practice, to find four phase-matched accelerometers, especially for high frequencies.

Also in practice, difficulties may arise for using the four-accelerometer probes, for examples (1) only one side of the test structure can be accessed; (2) less than four identical accelerometers or no/one signal sum/difference preamplifier may be available. The two-accelerometer probes are easier and more convenient to implement than the four-accelerometer probes. It is useful to estimate approximately the bias error before the measurement and then to decide which kind of intensity probe should be employed.

If the biaxial accelerometer technique is employed, from equations (25) and (26) the measured imaginary part of the cross spectrum can be expressed as

$$\operatorname{Im}\{G(a_{Rm}, a_{Bm})\} \approx (1-\alpha^2)\operatorname{Im}\{G(a^R, a)\} + (1-\alpha\alpha_0)\operatorname{Im}\{G(a_L, a)\} + (\alpha_0 - \alpha)\operatorname{Im}\{G(a^R, a_L)\}$$

Since the rotational acceleration a^R is usually smaller than translational acceleration a, and the quantity $|\alpha_0 - \alpha| = \pi\mu h/\lambda_L$ is less than 1, and the coherence between a and a_L should be equal to the coherence between a^R and a_L, the above equation can be further approximated to

$$\operatorname{Im}\{G(a_{Rm}, a_{Bm})\} \approx (1-\alpha^2)\operatorname{Im}\{G(a^R, a)\} + \operatorname{Im}\{G(a_L, a)\} \tag{27}$$

According to equation (7), the normalized bias error $\varepsilon_{L\alpha}$ can be expressed as

$$\varepsilon_{L\alpha} \approx -\alpha^2 + \frac{\operatorname{Im}\{G(a_L, a)\}}{\operatorname{Im}\{G(a^R, a)\}} = -\alpha^2 + \sqrt{\gamma^2 \frac{G(a_L)}{G(a^R)}}\,\frac{\sin\theta_{LB}}{\sin\theta_{12}} \tag{28}$$

where γ^2 is the coherence between the bending and longitudinal wave accelerations, a and a_L; $G(a)$ and $G(a_L)$ are the autospectra of the bending and longitudinal wave accelerations;

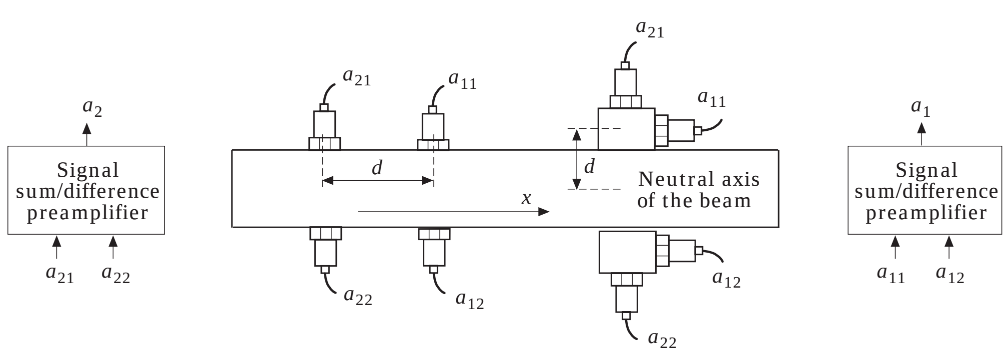

Figure 5. Diagram of the four-accelerometer probes used in the measurement of bending wave power flow on a uniform beam where longitudinal waves exist.

θ_{12} and θ_{LB} are the phases of the cross spectra, $G(a^R, a)$ and $G(a_L, a)$. The first term in the above equation represents the normalized bias error, ε_a, solely due to the non-zero transverse sensitivities of the biaxial accelerometer, the value of which is usually smaller than 0·0025 and can be neglected. The second term is the normalized bias error ε_L solely due to the presence of coherent longitudinal waves. If the bending and longitudinal waves are incoherent, i.e., $\gamma^2 = 0$, the normalized bias error ε_L is zero. This indicates that incoherent longitudinal waves will not affect the measurement accuracy of the biaxial accelerometer technique. If bending and longitudinal waves are coherent, the normalized bias error ε_L is proportional to the ratio of the imaginary parts of two cross spectra and could be greater than 1 if the coherence is high and the bending wave energy is smaller than the longitudinal wave energy.

For the measurement of bending wave power flow using the two-accelerometer array technique, the measured imaginary part of the cross spectrum can be expressed as

$$\begin{aligned}
\mathrm{Im}\,\{\mathrm{G}(\mathrm{a_{Bm1}}, \mathrm{a_{Bm2}})\} &= \mathrm{Im}\,\{G(a_1, a_2)\} + \alpha_0^2\,\mathrm{Im}\,\{G(a_{L1}, a_{L2})\} + \alpha_0\,\mathrm{Im}\,\{G(a_{L1}, a_2) \\
&\quad + G(a_1, a_{L2})\} + \alpha^2\,\mathrm{Im}\,\{G(a_1^R, a_2^R)\} + \alpha\,\mathrm{Im}\,\{G(a_1^R, a_2) \\
&\quad + G(a_1, a_2^R)\} + \alpha\alpha_0\,\mathrm{Im}\,\{G(a_{L1}, a_2^R) + G(a_1^R, a_{L2})\} \\
&= \mathrm{Im}\,\{G(a_1, a_2)\} + \alpha^2\,\mathrm{Im}\,\{G(a_1^R, a_2^R)\} + \alpha\,\mathrm{Im}\,\{G(a_1^R, a_2) \\
&\quad + G(a_1, a_2^R)\} + \alpha_0^2\,\mathrm{Im}\,\{G(a_{L1}, a_{L2})\} + \alpha_0\,\mathrm{Im}\,\{G(a_{L1}, a_2 + \alpha a_2^R) \\
&\quad + G(a_1 + \alpha a_1^R, a_{L2})\} \\
&\approx Im\{G(a_1, a_2)\} + \alpha^2\,\mathrm{Im}\,\{G(a_1^R, a_2^R)\} + \alpha\,\mathrm{Im}\,\{G(a_1^R, a_2) \\
&\quad + G(a_1, a_2^R)\} + \alpha_0^2\,\mathrm{Im}\,\{G(a_{L1}, a_{L2})\} + \alpha_0\,\mathrm{Im}\,\{G(a_{L1}, a_2) \\
&\quad + G(a_1, a_{L2})\}
\end{aligned} \tag{29}$$

For travelling bending waves given by equation (1), there are

$$\mathrm{Im}\,\{G(a_1^R, a_2^R)\} = (dk)^2\,\mathrm{Im}\,\{G(a_1, a_2)\}, \qquad \mathrm{Im}\,\{G(a_1^R, a_2)\} = -\mathrm{Im}\,\{G(a_1, a_2^R)\} \qquad (30, 31)$$

Combining equations (5)–(7) and (29)–(31) gives the normalized bias error $\varepsilon_{L\alpha}$ for the measurement of bending wave power flow using the two-accelerometer array technique

$$\varepsilon_{L\alpha} \approx (\alpha dk)^2 + \alpha_0^2 \beta \frac{P_L}{P_B} + \alpha_0\,\mathrm{Im}\{G(a_{L1}, a_2) + G(a_1, a_{L2})\}/\mathrm{Im}\{G(a_1, a_2)\} \tag{32}$$

For thin beams

$$\beta = (kh)^2/6 = 3{\cdot}6\, h/\lambda_L \tag{33}$$

For thick beams

$$\beta = (0{\cdot}32Bk_1^2 - 1{\cdot}32\rho J\omega^2)/(g - \rho J\omega^2) + (h^2/12)(k_1^2 - m_b\omega^2/g) \tag{34}$$

When only bending waves exist, equation (32) gives the normalized bias error, ε_α, solely due to the non-zero transverse sensitivities of the accelerometers

$$\varepsilon_\alpha = (\alpha dk)^2 \tag{35}$$

Since the ratio of the accelerometer separation to the bending wavelength, d/λ, should be at least smaller than 0·2, ε_α is usually less than 0·004 which is negligible. When the transverse sensitivities of the accelerometers are zero, equation (32) gives the normalized bias error, ε_L, solely due to the presence of longitudinal waves

$$\varepsilon_L \approx \beta\left(\frac{\pi\mu h}{\lambda_L}\right)^2 \frac{P_L}{P_B} \pm \frac{\pi\mu h}{\lambda_L} \frac{\text{Im}\,\{G(a_{L1}, a_2) + G(a_1, a_{L2})\}}{\text{Im}\,\{G(a_1, a_2)\}} \tag{36}$$

which increases with the increases of the longitudinal to bending wave power ratio and the coherence γ^2. If longitudinal waves are incoherent with bending waves ($\gamma^2 = 0$), the second term in the above equation is zero and the bias error is linearly proportional to the longitudinal to bending wave power ratio P_L/P_B.

For thin beams, the bending wavelength to the beam thickness ratio, λ/h, should be larger than 6 [9], which leads to $h/\lambda_L < 0{\cdot}05$. Therefore, the following equation will hold

$$|\varepsilon_L| \leqslant 4{\cdot}5 \times 10^{-4}\left|\frac{P_L}{P_B}\right| + 0{\cdot}05\left|\frac{\text{Im}\,\{G(a_{L1}, a_2) + G(a_1, a_{L2})\}}{\text{Im}\,\{G(a_1, a_2)\}}\right| \tag{37}$$

where the first term is negligible unless the longitudinal wave power flow is very much greater than the bending wave power flow. If bending and longitudinal waves are coherent, the signs of the imaginary parts of the two cross spectra, $G(a_{L1}, a_2)$ and $G(a_1, a_{L2})$, should be opposite, and the normalized bias error is small as long as the longitudinal wave energy is not much greater than the bending wave energy.

If the shear deformation and rotatory inertia are included, β should be calculated from equation (34) rather than (33), it increases with increasing frequency and approaches to 2/3 at frequencies close to f_0. The value of $\pi\mu h/\lambda_L$ is smaller than $1/\pi$ (seeing equation (14)). Therefore, the following equation will hold

$$|\varepsilon_L| \leqslant \tfrac{2}{30}|P_L/P_B| + (1/\pi)|\text{Im}\,\{G(a_{L1}, a_2) + G(a_1, a_{L2})\}/\text{Im}\,\{G(a_1, a_2)\}| \tag{38}$$

where the second term is zero or negligible if bending and longitudinal waves are incoherent or coherent but the magnitudes of the imaginary parts of the cross spectra, $G(a_{L1}, a_2)$ and $G(a_1, a_{L2})$, is approximately equal. For the case where bending waves are incoherent with longitudinal waves, ε_L is negligible unless the longitudinal wave power flow is much larger than the bending wave power flow. For the case where bending and longitudinal waves are highly coherent and the bending wave energy is smaller than the longitudinal wave energy, ε_L could be large. In practice, the longitudinal wave energy usually increase with increasing frequency. Therefore, the effect of coherent longitudinal waves could become important at high frequencies.

5. BIAS ERROR DUE TO THE PRESENCE OF BENDING WAVES

The presence of bending waves also affects the measurement of longitudinal wave power flow resulting in a bias error. When longitudinal and bending waves propagate in a uniform beam, the measured longitudinal wave acceleration a_{Lm} will include three components: the longitudinal wave acceleration a_L, the bending rotation a^R and the contribution from the vertical motions due to the non-zero transverse sensitivity of the acelerometer, that is

$$a_{Lm} = a_L + a^R + \alpha(a + a_{Lz}) = \alpha_1 a_L + \alpha a + a^R \tag{39}$$

where $\alpha_1 = 1 \pm \alpha\pi\mu h/\lambda_L$. The bending rotation can be eliminated by using two identical rotational accelerometers which are fixed on the same position but opposite sides of the test structure [6]. The intensity probe should consist of four identical rotational accelerometers and two signal sum/difference preamplifiers, as shown in Figure 6. In

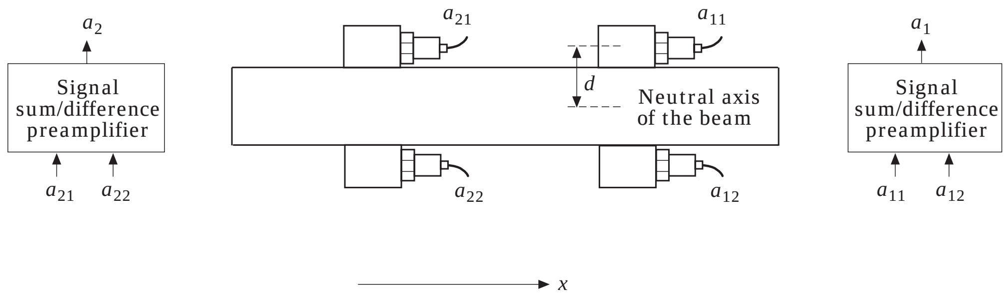

Figure 6. Diagram of the four-transducer probe used in the measurement of longitudinal wave power flow on a uniform beam where bending waves exist.

practice, however, there could be difficulties, similar to those discussed in the above section, to use or find the four-transducer probe.

If the two-transducer probe is used, the imaginary part of the measured cross-spectrum can be expressed as

$$\begin{aligned}\mathrm{Im}\,\{G(a_{Lm1}, a_{Lm2})\} &= \alpha_1^2\,\mathrm{Im}\,\{G(a_{L1}, a_{L2})\} + \alpha^2\,\mathrm{Im}\,\{G(a_1, a_2)\} + \mathrm{Im}\,\{G(a_1^R, a_2^R)\}\\ &\quad + \alpha\,\mathrm{Im}\,\{G(a_1, a_2^R) + G(a_1^R, a_2)\} + \alpha_1\,\mathrm{Im}\,\{G(a_{x1}, a_{L2}) + G(a_{L1}, a_{x2})\}\end{aligned}$$

where $a_{x1} = a_1^R + \alpha a_1$ and $a_{x2} = a_2^R + \alpha a_2$ are the measured rotational accelerations in the x direction due to the bending waves only. Combining the above equation and equations (5)–(7), (30) and (31) gives the normalized bias error, $\varepsilon_{B\alpha}$, for the measurement of longitudinal wave power due to the presence of bending waves and the non-zero transverse sensitivities of the transducers

$$\varepsilon_{B\alpha} = \pm\frac{2\alpha\pi\mu h}{\lambda_L} + \left(\frac{\alpha\pi\mu h}{\lambda_L}\right)^2 + \frac{(kd)^2 + \alpha^2}{\beta}\frac{P_B}{P_L} + \alpha_1\frac{\mathrm{Im}\,\{G(a_{x1}, a_{L2}) + G(a_{L1}, a_{x2})\}}{\mathrm{Im}\,\{G(a_{L1}, a_{L2})\}} \tag{40}$$

where β is given by equation (33) for thin beams or equation (34) for thick beams. If only longitudinal wave exist, the above equation gives the normalized bias error, ε_α, solely due to the non-zero transverse sensitivities of the transducers

$$\varepsilon_\alpha = \pm 2\alpha\pi\mu h/\lambda_L + (\alpha\pi\mu h/\lambda_L)^2 \approx \pm 2\alpha h/\lambda_L \tag{41}$$

which is usually negligible. When the transverse sensitivities of the transducers are zero, the normalized bias error, ε_B, solely due to the presence of bending waves is given by

$$\begin{aligned}\varepsilon_B &= \frac{(kd)^2}{\beta}\frac{P_B}{P_L} + \alpha_1\frac{\mathrm{Im}\,\{G(a_{x1}, a_{L2}) + G(a_{L1}, a_{x2})\}}{\mathrm{Im}\,\{G(a_{L1}, a_{L2})\}}\\ &\approx \frac{(kd)^2}{\beta}\frac{P_B}{P_L} + \frac{\mathrm{Im}\,\{G(a_{x1}, a_{L2}) + G(a_{L1}, a_{x2})\}}{\mathrm{Im}\,\{G(a_{L1}, a_{L2})\}}\end{aligned} \tag{42}$$

which shows that the bias error increases with the increase of the bending to longitudinal wave power ratio and the coherence γ^2. If the phase of the sum of the cross spectra, $G(a_{x1}, a_{L2})$ and $G(a_{L1}, a_{x2})$, is not negligible, the bias error increases with increasing the bending to longitudinal wave energy ratio. If bending and longitudinal waves are highly coherent, the second term in the above equation is greater than 1 when the bending wave energy is greater than the longitudinal wave energy. If bending and longitudinal waves are incoherent ($\gamma^2 = 0$), the second term in the above equation is zero, ε_B is linearly proportional to the bending to longitudinal wave power ratio and for thin beams given by

$$\varepsilon_B \approx 6(d/h)^2 \, P_B/P_L \tag{43}$$

which is proportional to the quantity $(d/h)^2$. The above equation shows that the bias error ε_B is more sensitive to the increase of the distance to beam thickness ratio d/h than the increase of the bending to longitudinal wave power ratio P_B/P_L. As the distance d cannot be smaller than half the beam thickness, the bias error is usually large even if the power ratio P_B/P_L is small. If the bending wave power is larger than the longitudinal wave power, the normalized bias error is at least larger than 1·5, therefore, the longitudinal wave power cannot be measured if the contribution from bending waves is not eliminated or reduced to a certain extent. If the distance d is larger than the beam thickness h, a small bending to longitudinal wave power ratio will result in a large bias error in the measurement of longitudinal wave power flow.

6. CONCLUSIONS

This paper has individually examined three bias errors in the measurement of structural wave power flow. Through the theoretical analysis the following conclusions can be drawn.

1. For a one-dimensional structure, there is a special frequency f_0 above which the decaying bending wave becomes a travelling bending wave and it is impossible to measure the power flow using two-accelerometer techniques. At frequencies below the frequency f_0, the bending wave power can be measured using two-transducer techniques; however, the estimate of bending wave power flow using the existing intensity theory, derived on the basis of the classical Bernoulli–Euler wave model, will result in an approximate error which increases with increasing frequency. At frequencies far below the frequency f_0, the shear deformation and rotatory inertia can be neglected, the approximate estimate error for using the existing intensity theory is small and can be neglected. At frequencies close to the frequency f_0, the shear deformation and rotatory inertia become important, the approximate estimate error is large. This error can be corrected during data processing.

2. For the measurement of bending and longitudinal wave powers using two-transducer techniques, the non-zero transverse sensitivities of the transducers results in a negligible bias error.

3. For the measurement of bending wave power flow using a biaxial accelerometer, the presence of incoherent longitudinal waves results in no bias error. When bending and longitudinal waves are coherent, the bias error increases with the increases of the longitudinal to bending wave energy ratio and the coherence, and could be greater than 1 if the longitudinal wave energy is much greater than the bending wave energy, which is the case at high frequencies.

4. For the measurement of bending wave power flow using the two-acelerometer array technique, the bias error due to the presence of longitudinal waves increases with increases in the longitudinal to bending wave power ratio and the coherence. If the coherence between bending and longitudinal waves is low, the bias error is usually negligible as long as the longitudinal wave power is not much greater than the bending wave power. If bending and longitudinal waves are highly coherent, the bias error could be large, especially for the case where the bending wave energy is smaller than the longitudinal wave energy.

5. For the measurement of longitudinal wave power, the presence of bending waves usually results in a large bias error even if bending waves are incoherent with longitudinal

waves. This bias error increases with the increases of the bending to longitudinal wave power ratio, the distance to beam thickness ratio and the coherence. If the bending wave power is greater than the longitudinal wave power, the normalized bias error is at least larger than 1·5. If the distance is larger than the beam thickness, a small bending to longitudinal wave power ratio will result in a large bias error. If the contribution from bending waves cannot be eliminated or reduced to a certain extent, the longitudinal wave power flow cannot be measured using the two-accelerometer array technique.

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