Journal of Sound and Vibration (1997) **204**(1), 73–84

CONSTRUCTION OF THE UNOBSERVED PART OF THE SQUARE DYNAMIC FLEXIBILITY MATRIX

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(*Received* 20 *April* 1995, *and in final form* 15 *January* 1997)

Four methods are proposed that allow construction of the unobserved part of the square dynamic flexibility matrix $\mathbf{\Gamma}(\omega)\in\mathbf{C}^{c,c}$ without performing a modal identification and based on the information contained in the known rectangular sub-matrix $\mathbf{\Gamma}_1(\omega)\in\mathbf{C}^{c,p}$, $p < c$. The formulations exploit the symmetry of the dynamic flexibility matrix and use specific decompositions. The proposed methods are illustrated by numerically simulated examples with different levels of damping.

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1. INTRODUCTION

In structural dynamics, continuous systems are often discretized and represented by models containing a finite number of degrees of freedom [1, 2]. One way to characterize a system in a given frequency domain by using an experimental approach is to measure its dynamic flexibility matrix at the degrees of freedom (dof) corresponding to the c sensors which have been optimally placed on the structure [3]. In order to measure completely this dynamic flexibility matrix it is necessary to excite the system with c linearly independent excitation forces. However, in practice there are often fewer than c independent excitations (for example, p excitations, $p < c$).

The proposed method allows the square dynamic flexibility matrix of order c to be constructed by using the information obtained from the measured rectangular sub-matrix of dimension (c, p). The originality of the proposed method lies in the fact that this dynamic flexibility matrix is constructed without exploiting the identified eigensolutions of the system. A similar principle has already been proposed [4] and the idea is completed and extended in the present article.

The method is based on the symmetry of the dynamic flexibility matrix and uses specific decompositions (Takagi factorization [5], and singular value decomposition [6]) of the square sub-matrix of order p of the above mentioned rectangular matrix.

To improve the robustness of the proposed method, an additional method is developed which expands the unobserved frequency data via two different techniques, one for the case

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when the frequency is close to resonance and the second for all other cases. Finally, the results obtained from numerically simulated examples are presented and the limits of the proposed methods are analyzed.

2. PROBLEM DEFINITION

2.1. USE OF THE DYNAMIC FLEXIBILITY MATRICES

In linear elastodynamics, the transfer functions can be used in several formulations to predict the behaviour of mechanical structures through dynamic sub-structuring or reanalysis of modified structures [7].

Consider, for example, the expression for the dynamic flexibility matrix $\hat{\mathbf{\Gamma}}_{nn}(\omega)$ of a modified structure,

$$\hat{\mathbf{\Gamma}}_{nn}(\omega) = [\hat{\mathbf{Z}}_{nn}(\omega)]^{-1} = [\mathbf{Z}_{nn}(\omega) + \Delta\mathbf{Z}_{nn}(\omega)]^{-1}, \tag{2.1}$$

where $\hat{\mathbf{\Gamma}}_{nn}(\omega) \in \mathrm{C}^{n,n}$, symmetric, is the dynamic flexibility matrix of the modified structure at the frequency ω, n is the total number of dof of the structure, $\mathbf{\Gamma}_{nn}(\omega) = [\mathbf{Z}_{nn}(\omega)]^{-1} \in \mathrm{C}^{n,n}$, symmetric, is the dynamic flexibility matrix of the initial structure, and $\Delta\mathbf{Z}_{nn}(\omega) \in \mathrm{C}^{n,n}$, symmetric, is the dynamic stiffness matrix of the modifications. (A list of principal notation is given in the Appendix.) In the case of "small" modifications, $\|\Delta\mathbf{Z}_{nn}(\omega)\| \ll \mathbf{Z}_{nn}(\omega)\|$ and thus $\hat{\mathbf{\Gamma}}_{nn}(\omega)$ can be written as

$$\begin{aligned}\hat{\mathbf{\Gamma}}_{nn}(\omega) &= \mathbf{\Gamma}_{nn}(\omega) - \mathbf{\Gamma}_{nn}(\omega)[\Delta\mathbf{Z}_{nn}(\omega)\mathbf{\Gamma}_{nn}(\omega)] \\ &\quad + \mathbf{\Gamma}_{nn}(\omega)[\Delta\mathbf{Z}_{nn}(\omega)\mathbf{\Gamma}_{nn}(\omega)][\Delta\mathbf{Z}_{nn}(\omega)\mathbf{\Gamma}_{nn}(\omega)] + \cdots.\end{aligned} \tag{2.2}$$

In practice, only a limited number c ($c \ll n$) of pickups is available. Let $\mathbf{\Gamma}(\omega) \in \mathrm{C}^{c,c}$ be the dynamic flexibility matrix at these c pickup dofs.

If it is assumed that the dofs which are involved in the modification are included among these c pickup dofs, then equation (2.2) becomes:

$$\hat{\mathbf{\Gamma}}(\omega) = \mathbf{\Gamma}(\omega) - \mathbf{\Gamma}(\omega)[\Delta\mathbf{Z}(\omega)\mathbf{\Gamma}(\omega)] + \mathbf{\Gamma}(\omega)[\Delta\mathbf{Z}(\omega)\mathbf{\Gamma}(\omega)][\Delta\mathbf{Z}(\omega)\mathbf{\Gamma}(\omega)] + \cdots, \tag{2.3}$$

where all the matrices are symmetric; $\hat{\mathbf{\Gamma}}(\omega) \in \mathrm{C}^{c,c}$ is the dynamic flexibility matrix of the modified structure relative to the c measured dofs. $\mathbf{\Gamma}(\omega) \in \mathrm{C}^{c,c}$ is the dynamic flexibility of the initial structure and $\Delta\mathbf{Z}(\omega) \in \mathrm{C}^{c,c}$ is the dynamic stiffness matrix characterizing the introduced modifications. The error in equation (2.3) is of the order of $[\Delta\mathbf{Z}(\omega)\mathbf{\Gamma}(\omega)]^3$ for an expansion limited to three terms. From a general point of view, an arbitrary precision for $\mathbf{\Gamma}(\omega)$ can be obtained by using a sufficient number of terms in the expansion, upon assuming that $\|\Delta\mathbf{Z}(\omega)\| \ll \|\mathbf{Z}(\omega)\|$ for all considered ω.

The global objective is to characterize the initial structure based on experimental observations. The elements of the matrix $\mathbf{\Gamma}(\omega)$ can be evaluated either from the identified eigensolutions or by the direct measurement of its $c(c+1)/2$ elements.

2.2. EXPANSION BASED ON THE IDENTIFIED EIGENSOLUTIONS

Consider a n dof structure whose behaviour is represented on the basis of its $2n$ complex modes. Its dynamic flexibility matrix $\mathbf{\Gamma}(\omega) \in \mathrm{C}^{n,n}$ is

$$\mathbf{\Gamma}_{nn}(\omega) = \mathbf{Y}(\mathrm{j}\omega\mathbf{I} - \mathbf{S})^{-1}\mathbf{Y}^{\mathrm{T}} + \bar{\mathbf{Y}}(\mathrm{j}\omega\mathbf{I} - \bar{\mathbf{S}})^{-1}\bar{\mathbf{Y}}^{\mathrm{T}}, \tag{2.4}$$

where $\mathbf{Y} \in \mathrm{C}^{n,n}$, $\mathbf{S} = \mathrm{Diag}\,[s_v] \in \mathrm{C}^{n,n}$, $\mathrm{Imag}\,(s_v) > 0$, $\forall v$ represent respectively the modal and spectral matrices of the structure and $\bar{\mathbf{Y}}$, $\bar{\mathbf{S}}$ are respectively the conjugate matrices of $\mathbf{Y}$ and $\mathbf{S}$.

One can introduce the submatrix partition

$$\mathbf{Y} = [\mathbf{Y}_1 \ \vdots \ \mathbf{Y}_2]; \quad \mathbf{S} = \begin{array}{|c|c|} \hline \mathbf{S}_1 & \\ \hline & \mathbf{S}_2 \\ \hline \end{array},$$

where $\mathbf{Y}_1 \in \mathbf{C}^{n,m}$, $\mathbf{S}_1 \in \mathbf{C}^{m,m}$ contain respectively the m eigenvectors and eigenvalues in the observed frequency band.

In this frequency band, equation (1.4) can be written as

$$\boldsymbol{\Gamma}_{nn}(\omega) = \mathbf{Y}_1[\mathrm{j}\omega\mathbf{I}_m - \mathbf{S}_1]^{-1}\mathbf{Y}_1^{\mathrm{T}} + \bar{\mathbf{Y}}_1[\mathrm{j}\omega\mathbf{I}_m - \bar{\mathbf{S}}_1]^{-1}\bar{\mathbf{Y}}_1^{\mathrm{T}} + \boldsymbol{\Gamma}_{nnr}(\omega), \tag{2.5}$$

where $\boldsymbol{\Gamma}_{nnr}(\omega)$ represents the contribution of the eigenmodes outside the observed band. In the following, the sub-matrix $\boldsymbol{\Gamma}(\omega) \in \mathrm{C}^{c,c}$ ($m \leqslant c \ll n$) of $\boldsymbol{\Gamma}_{nn}(\omega)$ relative to the c pickup dofs will be considered. The matrix $\boldsymbol{\Gamma}(\omega)$ is defined by:

$$\boldsymbol{\Gamma}(\omega) = \mathbf{Y}_{1c}[\mathrm{j}\omega\mathbf{I}_m - \mathbf{S}_1]^{-1}\mathbf{Y}_{1c}^{\mathrm{T}} + \bar{\mathbf{Y}}_{1c}[\mathrm{j}\omega\mathbf{I}_m - \bar{\mathbf{S}}_1]^{-1}\bar{\mathbf{Y}}_{1c}^{\mathrm{T}} + \boldsymbol{\Gamma}_r(\omega), \tag{2.6}$$

where $\mathbf{Y}_{1c} \in \mathrm{C}^{c,m}$ is the modal sub-matrix of $\mathbf{Y}_1$ at the c observed dofs and $\boldsymbol{\Gamma}_r(\omega)$ is the contribution of the eigenmodes outside the observed frequency band at the c observed dofs.

The construction of $\boldsymbol{\Gamma}_{cc}(\omega)$ requires the identification of $\mathbf{Y}_{1c}$, $\mathbf{S}_1$ and $\boldsymbol{\Gamma}_r(\omega)$. In order to identify $\mathbf{Y}_{1c}$ and $\mathbf{S}_1$ it is sufficient to determine p columns ($1 \leqslant p \leqslant c$) of $\boldsymbol{\Gamma}(\omega)$ by applying linearly independent excitations in the observed frequency band. In practice, the number p is much smaller than the number c of observed dofs. Equation (2.5) thus allows the matrix $\boldsymbol{\Gamma}(\omega)$ to be reconstructed from a small number of observed columns p among the c columns. Many modal identification methods have been developed for this purpose (see, for example, references [8, 9]). In order to avoid a costly modal identification of the three matrices $\mathbf{Y}_{1c}$, $\mathbf{S}_1$ and $\boldsymbol{\Gamma}_r(\omega)$, an alternative method [10] is proposed here which is based on the direct exploitation of a sub-matrix of $\boldsymbol{\Gamma}(\omega)$.

2.3. DIRECT EXPANSION OF THE DYNAMIC FLEXIBILITY MATRICES

In this case, the contributions of all the structural modes are taken into account. The complete knowledge of $\boldsymbol{\Gamma}(\omega)$ requires in principle c pickups and c excitations. For economical reasons, only a limited number p ($p < c$) of linearly independent excitation configurations is usually available.

The problem can be formulated as follows. If p columns of $\boldsymbol{\Gamma}(\omega)$ are known, denoted by the sub-matrix $\boldsymbol{\Gamma}_1(\omega) \in \mathrm{C}^{c,p}$, can the c–p remaining columns be determined without performing a modal identification?

In order to solve this problem, several methods are proposed which are based on specific decompositions of the dynamic flexibility matrix $\boldsymbol{\Gamma}(\omega)$.

2.4. NOTATIONS

The dynamic flexibility matrix $\boldsymbol{\Gamma}(\omega) \in \mathrm{C}^{c,c}$ is partitioned into submatrices as

$$\boldsymbol{\Gamma}(\omega) = \begin{array}{|c|c|} \hline \boldsymbol{\Gamma}_1(\omega) & \boldsymbol{\Gamma}_2(\omega) \\ \hline \end{array} = \begin{array}{|c|c|} \hline \boldsymbol{\Gamma}_{11}(\omega) & \boldsymbol{\Gamma}_{12}(\omega) \\ \hline \boldsymbol{\Gamma}_{21}(\omega) & \boldsymbol{\Gamma}_{22}(\omega) \\ \hline \end{array}, \tag{2.7}$$

where $\boldsymbol{\Gamma}_1(\omega) \in \mathrm{C}^{c,p}$ is observed, $\boldsymbol{\Gamma}_{11}(\omega) \in \mathrm{C}^{p,p}$ is a square submatrix of $\boldsymbol{\Gamma}_1(\omega)$, and $\boldsymbol{\Gamma}_2(\omega) \in \mathrm{C}^{c,c-p}$. It is assumed that the dynamic flexibility matrix $\boldsymbol{\Gamma}(\omega)$ is symmetric: $\boldsymbol{\Gamma}_{12}(\omega) = \boldsymbol{\Gamma}_{21}^{\mathrm{T}}(\omega)$, $\boldsymbol{\Gamma}_{11}(\omega) = \boldsymbol{\Gamma}_{11}^{\mathrm{T}}(\omega)$ and $\boldsymbol{\Gamma}_{22}(\omega) = \boldsymbol{\Gamma}_{22}^{\mathrm{T}}(\omega)$. The number of unknown elements contained in the matrix $\boldsymbol{\Gamma}_{22}$ is thus reduced to $(c - p)(c - p + 1)/2$.

3. EXPANSION METHODS FOR THE DYNAMIC FLEXIBILITY MATRIX

Given the symmetry of the complex flexibility matrix, an important part of this section is devoted to the symmetry decomposition of a symmetric matrix.

3.1. TAKAGI'S SYMMETRIC DECOMPOSITION [5]

Theorem. For each symmetric complex matrix $\mathbf{A}=\mathbf{A}^{\mathrm{T}}\in\mathrm{C}^{m,m}$, there exists a unitary matrix $\mathbf{U}\in\mathrm{C}^{m,m}$ ($\mathbf{U}^{\mathrm{H}}\mathbf{U}=\mathbf{I}$) and a non-negative diagonal matrix $\Sigma=\mathrm{Diag}\,[\sigma_v]\in\mathbf{R}^{m,m}$, $\sigma_v\geqslant 0$, $v=1,2,\ldots,m$ such that

$$\mathbf{A}=\mathbf{U}\Sigma\mathbf{U}^{\mathrm{T}}, \tag{3.1}$$

where the upper index H is the transpose complex conjugate ($\mathbf{U}^{\mathrm{H}}=\bar{\mathbf{U}}^{\mathrm{T}}$). The values σ_v are singular values of the matrix $\mathbf{A}$, i.e., the non-negative roots of the eigenvalues of the matrix $\mathbf{A}\mathbf{A}^{\mathrm{H}}=\mathbf{A}\bar{\mathbf{A}}$ and the columns of the matrix $\mathbf{U}$ are the corresponding eigenvectors of $\mathbf{A}\mathbf{A}^{\mathrm{H}}$.

Although the proof of Takagi's theorem (published in reference [5]) is beyond the scope of this article, we will derive its specific form for the matrices $\mathbf{A}$ with simple (non-multiple) singular values.

3.1.1. *Simple non-zero singular values of* A

If all non-zero singular values of the matrix $\mathbf{A}=\mathbf{A}^{\mathrm{T}}\in\mathrm{C}^{m,m}$ are simple (for the matrices with experimentally determined elements, this situation is nearly always the case), one can construct its symmetry decomposition with the singular values decomposition more easily than in general case.

Assume that the singular value decomposition of $\mathbf{A}$ [6] is:

$$\mathbf{A}=\mathbf{X}_1\Sigma_1\mathbf{W}_1^{\mathrm{H}}, \tag{3.2}$$

where $\mathbf{X}_1=[-\mathbf{x}_v-]$, $\mathbf{W}_1=[-\mathbf{w}_v-]\in\mathrm{C}^{m,q}$, $\mathbf{X}_1^{\mathrm{H}}\mathbf{X}_1=\mathbf{I}_q=\mathbf{W}_1^{\mathrm{H}}\mathbf{W}_1$ and the diagonal matrix $\Sigma_1\in\mathbf{R}^{q,q}$ contains q ($q\leqslant m$) non-zero singular values. If one defines the matrix $\mathbf{U}_1\in\mathrm{C}^{m,q}$ as

$$\mathbf{U}_1=\mathbf{X}_1\mathbf{D}, \tag{3.3}$$

where $\mathbf{D}=\mathrm{Diag}\,[d_v]\in\mathrm{C}^{q,q}$, $d_v\neq 0$, $d_v^2=\mathbf{x}_v^H\cdot\bar{\mathbf{w}}_v$, $v=1,2,\ldots,q$, the Takagi decomposition (3.1) of $\mathbf{A}$ is

$$\mathbf{A}=\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}},\ \text{where } \mathbf{U}_1^{\mathrm{H}}\mathbf{U}_1=\mathbf{I}_q. \tag{3.4}$$

In contrast to expression (3.1), only the columns of $\mathbf{U}$ corresponding to its non-zero singular values in Σ are used in expression (3.4).

The Takagi decomposition leads to a simple definition of the Moore–Penrose generalized inversion matrix $\mathbf{A}^+$ for the matrix $\mathbf{A}$:

$$\mathbf{A}^+=\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}}. \tag{3.5}$$

Indeed, the four expressions $\mathbf{A}\mathbf{A}^+\mathbf{A}=\mathbf{A}$, $\mathbf{A}^+\mathbf{A}\mathbf{A}^+=\mathbf{A}^+$, $(\mathbf{A}\mathbf{A}^+)^{\mathrm{H}}=\mathbf{A}\mathbf{A}^+$ and $(\mathbf{A}^+\mathbf{A})^{\mathrm{H}}=\mathbf{A}^+\mathbf{A}$ hold:

$$\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}}\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}}\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}}=\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}},$$

$$\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}}\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}}\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}}=\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}},$$

$$(\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}}\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}})^{\mathrm{H}}=(\mathbf{U}_1\mathbf{U}_1^{\mathrm{H}})^{\mathrm{H}}=\mathbf{U}_1\mathbf{U}_1^{\mathrm{H}}=\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}}\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}}$$

$$(\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}}\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}})^{\mathrm{H}}=(\bar{\mathbf{U}}_1\mathbf{U}_1^{\mathrm{T}})^{\mathrm{H}}=\bar{\mathbf{U}}_1\mathbf{U}_1^{\mathrm{T}}=\bar{\mathbf{U}}_1\Sigma_1^{-1}\mathbf{U}_1^{\mathrm{H}}\mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}}.$$

Proof of equation (3.4). Given equation (3.2), it follows that:

$$\mathbf{A}^{\mathrm{H}}\mathbf{A} = \mathbf{W}_1\Sigma_1^2\mathbf{W}_1^{\mathrm{H}} \tag{3.6}$$

and by symmetry of $\mathbf{A}$ one has

$$\mathbf{A}^{\mathrm{H}}\mathbf{A} = \bar{\mathbf{A}}^{\mathrm{T}}\mathbf{A} = \bar{\mathbf{A}}\mathbf{A}^{\mathrm{T}} = \overline{\mathbf{A}\mathbf{A}^{\mathrm{H}}} = \overline{\mathbf{X}_1\Sigma_1^2\mathbf{X}_1^{\mathrm{H}}} = \bar{\mathbf{X}}_1\Sigma_1^2\mathbf{X}_1^{\mathrm{T}}. \tag{3.7}$$

Post-multiplying equation (3.6) by $\mathbf{W}_1$ ($\mathbf{W}_1^{\mathrm{H}}\mathbf{W}_1 = \mathbf{I}_q$) and equation (3.7) by $\bar{\mathbf{X}}_1$ ($\mathbf{X}_1^{\mathrm{T}}\bar{\mathbf{X}}_1 = \overline{\mathbf{X}_1^{\mathrm{H}}\mathbf{X}_1} = \mathbf{I}_q$) gives

$$\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{W}_1 = \mathbf{W}_1\Sigma_1^2 \qquad \text{and} \qquad \mathbf{A}^{\mathrm{H}}\mathbf{A}\bar{\mathbf{X}}_1 = \bar{\mathbf{X}}_1\Sigma_1^2. \tag{3.8, 3.9}$$

Comparing equations (3.8) and (3.9) makes it evident that the vectors $\mathbf{w}_v \in C^m$, $\bar{\mathbf{x}}_v \in C^m$, columns of the matrices $\mathbf{W}_1$, $\bar{\mathbf{X}}_1$ respectively, are the eigenvectors of the same Hermitian matrix $\mathbf{A}^{\mathrm{H}}\mathbf{A}$ and Σ_1^2 contains its non-zero and simple eigenvalues. Since the eigenvectors corresponding to simple eigenvalues are defined in direction only, one has

$$\mathbf{W}_1 = \bar{\mathbf{X}}_1\bar{\mathbf{D}}^2, \tag{3.10}$$

with the complex diagonal matrix $\bar{\mathbf{D}}^2 = \mathrm{Diag}\,[\bar{d}_v^2] \in C^{q,q}$. Left multiplication of equation (3.10) by $\mathbf{X}_1^{\mathrm{T}}$ gives:

$$\mathbf{X}_1^{\mathrm{T}}\mathbf{W}_1 = \mathbf{X}_1^{\mathrm{T}}\bar{\mathbf{X}}_1\bar{\mathbf{D}}^2 = \bar{\mathbf{D}}^2, \tag{3.11}$$

where $d_v^2 = \mathbf{x}_v^{\mathrm{H}}\bar{\mathbf{w}}_v$, $v = 1, 2, \ldots, q$. If one substitutes equation (3.9) into equation (3.2) one obtains (3.4)-type decomposition

$$\mathbf{A} = \mathbf{X}_1\Sigma_1\mathbf{D}^2\mathbf{X}_1^{\mathrm{T}} = \mathbf{X}_1\mathbf{D}\Sigma_1\mathbf{D}\mathbf{X}_1^{\mathrm{T}} = \mathbf{U}_1\Sigma_1\mathbf{U}_1^{\mathrm{T}}, \tag{3.12}$$

where

$$\mathbf{U}_1 = \mathbf{X}_1\mathbf{D} \qquad \text{and} \qquad \mathbf{U}_1^{\mathrm{H}}\mathbf{U}_1 = \bar{\mathbf{D}}\mathbf{X}_1^{\mathrm{H}}\mathbf{X}_1\mathbf{D} = \bar{\mathbf{D}}\mathbf{D} = \mathrm{Diag}\,[|d_v|^2] = \mathbf{I}_q. \tag{3.13}$$

This last equation results from equation (3.10):

$$\mathbf{I}_q = \mathbf{W}_1^{\mathrm{H}}\mathbf{W}_1 = \mathbf{D}^2\mathbf{X}_1^{\mathrm{T}}\bar{\mathbf{X}}_1\bar{\mathbf{D}}^2 = (\mathbf{D}\bar{\mathbf{D}})^2 = \mathrm{Diag}\,[|d_v|^4],$$

because $|d_v|^4 = 1$ implies $|d_v|^2 = 1$.

3.1.2. *Matrix* $\mathbf{A}$ *with rank* ($\mathbf{A}$) = 1

In the special case where the numerical rank of $\mathbf{A} = \mathbf{A}^{\mathrm{T}} \in C^{m,m}$ is unity, the Takagi decomposition (3.4) of $\mathbf{A}$ reduces to:

$$\mathbf{A} = \mathbf{u}_1 \cdot \sigma_1 \cdot \mathbf{u}_1^{\mathrm{T}}, \tag{3.14}$$

where $\sigma_1 > 0$ is the non-zero singular value of $\mathbf{A}$ and $\mathbf{u}_1 \in C^m$ ($\mathbf{u}_1^{\mathrm{H}}\mathbf{u}_1 = 1$) is the eigenvector of the matrix $\mathbf{A}\mathbf{A}^{\mathrm{H}}$ corresponding to its eigenvalue σ_1^2.

Let $\mathbf{a}_v \in C^m$ denote the columns of $\mathbf{A} = [-\mathbf{a}_v-]$. Then an arbitrary vector $\mathbf{a}_v = [-a_{vk}-]^{\mathrm{T}}$ with non-zero component $a_{vv} \neq 0$ defines the vector $\mathbf{u}_1$ as:

$$\mathbf{u}_1 = \mathbf{a}_v \cdot (1/\|\mathbf{a}_v\|) \cdot \sqrt{|a_{vv}|a_{vv}}, \tag{3.15}$$

where

$$\mathbf{u}_1^{\mathrm{H}}\mathbf{u}_1 = \frac{\|\mathbf{a}_v\|^2}{\|\mathbf{a}_v\|^2}\sqrt{\frac{|a_{vv}|}{\bar{a}_{vv}} \cdot \frac{|a_{vv}|}{a_{vv}}} = 1.$$

Moreover, in this case the spectral norm of the matrix $\mathbf{A} = [A_{ik}] = [a_{ik}]$ is identical to its

Frobenius norm, and hence

$$\sigma_1 = \|\mathbf{A}\| = \|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m}\sum_{k=1}^{m} |a_{ik}|^2} = \|a_v\|^2/|a_{vv}|. \tag{3.16}$$

The latter expression follows from the fact that every unit-rank symmetric matrix $\mathbf{A}$, can be written in the form:

$$\mathbf{A} = \frac{\mathbf{a}_v}{\sqrt{a_{vv}}} \cdot \frac{\mathbf{a}_v^{\mathrm{T}}}{\sqrt{a_{vv}}} = \frac{\mathbf{a}_v \cdot \mathbf{a}_v^{\mathrm{T}}}{a_{vv}}, \tag{3.17}$$

through comparison of equation (3.17) with equations (3.14) and (3.15).

The equality of the spectral and Frobenius norms can be demonstrated as follows:

$$\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\mathrm{Tr}\,(\mathbf{A}^{\mathrm{H}}\mathbf{A})} = \sqrt{\frac{\|\mathbf{a}_v\|^2}{|a_{vv}|^2}\,\mathrm{Tr}\,(\bar{\mathbf{a}}_v \mathbf{a}_v^{\mathrm{T}})}, \qquad \|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\frac{\|\mathbf{a}_v\|^2}{|a_{vv}|^2} \cdot \|\mathbf{a}_v\|^2} = \frac{\|\mathbf{a}_v\|}{|a_{vv}|},$$

where Tr is the trace of a matrix.

In the definition of $\mathbf{u}_1$ given by equation (3.15), the column $\mathbf{a}_v$ of $\mathbf{A}$ with maximal $|a_{vv}|$ is chosen for numerical reasons.

3.2. FIRST METHOD OF EXPANSION OF THE FLEXIBILITY MATRIX $\mathbf{\Gamma}(\omega)$

3.2.1. *Expression of the sub-matrix* $\mathbf{\Gamma}_{22}(\omega)$

According to the partitioning (2.7) of the symmetric flexibility matrix $\mathbf{\Gamma}(\omega) = \mathbf{\Gamma}^{\mathrm{T}}(\omega) \in \mathrm{C}^{c,c}$ into sub-matrices $\mathbf{\Gamma}_{11}(\omega) \in \mathrm{C}^{p,p}$, $\mathbf{\Gamma}_{21}(\omega)\mathrm{C}^{c,p}$, $\mathbf{\Gamma}_{12}(\omega) = \mathbf{\Gamma}_{21}^{\mathrm{T}}(\omega) \in \mathrm{C}^{p,c}$ and $\mathbf{\Gamma}_{22}(\omega) \in \mathrm{C}^{c-p,c-p}$, one knows from dynamic experiments the matrices $\mathbf{\Gamma}_{11}(\omega)$, $\mathbf{\Gamma}_{12}(\omega)$ and one is searching for $\mathbf{\Gamma}_{22}(\omega)$. Because in this section one works with $\mathbf{\Gamma}(\omega)$ for any fixed frequency ω only, this notation will be omitted in the following text.

Suppose that $1 \leqslant \mathrm{rank}\,(\mathbf{\Gamma}) = \mathrm{rank}\,(\mathbf{\Gamma}_{11}) = q \leqslant p < c$. The Takagi decomposition (3.4) of $\mathbf{\Gamma}_{11}$ is:

$$\mathbf{\Gamma}_{11} = \mathbf{U}_{11} \cdot \Sigma_1 \cdot \mathbf{U}_{11}^{\mathrm{T}} = (\mathbf{U}_{11} \cdot \Sigma_{11}^{1/2})(\mathbf{U}_{11} \cdot \Sigma_{11}^{1/2})^{\mathrm{T}}, \tag{3.18}$$

where $\mathbf{U}_{11} \in \mathrm{C}^{p,q}$, $(\mathbf{U}_{11}^{\mathrm{H}}\mathbf{U}_{11} = \mathbf{I}_q = \mathbf{U}_{11}^{\mathrm{T}}\bar{\mathbf{U}}_{11})$ and the non-zero singular value of $\mathbf{\Gamma}_{11}$ form the diagonal matrix $\Sigma_{11} \in \mathbf{R}^{q,q}$. Following this symmetric decomposition of $\mathbf{\Gamma}_{11}$, a decomposition of the sub-matrix $\mathbf{\Gamma}_{21}$ is sought in the form

$$\mathbf{\Gamma}_{21} = (\mathbf{V}_{21} \cdot \Sigma_{11}^{1/2}) \cdot (\mathbf{U}_{11}\Sigma_{11}^{1/2})^{\mathrm{T}}, \tag{3.19}$$

where $\mathbf{V}_{21} \in \mathrm{C}^{c-p,q}$. Equation (3.19) leads to:

$$\mathbf{V}_{21} = \mathbf{\Gamma}_{21} \cdot \bar{\mathbf{U}}_{11} \cdot \Sigma_{11}^{-1}. \tag{3.20}$$

For symmetry reasons, one has

$$\mathbf{\Gamma}_{21}^{\mathrm{T}} = (\mathbf{U}_{11}\Sigma_{11}^{1/2})(\mathbf{V}_{21} \cdot \Sigma_{11}^{1/2})^{\mathrm{T}} = (\mathbf{U}_{11}\Sigma_{11}^{1/2})(\Sigma_{11}^{1/2}\mathbf{V}_{21}^{\mathrm{T}}) = \Gamma_{12}. \tag{3.21}$$

Equations (3.18) and (3.19) are regrouped in the form

$$\mathbf{\Gamma}_1 = \begin{bmatrix} \mathbf{\Gamma}_{11} \\ \mathbf{\Gamma}_{21} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_{11}\Sigma_{11}^{1/2} \\ \mathbf{V}_{21}\Sigma_{11}^{1/2} \end{bmatrix} [\Sigma_{11}^{1/2} \quad \mathrm{U}_{11}^{\mathrm{T}}]. \tag{3.22}$$

The first p columns of the matrix $\mathbf{\Gamma}$ are given by the linear combinaton of q columns of

$$\begin{bmatrix} \mathbf{U}_{11}\mathbf{\Sigma}_{11}^{1/2} \\ \mathbf{V}_{21}\mathbf{\Sigma}_{11}^{1/2} \end{bmatrix}$$

with the coefficient matrix $\mathbf{\Sigma}_{11}^{1/2}\mathbf{U}_{11}^{\mathrm{T}}$. From the above, one knows that for the symmetric complex matrix $\mathbf{\Gamma}_{22}$, there exists a symmetric decomposition (3.4). Since the submatrix $\mathbf{\Gamma}_{12}$ (the first parts of the (c–p) remaining columns of $\mathbf{\Gamma}$) is given in equation (3.21) by the linear combination of q columns of $\mathbf{U}_{11}\mathbf{\Sigma}_{11}^{1/2}$ with coefficient matrix $\mathbf{\Sigma}_{11}^{1/2}\mathbf{V}_{21}^{\mathrm{T}}$, the submatrix $\mathbf{\Gamma}_{22}$ will be given by the linear combination of $[\mathbf{V}_{21}\mathbf{\Sigma}_{11}^{1/2}]$ with the same coefficient matrix $\mathbf{\Sigma}_{11}^{1/2}\mathbf{V}_{21}^{\mathrm{T}}$:

$$\mathbf{\Gamma}_{22} = (\mathbf{V}_{21}\mathbf{\Sigma}_{11}^{1/2})(\mathbf{\Sigma}_{11}^{1/2}\mathbf{V}_{21}^{\mathrm{T}}) = \mathbf{V}_{21}\mathbf{\Sigma}_{11}\mathbf{V}_{21}^{\mathrm{T}}. \tag{3.23}$$

Introducing into expression (3.23) the matrix $\mathbf{V}_{21}$ from equation (3.20), one obtains

$$\mathbf{\Gamma}_{22} = \mathbf{\Gamma}_{21}\bar{\mathbf{U}}_{11}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{11}\mathbf{\Sigma}_{11}^{-1}\mathbf{U}_{11}^{\mathrm{H}}\cdot\mathbf{\Gamma}_{21}^{\mathrm{T}} = \mathbf{\Gamma}_{21}\bar{\mathbf{U}}_{11}\mathbf{\Sigma}_{11}^{-1}\mathrm{U}_{11}^{\mathrm{H}}\mathbf{\Gamma}_{21} = \mathbf{\Gamma}_{21}\mathbf{\Gamma}_{11}^{+}\cdot\mathbf{\Gamma}_{21}^{\mathrm{T}}, \tag{3.24}$$

where

$$\mathbf{\Gamma}_{11}^{+} = \bar{\mathbf{U}}_{11}\mathbf{\Sigma}_{11}^{-1}\mathrm{U}_{11}^{\mathrm{H}} \tag{3.25}$$

is the generalized Moore–Penrose inverse of the matrix $\mathbf{\Gamma}_{11}$, as follows from equation (3.5).

More, because $\mathbf{\Gamma}_{11} = \mathbf{\Gamma}_{11}\mathbf{\Gamma}_{11}^{+}\mathbf{\Gamma}_{11} = \mathbf{\Gamma}_{11}\mathbf{\Gamma}_{11}^{+}\mathbf{\Gamma}_{11}^{\mathrm{T}}$, $\mathbf{\Gamma}_{21} = \mathbf{\Gamma}_{21}\mathbf{\Gamma}_{11}^{+}\mathbf{\Gamma}_{11} = \mathbf{\Gamma}_{21}\mathbf{\Gamma}_{11}^{+}\mathbf{\Gamma}_{11}^{\mathrm{T}}$ and $\mathbf{\Gamma}_{12} = \mathbf{\Gamma}_{21}^{\mathrm{T}} = \mathbf{\Gamma}_{11}(\mathbf{\Gamma}_{11}^{+})^{\mathrm{T}}\mathbf{\Gamma}_{21}^{\mathrm{T}} = \mathbf{\Gamma}_{11}\mathbf{\Gamma}_{11}^{+}\mathbf{\Gamma}_{21}^{\mathrm{T}}$, one can directly write the symetric decomposition of the complete matrix $\mathbf{\Gamma}$ as:

$$\mathbf{\Gamma} = \mathbf{\Gamma}_{1}\cdot\mathbf{\Gamma}_{11}^{+}\cdot\mathbf{\Gamma}_{1}^{\mathrm{T}}, \tag{3.26}$$

where $\mathbf{\Gamma}_{1}\in\mathrm{C}^{c\cdot p}$. Note that in the case of rank ($\mathbf{\Gamma}$) = 1, equation (3.17) corresponds to a particular case of equation (3.26).

3.2.2. *Suppression of the small singular values*

It is evident that to obtain the Moore–Penrose generalized inversion of $\mathbf{\Gamma}_{11}^{+}$ one can use either equation (3.25) via Takagi's symmetric decomposition or a classical (‘‘non-symmetric’’) singular value decomposition (e.g., for the matrix $\mathbf{A}$ decomposed as in equation (3.2), it is given only by $\mathbf{A}^{+} = \mathbf{W}_{1}\Sigma_{1}^{-1}\mathbf{X}_{1}^{\mathrm{H}}$). In both cases one must invert the diagonal matrix $\mathbf{\Gamma}_{1} = \mathrm{Diag}\,[\sigma_{v}]$, $v = 1, 2, \ldots, q$ of singular values of matrix $\mathbf{\Gamma}_{11}$.

In an experimental context, the inversion of small singular values (determined more by the uncertainties in the measurements than by dynamical reasons) can cause difficulties. For this reason, the smallest singular values σ_{v} such that:

$$\sigma_{v} \leqslant \|\varDelta\mathbf{\Gamma}_{11}\|, \tag{3.27}$$

are neglected, where $\varDelta\mathbf{\Gamma}_{11}$ is an estimation of experimental errors of $\mathbf{\Gamma}_{11}$. The other ‘‘non-zero’’ singular values in number $q \leqslant p$ define the numerical rank of the matrix $\mathbf{\Gamma}_{11}$ and can be inverted.

The above results are based on the condition rank ($\mathbf{\Gamma}$) = rank ($\mathbf{\Gamma}_{11}$) = q. If this condition is not satisfied, one must either choose other positions for the excitations or increase the number of excitations p.

3.3. SECOND METHOD: LINEAR COMBINATION OF THE RESONANCE FLEXIBILITY MATRICES $\mathbf{\Gamma}(\omega_{v})$

In the verification of the method proposed in section 3.2 by numerical simulations, it was noticed that in the low amplitude regions (in the neighbourhoods of antiresonances) it is relatively difficult to decide which small singular values one can consider as zero. These uncertainties increase the expansion errors. The expansion at resonance frequencies is much better. This leads to the idea of expressing the dynamic flexibility at a given frequency

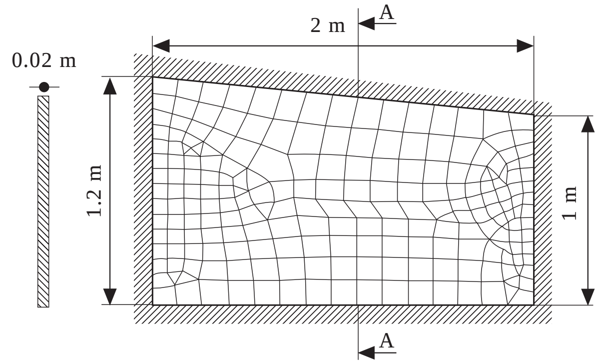

Figure 1. Clamped plate.

ω in the observed band, by a linear combination of the dynamic flexibilities at the resonance frequencies.

The proposed procedure has three steps.

(a) Search for resonance frequencies ω_v, $v = 1, 2, \ldots, r$ (e.g., a local maxima of the function $\|\mathbf{\Gamma}_1(\omega)\|$).

(b) Expansion of the flexibility matrices $\mathbf{\Gamma}(\omega_v)$, $v = 1, 2, \ldots, r$ by the method proposed in section 2.2.

(c) Expansion of the flexibility matrices $\mathbf{\Gamma}(\omega)$ for the entire analyzed frequency band as linear combinations of $\mathbf{\Gamma}(\omega_v)$.

In the third step, for $\omega \neq \omega_v$ one first finds the complex scalars $\alpha_v(\omega) \in \mathrm{C}$, $v = 1, 2, \ldots, r$ such that

$$\mathbf{\Gamma}_1(\omega) = \sum_{v=1}^{r} \alpha_v(\omega) \mathbf{\Gamma}_1(\omega_v). \tag{3.28}$$

The scalars $\alpha_v(\omega)$ are determined from equation (3.28) by using a least squares technique.

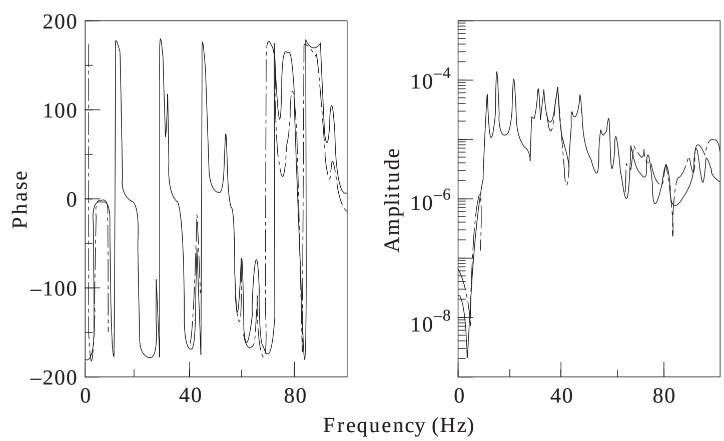

Figure 2. Expansion of the element Γ_{ij} of the matrix $\mathbf{\Gamma}$ by the first method, with $a_v = 0{\cdot}001$. ·—·—, Calculated; ——, exact.

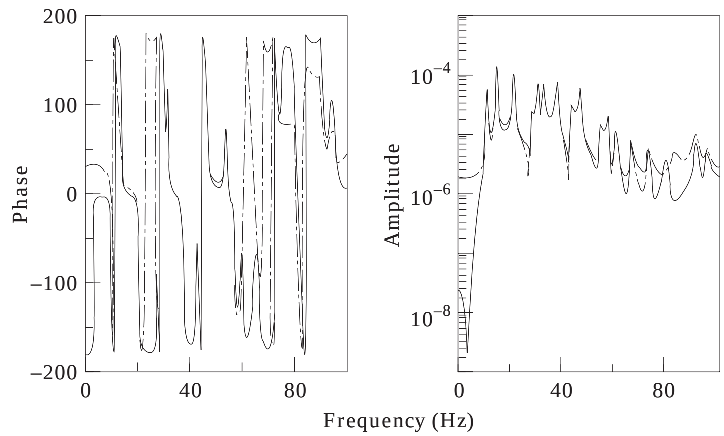

Figure 3. Expansion of the element Γ_{ij} of the matrix $\mathbf{\Gamma}$ by the second method, with $a_v = 0{\cdot}01$. ·—·–, Calculated; ——, exact.

The equation (3.28) can be rewritten in matrix form as

$$\mathbf{G}\boldsymbol{\alpha} = \mathbf{g}, \tag{3.29}$$

where $\mathbf{G} \in C^{c,p,r}$; $\mathbf{g} \in C^{c,p}$, $\boldsymbol{\alpha} = [-\alpha_v(\omega)-]^{\mathrm{T}} \in C^r$. Assuming that $c, p > r$ and rank $(\mathbf{G}) = r$, one has

$$\boldsymbol{\alpha} = \mathbf{G}^+ \cdot \mathbf{g}, \tag{3.30}$$

where $\mathbf{G}^+ = (\mathbf{G}^{\mathrm{H}}\mathbf{G})^{-1}\mathbf{G}^{\mathrm{H}}$.

The expanded dynamic flexibility matrix $\mathbf{\Gamma}(\omega)$ at frequency ω is then given by the same linear combination of $\mathbf{\Gamma}(\omega_v)$ as in equation (3.28):

$$\Gamma(\omega) = \sum_{v=1}^{r} \alpha_v(\omega)\Gamma(\omega_v). \tag{3.31}$$

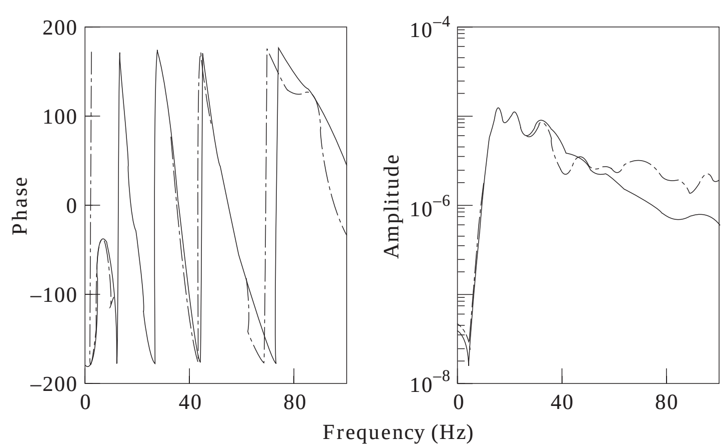

Figure 4. Expansion of the element Γ_{ij} of the matrix $\mathbf{\Gamma}$ by the first method, with $a_v = 0{\cdot}1$. ·—·–, Calculated; ——, exact.

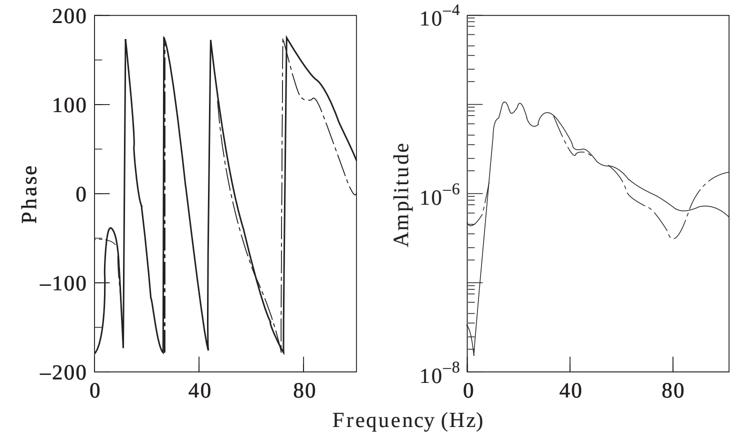

Figure 5. Expansion of the element Γ_{ij} of the matrix $\mathbf{\Gamma}$ by the second method, with $a_v = 0{\cdot}1$. ·—·–, Calculated; ——, exact.

4. NUMERICAL SIMULATION

The preceding methods will be illustrated by a test case consisting of a clamped plate (see Figure 1) in which the plate has the following physical and geometrical characteristics: $E = 2{\cdot}1 \times 10^{11}$ N/m^2, $\rho = 7800$ kg/m^3, $\nu = 0{\cdot}291$.

This structure is modelled by using the finite element code ANSYS. This model contains 236 nodes with six dofs per node. A proportional damping ($\mathbf{B}\ \mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\ \mathbf{B}$) is introduced and the dynamic flexibility matrix $\mathbf{\Gamma}(\omega)$ is constructed at each frequency in the analyzed band by using the eigenmodes of the dissipative structure. The frequency band under consideration [0, 100 Hz] contains the first 21 eigenmodes of the structure. A total of $c = 50$ pickups are arbitrarily chosen along with $p = 8$ arbitrarily selected excitation points. The remaining 42 columns of the matrix $\mathbf{\Gamma}(\omega)$ are determined on the basis of the 8 measured columns, yielding a total of 903 unknown elements if symmetry is taken into account. The evolution of the amplitude and the phase of one unknown element Γ_{ij} is plotted as a function of ω and compared with the exact values.

The results obtained with the first and second methods will be compared. For Figures 2 and 3, the first and second formulations, respectively, were used to expand the element $\mathbf{\Gamma}_{ij}$ with a modal damping factor $a_v = |\mathrm{Re}\,(s_v)|/\mathrm{Im}\,(s_v) = 0{\cdot}01$. The expansion is nearly perfect in the neighbourhood of the resonances but remains mediocre in the low amplitude regions, especially near the antiresonances. A certain number of parasitic resonances also appear in the expanded component of $\mathbf{\Gamma}(\omega)$. These peaks correspond to the resonance frequencies of the structure and though they do not appear on all the elements of the exact flexibility matrix $\mathbf{\Gamma}(\omega)$, they can appear in the homologous expanded elements. These parasitic peaks can be attenuated by judiciously choosing the rank of $\mathbf{\Gamma}_{11}(\omega)$. Generally the results obtained by the second method are better than those of the first one.

The effect of damping is illustrated in Figures 4 and 5 for the same element of $\mathbf{\Gamma}(\omega)$ as before except that the damping coefficients are now $a_v = 0{\cdot}1$. The results are of lower quality in certain regions but in general remain acceptable in the frequency band [0, 50 Hz].

Generally the quality of the expansion is a function of (a) the positions of the excitation points, (b) the number of known columns of $\mathbf{\Gamma}$ (it is clear that increasing p will improve the quality of the results) and (c) the damping in the structure.

5. CONCLUSION

Two methods were proposed for expanding the dynamic flexibility matrix. The quality of the expansion is seen to depend on several factors. It is intimately related to the numerical rank of either the matrix $\mathbf{\Gamma}_1(\omega)$ or of the matrix $\mathbf{\Gamma}_{11}(\omega)$. Two major problems can be encountered in practice.

(1) If the numerical rank of the exact matrix $\mathbf{\Gamma}(\omega)$ is superior to the numerical rank of the matrix $\mathbf{\Gamma}_1(\omega)$ and if $p \geqslant$ numerical rank of $\mathbf{\Gamma}(\omega)$, then the quality of the expansion can be improved by a better choice of the excitation degrees of freedom.

(2) If the numerical rank of the exact matrix $\mathbf{\Gamma}(\omega)$ is superior or equal to p, then the number of excitation points must be increased if a good expansion is to be obtained.

Methods for this type are required in the medium frequency domain where classical modal analysis techniques are practically unusable.

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APPENDIX: PRINCIPAL NOTATIONS

n	number of degrees of freedom of the discrete structural model
$\mathbf{\Gamma}(\omega) \in \mathbf{C}^{n,n}$	symmetric dynamic flexibility matrix of the initial structure at the frequency ω
$\hat{\mathbf{\Gamma}}(\omega) \in \mathbf{C}^{n,n}$	symmetric dynamic flexibility matrix of the modified structure at the frequency ω
$\mathbf{Y} \in \mathbf{C}^{n,n}$, $\mathbf{S} \in \mathbf{C}^{n,n}$	the modal and diagonal spectral matrices of the structure
$\mathbf{Y}_1 \in \mathbf{C}^{n,m}$, $\mathbf{S}_1 \in \mathbf{C}^{m,m}$	known modal sub-basis and spectral sub-basis
$\Delta\mathbf{Z}_{nn}(\omega) \in \mathbf{C}^{n,n}$	symmetric dynamic stiffness matrix of the modifications
$\mathbf{\Gamma}(\omega) \in \mathbf{C}^{c,c}$	symmetric dynamic flexibility matrix of the initial structure relative to the c measured degrees of freedom
$\hat{\mathbf{\Gamma}}(\omega) \in \mathbf{C}^{c,c}$	symmetric dynamic flexibility matrix of the modified structure relative to the c measured degrees of freedom
$\Delta\mathbf{Z}(\omega) \in \mathbf{C}^{c,c}$	symmetric dynamic stiffness matrix characterizing the introduced modifications at c pickup degrees of freedom

$\mathbf{Y}_{1c} \in \mathrm{C}^{c,m}$	identified modal sub-basis at the c pickup degrees of freedom
ω_v	the angular eigenfrequency of the vth mode
$\mathbf{K}, \mathbf{M}, \mathbf{B} \in \mathrm{R}^{n,n}$	Stiffness, mass and damping matrices of the discrete model of the initial structure. $\mathbf{K}$ and $\mathbf{M}$ are symmetric and positive definite. $\mathbf{B}$ is symmetric and positive semi-definite.
$\mathbf{A}$	symmetric complex matrix
$\bar{\mathbf{A}}$	complex conjugate of $\mathbf{A}$
$\mathbf{A}^{\mathrm{T}}$	transpose of $\mathbf{A}$
$\mathbf{A}^{\mathrm{H}}$	transpose complex conjugate of $\mathbf{A}$ ($\mathbf{A}^{\mathrm{H}} = \bar{\mathbf{A}}^{\mathrm{T}}$)
$\Vert\mathbf{A}\Vert, \Vert\mathbf{A}\Vert_{\mathrm{F}}$	spectral and Frobenius norm of $\mathbf{A}$
$\mathrm{Tr}\,(\mathbf{A})$	trace of a matrix $\mathbf{A}$
$\mathrm{rank}\,(\mathbf{A})$	rank (numerical rank) of a matrix $\mathbf{A}$
$\mathbf{A}^{+}$	Moore–Penrose generalized inversion of $\mathbf{A}$
$\mathbf{I}, \mathbf{I}_m, \mathbf{I}_q$	identity matrices
$\mathbf{U}, \mathbf{X}_1, \mathbf{W}_1, \mathbf{U}_1, \mathbf{U}_{11}$	Unitary matrices
Σ, Σ_{11}	Diagonal matrices on non-negative singular values σ_v
$\mathbf{D}$	Diagonal matrix with non-zero elements d_v