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VIBRATION OF INITIALLY STRESSED REDDY PLATES ON A WINKLER–PASTERNAK FOUNDATION

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Treated herein is the vibration of isotropic Reddy plates. The plates considered are of general polygonal shape and their edges are all simply supported. Complicating effects such as the presence of initial stresses and a Winkler–Pasternak foundation are also considered. It is shown herein that the vibration solution can be readily obtained from the classical Kirchhoff plate vibration results because of the mathematical similarity of the two kinds of problems. The mathematical analogy permits the development of an exact relationship that links the natural frequencies of the initially stressed Reddy plates resting on a Winkler–Pasternak foundation to the corresponding classical Kirchhoff plate solutions without the presence of complicating effects.

1. INTRODUCTION

The simplest plate theory is based on the Kirchhoff [1] assumptions that the normals remain straight and normal to the deformed plate midsurface. This classical plate theory, however, does not allow for the effect of transverse shear deformation because of the normality assumption. The effect becomes important when considering thick plates or when accurate vibration solutions corresponding to higher modes are to be determined. A more refined plate theory to allow for the effect of transverse shear deformation has been proposed by Reissner [2, 3], Hencky [4] and Mindlin [5]. This first-order shear deformation plate theory, commonly referred to as the Mindlin plate theory, relaxes the normality assumption by allowing the normal to have a rotation with respect to the midsurface of the plate. Owing to the assumption, a constant shear stress results through the plate thickness, thus violating the vanishing shear stress requirement at the free surfaces of the plate. Mindlin [5] suggested using a shear correction factor to be multiplied to the transverse shear moduli to compensate for this error. Further higher order theories expand the displacement field in terms of the thickness co-ordinate up to any desired degree with the view to better represent the kinematics and to yield better stress distributions. Also higher-order theories do away with the need for a shear correction factor. Of these higher order theories, the third-order plate theory proposed by Reddy [6, 7] is well received. The Reddy plate theory suffices in capturing the kinematics and stress distributions of the plate without thrusting into too complicated computational procedure for solutions.

In 1985, Irschik [8] derived an exact relationship between the vibration frequencies of initially stressed, simply supported, polygonal Mindlin plates on a Winkler–Pasternak foundation and that of the corresponding prestressed membranes. Also the studies by Xiang *et al*. [9] and Wang [10, 11] showed the vibration relationship between the Mindlin

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and Kirchhoff plates. Such a relationship is useful as very accurate Mindlin solutions can be readily obtained upon supplying the Kirchhoff solutions. The benchmark Mindlin results obtained may be used to check the validity, convergence and accuracy of numerical results computed from thick plate software packages.

The present paper presents an exact relationship between the natural frequencies of Reddy plates with that of the classical Kirchhoff plates. This relationship is valid for any general polygonal, isotropic plates with simply supported edges. The complicating effects of initial stresses and Winkler–Pasternak foundations can also be accounted for in the relationship. Note that for Reddy solutions with these complicating effects, one simply needs the Kirchhoff solutions without any complicating effects to be used with the relationship. Further references on the subject of vibrating, initially stressed plates on Pasternak foundation may be found in the recent paper of Xiang *et al*. [12].

2. EQUATIONS OF MOTION OF REDDY PLATE THEORY

The Reddy third order plate theory is based on the following displacement field [6, 7]:

$$u(x, y, z, t) = z\phi_x(x, y, t) - \frac{4z^3}{3h^2}\left[\phi_x(x, y, t) + \frac{\partial w_0(x, y, t)}{\partial x}\right], \tag{1a}$$

$$v(x, y, z, t) = z\phi_y(x, y, t) - \frac{4z^3}{3h^2}\left[\phi_y(x, y, t) + \frac{\partial w_0(x, y, t)}{\partial y}\right], \tag{1b}$$

$$w(x, y, z, t) = w_0(x, y, t), \tag{1c}$$

where (u, v, w) are the displacement components along the (x, y, z) co-ordinate directions, respectively, w_0 is the transverse displacement component of a point on the plate midsurface, (ϕ_x, ϕ_y) are the rotations of the normals in the x and y-directions, respectively, h is the plate thickness and t denotes time.

Based on the foregoing displacement field in equation (1), Reddy [6, 7] used a consistent variational formulation to obtain the governing equations of motion for laminated plates. When specialized for isotropic plates, these equations are as follows:

$$\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = -\frac{17\rho h^3}{315}\,\omega_R^2\phi_x + \frac{4\rho h^3}{315}\,\omega_R^2\,\frac{\partial w_0}{\partial x}, \tag{2}$$

$$\frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = -\frac{17\rho h^3}{315}\,\omega_R^2\phi_y + \frac{4\rho h^3}{315}\,\omega_R^2\,\frac{\partial w_0}{\partial y}, \tag{3}$$

$$\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + \frac{4}{3h^2}\left(\frac{\partial^2 P_{xx}}{\partial x^2} + 2\,\frac{\partial^2 P_{xy}}{\partial x\,\partial y} + \frac{\partial^2 P_{yy}}{\partial y^2}\right)$$
$$= \left(\sigma h + \frac{\rho h^3}{252}\,\omega_R^2\right)\nabla^2 w_0 - \rho h\omega_R^2 w_0 - \frac{4\rho h^3}{315}\,\omega_R^2\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) + p, \tag{4}$$

where ρ is the mass density of the plate, ω_R the circular frequency of the plate, σ the uniform initial stress in the plate, $\nabla^2(\cdot) = \partial^2(\cdot)/\partial x^2 + \partial^2(\cdot)/\partial y^2$ is the Laplacian operator, and p the foundation interface pressure.

For the elastic foundation, one assumes the two-parameter elastic foundation model proposed by Pasternak [13]. In addition to the well-known Winkler foundation springs, the Pastermak model takes into account the shear interaction between the spring elements. This is accomplished by connecting the ends of the springs to the plate with incompressible vertical elements that deform only by transverse shear as shown in Figure 1. The foundation medium is assumed to be linear, homogenous and isotropic. The bonding between the plate and the foundation is perfect and frictionless. If the effects of damping and inertia force in the foundation are neglected, the foundation interface pressure p may be expressed as

$$p = kw_0 - G_b \nabla^2 w_0 \tag{5}$$

where k is the modulus of subgrade reaction for the foundation and G_b the shear modulus of the subgrade. Note that by setting $G_b = 0$, the Pasternak model becomes that of the Winkler foundation model.

The stress-resultants based on the Reddy plate theory are given by

$$\bar{M}_{\alpha\beta} = M_{\alpha\beta} - \left(\frac{4}{3h^2}\right) P_{\alpha\beta} \tag{6}$$

$$\bar{Q}_{\alpha} = Q_{\alpha} - \left(\frac{4}{h^2}\right) R_{\alpha} \tag{7}$$

with

$$M_{xx} = \frac{4D}{5}\left(\frac{\partial \phi_x}{\partial x} + v\frac{\partial \phi_y}{\partial y}\right) - \frac{D}{5}\left(\frac{\partial^2 w_0}{\partial x^2} + v\frac{\partial^2 w_0}{\partial y^2}\right), \tag{8a}$$

$$P_{xx} = \frac{4h^2 D}{35}\left(\frac{\partial \phi_x}{\partial x} + v\frac{\partial \phi_y}{\partial y}\right) - \frac{h^2 D}{28}\left(\frac{\partial^2 w_0}{\partial x^2} + v\frac{\partial^2 w_0}{\partial y^2}\right); \tag{8b}$$

$$M_{yy} = \frac{4D}{5}\left(v\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) - \frac{D}{5}\left(v\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right), \tag{9a}$$

$$P_{yy} = \frac{4h^2 D}{35}\left(v\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) - \frac{h^2 D}{28}\left(v\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right); \tag{9b}$$

$$M_{xy} = \left(\frac{1-v}{2}\right)\left[\frac{4D}{5}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - \frac{D}{5}\left(2\frac{\partial^2 w_0}{\partial x\,\partial y}\right)\right], \tag{10a}$$

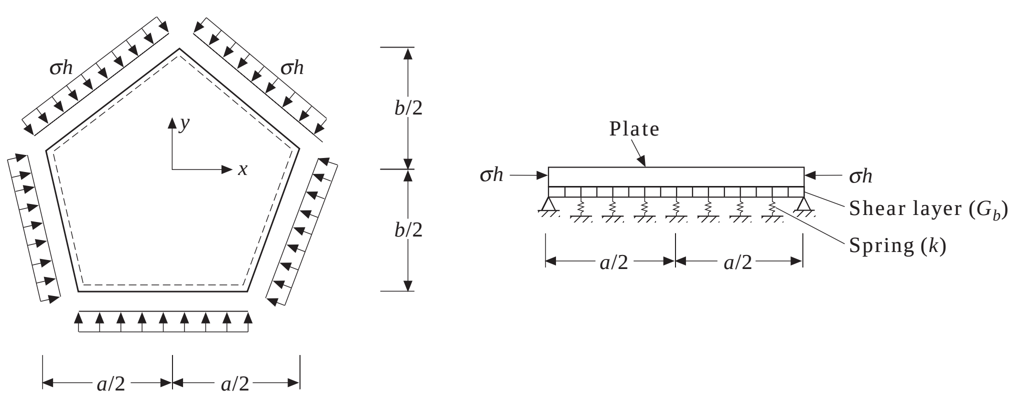

Figure 1. Initially stressed, simply supported, polygonal Reddy plates on a Pasternak foundation.

$$P_{xy} = \left(\frac{1-v}{2}\right)\left[\frac{4h^2D}{35}\left(\frac{\partial\phi_x}{\partial y} + \frac{\partial\phi_y}{\partial x}\right) - \frac{h^2D}{28}\left(2\,\frac{\partial^2 w_0}{\partial x\,\partial y}\right)\right]; \tag{10b}$$

$$Q_x = \frac{2hG}{3}\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \qquad R_x = \frac{h^3G}{30}\left(\phi_x + \frac{\partial w_0}{\partial x}\right); \tag{11a, b}$$

$$Q_y = \frac{2hG}{3}\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \qquad R_y = \frac{h^3G}{30}\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \tag{12a, b}$$

where $D = Eh^3/[12(1-v^2)]$ is the flexural rigidity of the plate, $G = E/[2(1+v)]$ the shear modulus, E the modulus of elasticity and v Poisson's ratio. Note that (P_{xx}, P_{yy}, P_{xy}) are the higher order moments of the Reddy theory and (R_x, R_y) are the higher order shear forces.

By substituting equations (2), (3), (5) into equation (4), the governing equation of motion may be expressed as

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} = \left(\sigma h - G_b + \frac{\rho h^3}{60}\,\omega_R^2\right)\nabla^2 w_0 - (\rho h\omega_R^2 - k)w_0 - \frac{\rho h^3}{15}\,\omega_R^2\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right) \tag{13}$$

Here, the moment sum M^{R} is introduced and defined by

$$\mathsf{M}^{\mathsf{R}} = \frac{M_{xx} + M_{yy}}{1+v} = \frac{4D}{5}\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right) - \frac{D}{5}\,\nabla^2 w_0. \tag{14}$$

In view of this moment sum and the moment expressions in equations (8a), (9a) and (10a), equation (13) may be written as

$$\nabla^2\mathsf{M}^{\mathsf{R}} = \left(\sigma h - G_b + \frac{\rho h^3}{60}\,\omega_R^2\right)\nabla^2 w_0 - (\rho h\omega_R^2 - k)w_0 - \frac{\rho h^3}{15}\,\omega_R^2\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right). \tag{15}$$

The substitution of equations (5), (8b), (9b), (10b), (11), (12) and (14) into equation (4) leads to

$$K\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y}\right) = \left(\sigma h - G_b + \frac{\rho h^3}{252}\,\omega_R^2 - \frac{8Gh}{15}\right)\nabla^2 w_0 - (\rho h\omega_R^2 - k)w_0 - \frac{4}{21}\,\nabla^2\mathsf{M}^{\mathsf{R}} + \frac{D}{105}\,\nabla^4 w_0, \tag{16}$$

where

$$K = 8Gh/15 + (4\rho h^3/315)\omega_R^2. \tag{17}$$

By substituting equation (16) into (15), one obtains

$$J\nabla^2\mathsf{M}^{\mathsf{R}} = -(\rho h\omega_R^2 - k)\left(1 - \frac{\rho h^3}{15K}\,\omega_R^2\right)w_0 - \frac{D}{105}\,\frac{\rho h^3}{15K}\,\omega_R^2\nabla^4 w_0 + L\nabla^2 w_0 \tag{18}$$

where

$$J = 1 - \frac{4\rho h^3}{315K}\omega_R^2, \qquad L = \left(\sigma h - G_b + \frac{\rho h^3}{60}\omega_R^2\right) - \frac{\rho h^3}{15K}\omega_R^2\left(\sigma h - G_b + \frac{\rho h^3}{252}\omega_R^2 - \frac{8Gh}{15}\right). \tag{19, 20}$$

Also be substituting equation (19) into equation (17), one obtains

$$\mathsf{M}^{\mathsf{R}} = -\frac{4D}{5K}(\rho h\omega_R^2 - k)w_0 - \frac{16D}{105K}\boldsymbol{\nabla}^2\mathsf{M}^{\mathsf{R}} + \frac{4D}{5K}\frac{D}{105}\boldsymbol{\nabla}^4 w_0$$
$$+ \left[-\frac{D}{5} + \frac{4D}{5K}\left(\sigma h - G_b + \frac{\rho h^3}{252}\omega_R^2 - \frac{8Gh}{15}\right)\right]\boldsymbol{\nabla}^2 w_0. \tag{21}$$

In view of equation (18), equation (21) may be expressed as

$$\mathsf{M}^{\mathsf{R}} = \left[-\frac{4D}{5K}(\rho h\omega_R^2 - k) + \frac{16D}{105KJ}(\rho h\omega_R^2 - k)\left(1 - \frac{\rho h^3}{15K}\omega_R^2\right)\right]w_0$$
$$+ \left[-\frac{D}{5} + \frac{4D}{5K}\left(\sigma h - G_b + \frac{\rho h^3}{252}\omega_R^2 - \frac{8Gh}{15}\right) - \frac{16DL}{105KJ}\right]\boldsymbol{\nabla}^2 w_0$$
$$+ \left[\frac{4D}{5K}\frac{D}{105} + \frac{16D}{105KJ}\frac{D}{105}\frac{\rho h^3}{15K}\omega_R^2\right]\boldsymbol{\nabla}^4 w_0. \tag{22}$$

The substitution of equation (22) into equation (18) furnishes the following sixth order governing differential equation in terms of w_0:

$$a_1\boldsymbol{\nabla}^6 w_0 + a_2\boldsymbol{\nabla}^4 w_0 + a_3\boldsymbol{\nabla}^2 w_0 + a_4 w_0 = 0, \tag{23}$$

where

$$a_1 = \frac{4D}{5K}\frac{D}{105}\left[1 + \frac{4}{21J}\frac{\rho h^3}{15K}\omega_R^2\right], \tag{24}$$

$$a_2 = -\frac{D}{5}\left[1 - \frac{4}{K}\left(\sigma h - G_b + \frac{\rho h^3}{252}\omega_R^2 - \frac{8Gh}{15}\right) + \frac{16L}{21KJ} - \frac{\rho h^3}{315KJ}\omega_R^2\right], \tag{25}$$

$$a_3 = -\frac{4D}{5K}(\rho h\omega_R^2 - k) + \frac{16D}{105KJ}(\rho h\omega_R^2 - k)\left(1 - \frac{\rho h^3}{15K}\omega_R^2\right) - \frac{L}{J}, \tag{26}$$

$$a_4 = \frac{(\rho h\omega_R^2 - k)}{J}\left(1 - \frac{\rho h^3}{15K}\omega_R^2\right). \tag{27}$$

The governing equation (23) may be factored to give

$$(\boldsymbol{\nabla}^2 + \lambda_1)(\boldsymbol{\nabla}^2 + \lambda_2)(\boldsymbol{\nabla}^2 + \lambda_3)w_0 = 0, \tag{28}$$

where

$$\lambda_1 = -2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) + \frac{a_2}{3a_1}, \qquad \lambda_2 = -2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + \frac{a_2}{3a_1}, \quad (29, 30)$$

$$\lambda_3 = -2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) + \frac{a_2}{3a_1} \quad (31)$$

and

$$\cos\theta = \frac{R}{\sqrt{(-Q)^3}}, \qquad Q = \frac{a_3}{3a_1} - \left(\frac{a_2}{3a_1}\right)^2, \qquad R = \frac{a_2 a_3}{6a_1^2} - \frac{a_4}{2a_1} - \left(\frac{a_2}{3a_1}\right)^3, \quad (32\text{–}34)$$

For a simply supported polygonal Reddy plate, the deflection and the moment sum are zero at the boundary: i.e.,

$$w_0 = 0, \qquad \mathsf{M}^{\mathrm{R}} = 0 \Rightarrow \nabla^2 w_0 = 0. \quad (35)$$

It can be seen from equation (28) that the governing equation of the considered Reddy plate may be written as

$$(\nabla^2 + \lambda_i) w_0 = 0, \qquad \text{where} \qquad i = 1, 2 \text{ or } 3. \quad (36)$$

3. KIRCHHOFF PLATE EQUATIONS

The equation of motion for a vibrating Kirchhoff plate is given by [14]

$$(\nabla^4 - \lambda_K^2) w_K = (\nabla^2 + \lambda_K)(\nabla^2 - \lambda_K) w_K = 0, \quad (37)$$

where

$$\lambda_K^2 = (\rho h / D)\omega_K^2 \quad (38)$$

and the subscript K denotes quantities belonging to the Kirchhoff plate.

For a simply supported polygonal Kirchhoff plate, the deflection and the moment sum are zero at the boundary [14]: i.e.,

$$w_K = 0 \qquad \text{and} \qquad \nabla^2 w_K = 0 \qquad \text{on the boundary.} \quad (39)$$

Since the equation $(\nabla^2 - \lambda_K) w_K = 0$ produces imaginary frequencies, the vibration of the Kirchhoff plate is thus governed by

$$(\nabla^2 + \lambda_K) w_K = 0 \quad (40)$$

4. FREQUENCY RELATIONSHIP

In view of the mathematical similarity of equations (35), (36), (39) and (40), it may be deduced that Reddy plate frequency ω_R is related to the Kirchhoff plate frequency ω_K by

$$\lambda_i = \lambda_K, \qquad \text{where} \qquad i = 1, 2 \text{ or } 3. \quad (41a)$$

From observation, it was found that λ_1 is always negative which therefore is not feasible. It was also observed that Reddy plate frequency determined from equation (41a) is smaller when $\lambda_i = \lambda_2$ as compared to the case when $\lambda_i = \lambda_3$. As such, equation (41a) may now be

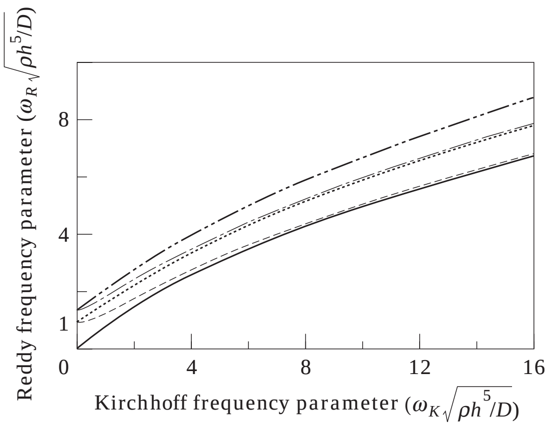

Figure 2. Relationship between frequencies of Reddy plates on elastic foundation and those of corresponding Kirchhoff plates for various foundation parameters ($\bar{\mu}_k = kh^4/D$, $\bar{\mu}_g = G_b h^2/D$). $\bar{\mu}_g$ and $\bar{\mu}_k$ values respectively: —··—, 2, 2; —·—·, 1, 2; ·····, 1, 1; - - - -, 0, 1; ——, 0, 0.

written as

$$\lambda_2 = \lambda_K \qquad \text{or} \qquad -2\sqrt{-Q}\cos(\theta/3 + 2\pi/3) + a_2/3a_1 = \sqrt{(\rho h/D)}\omega_K. \tag{41b}$$

Upon supplying the Kirchhoff plate frequencies, the foregoing exact relationship given in equation (41b) can be used to compute the Reddy plate frequencies. The transcendental equation (41b) may be solved using the false position method.

It can also be readily shown that the vibration frequency $\bar{\omega}_K$ of Kirchhoff plate with initial stresses and resting on a Pasternak foundation is related to the corresponding Kirchhoff plate solution ω_K without these complicating effects by

$$\bar{\omega}_K^2 = \omega_K^2 + (G_b - \sigma h)\omega_K/\sqrt{\rho h D} + k/\rho h. \tag{42}$$

A graphical representation of equation (41b) for any polygonal plate with simply supported edges is given in Figure 2 without the presence of initial stresses, i.e., $\sigma = 0$. This figure shows clearly the effect of the elastic foundation on the frequencies. The frequencies increase with greater values of foundation modulus parameters ($\bar{\mu}_k = kh^4/D$, $\bar{\mu}_g = G_b h^2/D$).

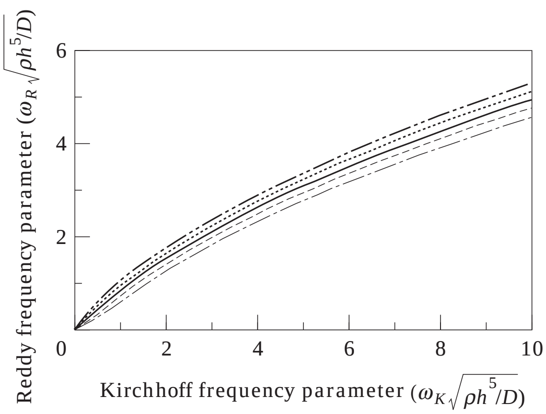

Figure 3. Relationship between frequencies of initially stressed Reddy plates and those of corresponding Kirchhoff plates for various stress parameters $\Lambda(=\sigma h^3/D)$ values: —··—·, $-0{\cdot}4$; ·····, $-0{\cdot}2$; ——, 0·0; - - - -, 0·2; —·—, 0·4.

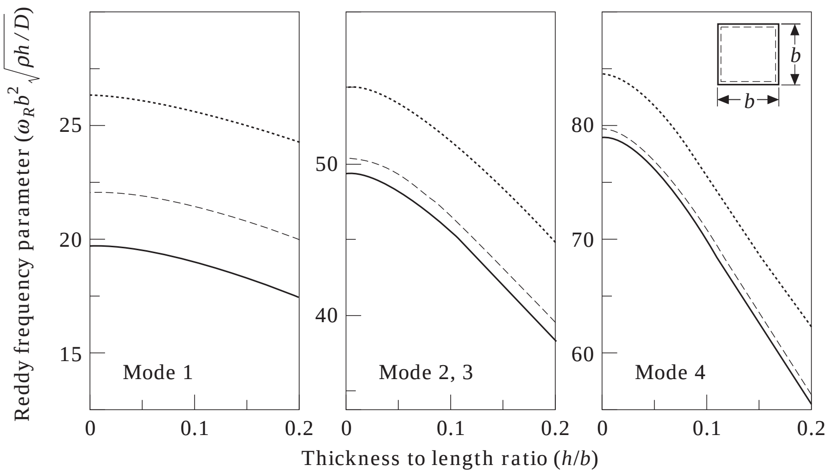

Figure 4. Variations of Reddy plate frequencies with respect to thickness-to-length ratio for square plates on an elastic foundation. μ_g and μ_k values respectively: ·····, 10, 100; - - - -, 0, 100; ——, 0, 0.

The foundation effect is more pronounced for thicker plates and also for higher vibration modes that correspond to higher frequency values. Note that both of these cases imply that one is moving further away from the origin along the curves. It is interesting to note that when the Kirchhoff frequency parameter $\omega_K b^2\sqrt{\rho h/D}$ is zero, the Reddy frequency parameter $\omega_R b^2\sqrt{\rho h/D}$ takes on the value of $\sqrt{kb^4/D}$ as can be deduced from equation (42).

On the other hand, Figure 3 shows the influence of initial stresses on the frequencies. When the initial stresses are compressive in nature (denoted by positive signs for the stress parameter $\Lambda = \sigma h^3/D$), the plate frequencies decrease as a result of a lowering of plate stiffness. The reverse occurs when the initial stresses are tensile in nature (denoted by the

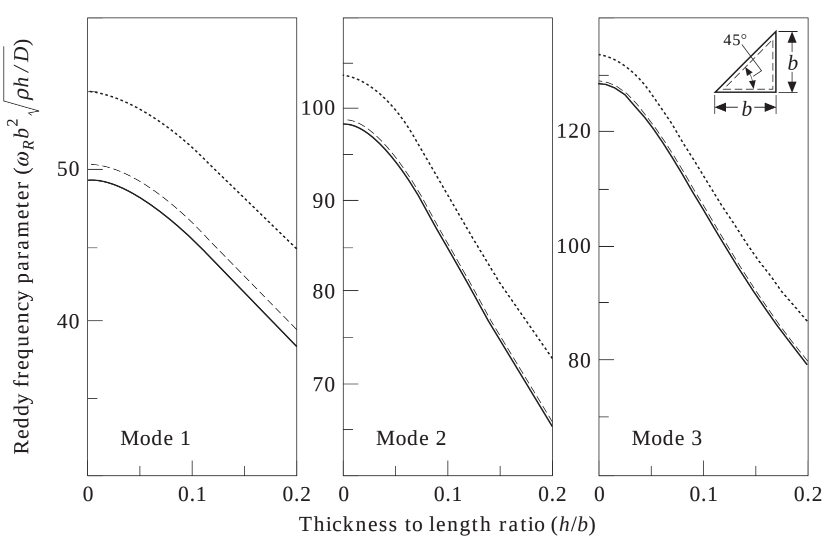

Figure 5. Variations of Reddy plate frequencies with respect to thickness-to-length ratio for right-angle isosceles triangular plates on an elastic foundation. Key as Figure 4.

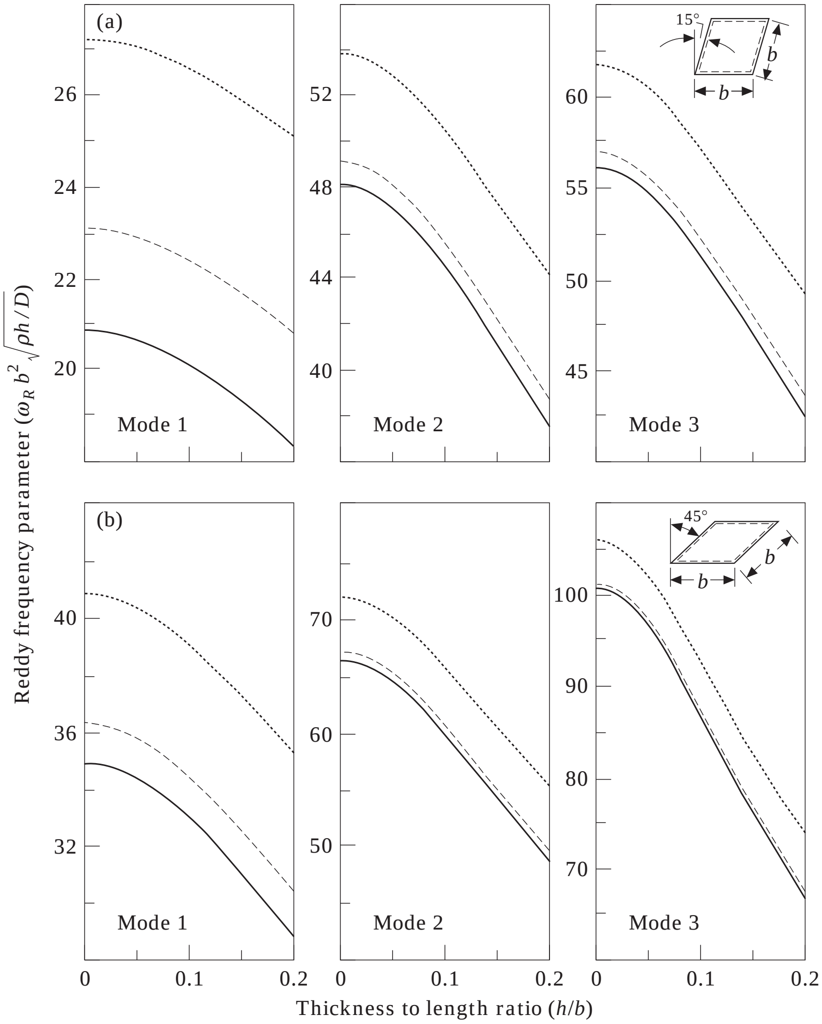

Figure 6. Variations of Reddy plate frequencies with respect to thickness-to-length ratio for rhombic plates on elastic foundation with skew angles of (a) 15° and (b) 45°. Key as Figure 4.

negative signs for the stress parameter). Again, the effect of initial stresses is more pronounced for thicker plates and higher frequency modes.

The relationship given in equation (41b) may be used to develop design charts for the vibration frequencies of simply supported, polygonal plates on Pasternak foundations. Figures 4–6 give typical design charts for square plates, right-angle isosceles triangular plate and rhombic plates, respectively. The charts furnish the first three frequencies of Reddy plates for a given thickness to length ratio.

5. CONCLUDING REMARKS

An exact relationship between the natural frequencies of Reddy plates and Kirchhoff plates has been derived. The relationship is valid for any general polygonal plates with simply supported edges. The complicating effects of initial stress and a Winkler–Pasternak foundation are also captured in the relationship. Once the Kirchhoff vibration solutions are known, the Reddy solutions may be readily calculated from this relationship. Unlike the Mindlin–Kirchhoff relationship derived earlier by Wang [10, 11], the present Reddy–Kirchhoff plate relationship does not need a shear correction factor. This feature is advantageous when considering laminated plates where the shear correction factor is not available.

The relationship may also be used as a basic form to develop approximate formulas for the vibration frequencies of other plate shapes and boundary conditions. Modification factors may be introduced into the relationship to adjust the solutions to within the desired accuracy.

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