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EXACT STATIONARY SOLUTIONS OF AVERAGED EQUATIONS OF STOCHASTICALLY AND HARMONICALLY EXCITED MDOF QUASI-LINEAR SYSTEMS WITH INTERNAL AND/OR EXTERNAL RESONANCES

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The exact stationary solutions of the averaged equations of stochastically and harmonically excited n-degree-of-freedom quasi-linear systems with m internal and/or external resonances are obtained as functions of both n independent amplitudes and m combinations of phase angles. To make the solutions more general, the equivalent stochastic systems of the averaged equations are obtained by using the differential forms and exterior differentiation. By considering the periodic boundary conditions with respect to m combinations of phase angles, the probability potentials of the exact stationary solutions of the equivalent stochastic systems are expanded into an m-fold harmonic series of m combinations of phase angles, and the exact stationary solutions are obtained for the case where the averaged equations belong to the class of stationary potential. Two examples are given to illustrate the application of the proposed procedure.

1. INTRODUCTION

The stochastic averaging method was first proposed by Stratonovich [1] and widely accepted by random vibration community after the stochastic averaging theorems were established by Khasminskii [2] and Papanicolaou and Kohler [3]. In the past three decades, the stochastic averaging method has been extensively applied to prediction of response, decision of stability and estimation of reliability of non-linear systems subject to wide band random excitations. Comprehensive reviews attesting to the success of the stochastic averaging method in random vibration have been written by Roberts and Spanos [4] and Zhu [5, 6].

The stochastic averaging method was mostly successfully applied to single-degree-of-freedom (SDOF) quasi-linear systems without resonance. In this case, the averaged Fokker–Planck–Kolmogorov (FPK) equation is one-dimensional and can always be solved to yield the stationary probability density of the response. In other cases, i.e., for multi-degree-of-freedom (MDOF) quasi-linear stochastic systems and for SDOF quasi-linear stochastic systems with external resonance, the stochastic averaging method was not so successful, since in these cases it is difficult to obtain the stationary solutions of the averaged FPK equations.

In the case of combined harmonic and random excitations, the stochastic averaging method was usually used for obtaining the stability conditions for statistical moments of stochastic systems [7–11]. Recently, the exact stationary solution to the averaged FPK

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equations of SDOF quasi-linear stochastic systems with external resonance was obtained by Lin and Cai [12, 13].

On the other hand, the stochastic averaging method was generalized recently by Zhu *et al*. [6, 14] to the case of MDOF quasi-integrable Hamiltonian systems, of which MDOF quasi-linear systems are a subclass. It has been shown that the dimension of the averaged equations depends upon the number of resonant relations in the Hamiltonian systems. For an n-DOF non-resonant Hamiltonian system, the dimension of the averaged equations is n, i.e., n equations for n action variables. In this case, the averaged equations may belong to the stationary potential and the stationary solutions may be obtained rather easily. For an n-DOF resonant Hamiltonian system with m resonant relations, the dimension of the averaged equations is $n+m$, i.e., n equations for n action variables and m equations for m combinations of angle variables. The stationary solutions of the averaged equations in this case remain to be obtained.

MDOF stochastic systems with internal and/or external resonances often occur in engineering. In the present paper, stochastically and harmonically excited MDOF quasi-linear stochastic systems with internal and/or external resonances are examined. First, the averaged equations are derived. Then the equivalent stochastic systems of the averaged equations are obtained by using differential forms and exterior differentiation. After that, the exact stationary solutions to the equivalent systems are obtained. Finally, two examples are shown to illustrate the application of the proposed procedure.

2. AVERAGED STOCHASTIC SYSTEMS

Consider an n-DOF quasi-linear stochastic system whose equations of motion are of the following form

$$\ddot{X}_i + \bar{\omega}_i^2 X_i = \varepsilon q_i(\mathbf{x}, \dot{\mathbf{x}}) + \varepsilon f_{ij}(\mathbf{x}, \dot{\mathbf{x}}) \sin(\nu_j t + \theta_j) + \varepsilon^{1/2} g_{ik}(\mathbf{x}, \dot{\mathbf{x}})\xi_k(t)$$

$$i = 1, 2, \ldots, n; \qquad j = 1, 2, \ldots, s; \qquad k = 1, 2, \ldots, r, \tag{1}$$

where X_i denote the normal co-ordinates of free vibration of the system; $\bar{\omega}_i$ are the natural frequencies; ε is a positive small parameter; $\xi_k(t)$ are weakly stationary wide band random processes with zero mean value and correlation functions $E[\xi_k(t)\xi_l(t+\tau)] = R_{kl}(\tau)$ or spectral densities $S_{kl}(\omega)$; $\varepsilon q_i(\mathbf{x}, \dot{\mathbf{x}})$ represent linear and/or non-linear damping forces; $\varepsilon f_{ij}(\mathbf{x}, \dot{\mathbf{x}})$ and $\varepsilon^{1/2} g_{ik}(\mathbf{x}, \dot{\mathbf{x}})$ are the amplitudes of harmonic and random excitations, respectively. It is assumed that the damping and harmonic excitations are of the order of ε and the random excitations are of the order of $\varepsilon^{1/2}$. Thus, their contributions to the system response are of the same order. In this paper, the Einstein convention on repeated indices is adopted.

When new parameters ω_i are introduced, where $\omega_i - \bar{\omega}_i = o(\varepsilon)$, equation (1) can be rewritten as follows

$$\ddot{X}_i + \omega_i^2 X_i = (\omega_i^2 - \bar{\omega}_i^2)X_i + \varepsilon q_i(\mathbf{x}, \dot{\mathbf{x}}) + \varepsilon f_{ij}(\mathbf{x}, \dot{\mathbf{x}}) \sin(\nu_j t + \theta_j) + \varepsilon^{1/2} g_{ik}(\mathbf{x}, \dot{\mathbf{x}})\xi_k(t)$$

$$i = 1, 2, \ldots, n; \qquad j = 1, 2, , \ldots, s; \qquad k = 1, 2, \ldots, r. \tag{2}$$

Introduce van der Pol transformations

$$X_i = A_i \cos \Phi_i, \qquad \dot{X}_i = -\omega_i A_i \sin \Phi_i, \qquad \Phi_i = \omega_i t + \varphi_i, \qquad i = 1, 2, \ldots, n, \tag{3}$$

where A_i and φ_i are the amplitude and phase of the normal co-ordinate X_i, respectively.

When equation (3) is substituted into equation (2), equation (2) becomes

$$\dot{A}_i = -(1/\omega_i)\{(\omega_i^2 - \bar{\omega}_i^2)A_i \cos \Phi_i + \varepsilon[q_i(\mathbf{A}, \Phi) + f_{ij}(\mathbf{A}, \Phi) \sin (\nu_j t + \theta_j)] + \varepsilon^{1/2} g_{ik}(\mathbf{A}, \Phi)\xi_k(t)\} \sin \Phi_i$$

$$\dot{\varphi}_i = -(1/A_i\omega_i)\{(\omega_i^2 - \bar{\omega}_i^2)A_i \cos \Phi_i + \varepsilon[q_i(\mathbf{A}, \Phi) + f_{ij}(\mathbf{A}, \Phi) \sin (\nu_j t + \theta_j)] + \varepsilon^{1/2} g_{ik}(\mathbf{A}, \Phi)\xi_k(t)\} \cos \Phi_i, \qquad i = 1, 2, \ldots, n, \tag{4}$$

where

$$q_i(\mathbf{A}, \Phi) = q_i(A_1 \cos \Phi_1, -\omega_1 A_1 \sin \Phi_1, \ldots, A_n \cos \Phi_n, -\omega_n A_n \sin \Phi_n),$$

$$f_{ij}(\mathbf{A}, \Phi) = f_{ij}(A_1 \cos \Phi_1, -\omega_1 A_1 \sin \Phi_1, \ldots, A_n \cos \Phi_n, -\omega_n A_n \sin \Phi_n),$$

$$g_{ik}(\mathbf{A}, \Phi) = g_{ik}(A_1 \cos \Phi_1, -\omega_1 A_1 \sin \Phi_1, \ldots, A_n \cos \Phi_n, -\omega_n A_n \sin \Phi_n).$$

The averaged Itô equations can be obtained by applying the stochastic and deterministic averaging to equation (4). The dimension and form of the averaged equations depend upon whether internal and/or external resonances are present in the system.

2.1. WITHOUT INTERNAL AND/OR EXTERNAL RESONANCE

In this case, the harmonic excitations can be ignored. Based on the Stratonovich–Khasiminskii limit theorem, vector $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ converges in probability to an n-dimensional diffusion Markov process as $\varepsilon \to 0$ in a time interval $0 \leqslant t \leqslant T$, where $T \sim 0(\varepsilon^{-1})$. The averaged Itô equations for $A_i (i = 1, 2, \ldots, n)$ are of the following form

$$\mathrm{d}A_i = \varepsilon m_i(\mathbf{A})\, \mathrm{d}t + \varepsilon^{1/2} s_{ik}(\mathbf{A})\, \mathrm{d}B_k(t), \qquad i = 1, 2, \ldots, n, \qquad k = 1, 2, \ldots, l, \tag{5}$$

where $B_i(t)$ are independent unit Wiener processes.

2.2. WITH INTERNAL AND/OR EXTERNAL RESONANCE

Internal and/or external resonances may occur in the system when some of the natural frequencies ω_j and harmonic excitation frequencies ν_j satisfy the resonant relationships

$$l_{ij}\omega_j + n_{ik}\nu_k = 0 \qquad \text{or} \qquad l_{ij}\bar{\omega}_j + n_{ik}\nu_k = 0(\varepsilon),$$
$$i = 1, 2, \ldots, m, \qquad j = 1, 2, \ldots, n, \qquad k = 1, 2, \ldots, s, \tag{6}$$

where l_{ij} and n_{ik} are positive or negative integers.

Internal resonances do occur in the system when some of the natural frequencies $\bar{\omega}_j$ satisfy the resonant relationships (6) with $n_{ik} = 0$ and the corresponding damping terms $\varepsilon q_i(\mathbf{x}, \dot{\mathbf{x}})$ satisfy certain conditions. For example, internal resonance occurs when $\bar{\omega}_i - \bar{\omega}_j = 0(\varepsilon)$, $q_i(\mathbf{x}, \dot{\mathbf{x}}) = ax_j + b\dot{x}_j$ and $q_j(\mathbf{x}, \dot{\mathbf{x}}) = cx_i + d\dot{x}_i (a, b, c, d$ are constants). External resonances do occur when one of the harmonic excitation frequencies is near certain linear combinations of the natural frequencies and the corresponding coefficients $f_{ij}(\mathbf{x}, \dot{\mathbf{x}})$ satisfy certain conditions. For examples, external resonance occurs when $\bar{\omega}_i - \nu_j = 0(\varepsilon)$ and $f_{ij}(\mathbf{x}, \dot{\mathbf{x}})$ is constant, or $\bar{\omega}_i - 2\nu_j = 0(\varepsilon)$ and $f_{ij}(\mathbf{x}, \dot{\mathbf{x}}) = ax_i + b\dot{x}_i$ where a and b are constants.

Assume that there are m resonant relations of the form of equation (6) and $q_i(\mathbf{x}, \dot{\mathbf{x}})$ and $f_{ij}(\mathbf{x}, \dot{\mathbf{x}})$ satisfy the corresponding conditions. So the resonances do occur in the system. Introduce m combinations of phase angles

$$\Psi_u = l_{uj}\varphi_j + n_{uk}\theta_k, \qquad u = 1, 2, \ldots m, \qquad j = 1, 2, \ldots, n, \qquad k = 1, 2, \ldots, s. \tag{7}$$

It can be shown that, in this case, A and Ψ converge to a $(n+m)$-dimensional diffusion Markov process as $\varepsilon \to 0$ in a time interval $0 \leqslant t \leqslant T$, where $T \sim 0(\varepsilon^{-1})$. The averaged Itô equations are of the following form

$$\mathrm{d}A_i = \varepsilon m_i(\mathbf{A}, \Psi)\,\mathrm{d}t + \varepsilon^{1/2} s_{ik}(\mathbf{A}, \Psi)\,\mathrm{d}B_k(t),$$
$$\mathrm{d}\Psi_u = \varepsilon m_u(\mathbf{A}, \Psi)\,\mathrm{d}t + \varepsilon^{1/2} s_{uk}(\mathbf{A}, \Psi)\,\mathrm{d}B_k(t),$$
$$i = 1, 2, \ldots, n, \qquad u = 1, 2, \ldots, m, \qquad k = 1, 2, \ldots, 2l. \tag{8}$$

The reduced averaged FPK equation governing stationary probability density $p(\mathbf{a}, \boldsymbol{\psi})$ is then given by

$$-\frac{\partial}{\partial a_i}(m_i p) - \frac{\partial}{\partial \psi_u}(m_u p) + \frac{1}{2}\frac{\partial^2}{\partial a_i \partial a_j}(b_{ij} p) + \frac{\partial^2}{\partial a_i \partial \psi_u}(b_{iu} p) + \frac{1}{2}\frac{\partial^2}{\partial \psi_u \partial \psi_v}(b_{uv} p) = 0 \tag{9}$$

where

$$b_{ij} = s_{ik} s_{jk}, \qquad b_{iu} = s_{ik} s_{uk}, \qquad b_{uv} = s_{uk} s_{vk}. \tag{10}$$

3. EQUIVALENT FPK EQUATIONS

Two reduced FPK equations are said to be equivalent if they have the same stationary solution. To obtain a more general exact stationary solution of equation (9), one first identifies a group of equivalent FPK equations which are equivalent to the system in equation (9) by using exterior differentiation [15, 16]. equation (9) can be rewritten as

$$(\partial/\partial y_l) J_l = 0 \tag{11}$$

where y_l and J_l stand for a_i and J_i for $l = 1, 2, \ldots, n$ and, for ψ_u and J_u for $l = n+1, n+2, \ldots, n+m$, and

$$J_i = m_i p - \tfrac{1}{2}(\partial/\partial a_j)(b_{ij} p) - \tfrac{1}{2}(\partial/\partial \psi_u)(b_{iu} p), \qquad J_u = m_u p - \tfrac{1}{2}(\partial/\partial a_i)(b_{iu} p) - \tfrac{1}{2}(\partial/\partial \psi_v)(b_{uv} p),$$
$$i = 1, 2, \ldots, n; \qquad u = 1, 2, \ldots m. \tag{12}$$

Equation (11) can be further rewritten as

$$\mathrm{d}\alpha = 0 \tag{13}$$

where d denotes the exterior differential operation and

$$\alpha = \sum_{l=1}^{n+m} J_l (-1)^{l-1}\,\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}\hat{y}_l \wedge \cdots \wedge \mathrm{d}y_{n+m} \tag{14}$$

is called the $(n+m-1)$ form. In equation (14), $\wedge$ denotes the Wedig product and $\mathrm{d}\hat{y}_l$ means that this term vanishes. The Poincaré lemma for differential forms is of the form [16]

$$\mathrm{d}\mathrm{d}\beta = 0, \tag{15}$$

where β is an arbitrary r form. Let β be an arbitrary $(n+m-2)$ form. Adding equation (15) to equation (13) leads to

$$\mathrm{d}(\alpha + \mathrm{d}\beta) = 0, \tag{16}$$

where

$$\beta = \sum_{i<l}\left(\sum_{l=1}^{n+m}\beta_{il}(-1)^{i+l-3}\,\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}\hat{y}_i\cdots\wedge\mathrm{d}\hat{y}_l\wedge\cdots\wedge\mathrm{d}y_{n+m}\right),$$

$$\mathrm{d}\beta = \sum_{l=1}^{n+m}\left(\sum_{i=1}^{n+m}\frac{\partial\beta_{il}^*}{\partial y_i}\right)(-1)^{l-1}\,\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}\hat{y}_l\wedge\cdots\wedge\mathrm{d}y_{n+m},\qquad \beta_{il}^* = -\beta_{li}^* = -\beta_{il}. \tag{17}$$

Let

$$\beta_{il}^* = -\gamma_{il}p/2 \tag{18}$$

where γ_{il} is an arbitrary antisymmetric functional matrix. Then one obtains a group of reduced FPK equations

$$(\partial/\partial y_l)J_l^* = 0, \tag{19}$$

which are equivalent to equation (11). In equation (19)

$$J_l^* = J_l - \tfrac{1}{2}(\partial/\partial y_i)(\gamma_{il}p),\qquad l = 1, 2, , \ldots, n+m;\qquad i = 1, 2, \ldots, n+m. \tag{20}$$

Equation (19) is solved subject to the following boundary conditions

$$\begin{aligned} &J_j^*(\mathbf{a}, \boldsymbol{\psi}) = 0,\qquad a_1 + a_2 + \cdots + a_n \to \infty,\qquad (j = 1, 2, \ldots, n),\\ &J_u^*(\mathbf{a}, \boldsymbol{\psi}_1 + k_1T_1, \ldots, \psi_m + k_mT_m) = J_u^*(\mathbf{a}, \boldsymbol{\psi}),\qquad (u = 1, \ldots, m), \end{aligned} \tag{21}$$

where T_u are periods of ψ_u and k_u are integers.

Note that the reduced FPK equations (11) and (19) are essentially the same since $\gamma_{il} = -\gamma_{li}$. Instead of reduced FPK equation (11), the equivalent FPK equation (19) is solved in the following section.

4. STATIONARY SOLUTION

Noting the non-negativity of probability density p and the boundary conditions in equation (21), the solution to reduced FPK equation (19) is assumed to be of the form

$$p = C\,\mathrm{e}^{-\lambda(a_1,a_2,\ldots,a_n,\psi_1,\psi_2,\ldots,\psi_m)}, \tag{22}$$

where C is a normalization constant and λ is the probability potential. Assume that m_i, m_u, b_{ij}, b_{iu}, b_{uv}, γ_{il} in equation (19) and λ can be expanded into an m-fold Fourier series with respect to ψ_u. A representative of these expansions is of the form

$$f = f_0(a_1, \ldots, a_n) + \sum_{r=1}^{\infty}\sum_{|p|=r}[f_P(a_1, \ldots, a_n)\cos(P, \varphi) + \bar{f}_P(a_1, \ldots, a_n)\sin(P, \varphi)], \tag{23}$$

where f stands for m_i, m_u, b_{ij}, b_{iu}, b_{uv}, γ_{il} and λ. $P = (p_1, p_2, \ldots, p_m)$, $p_j, j = 1, m$ are positive or negative integers, $|P| = \Sigma_{j=1}^m|p_j|$ and $(P, \varphi) = \Sigma_{j=1}^m p_j\varphi_j$.

Inserting the Fourier expansions m_i, m_u, b_{ij}, b_{iu}, b_{uv}, γ_{il} and λ of the form of equation (23) into equation (19), one obtains equations for the coefficients λ_p and $\bar{\lambda}_p$ of the Fourier expansions of probability potential λ. In the special case of $b_{ij} = b_{ij0}$, $b_{iu} = b_{iu0}$, $b_{uv} = b_{uv0}$ and $\gamma_{il} = \gamma_{il0}$, one obtains the equations of the form:

$$(b_{ij0} + \gamma_{ij0})\frac{\partial \lambda_0}{\partial a_j} = \frac{\partial (b_{ij0} + \gamma_{ij0})}{\partial a_j} - 2m_{i0}, \qquad (b_{ij0} + \gamma_{ij0})\frac{\partial \lambda_P}{\partial a_j} + p_u \overline{\lambda}_P (b_{iu0} + \gamma_{iu0}) = -2m_P,$$

$$(b_{ij0} + \gamma_{ij0})\frac{\partial \overline{\lambda}_P}{\partial a_j} - p_u \lambda_P (b_{iu0} + \gamma_{iu0}) = -2\bar{m}_P, \qquad \sum_{u=1}^{m}\left(\frac{\partial a_u^I}{\partial \psi_u} - a_u^I \frac{\partial \lambda}{\partial \psi_u}\right) = 0,$$

$$i = 1, 2, \ldots, n; \qquad u = 1, 2, \ldots, m; \qquad |P| = 1, 2, \ldots, \infty, \tag{24}$$

where

$$a_u^I = m_u + \tfrac{1}{2}(b_{iu0} + \gamma_{iu0})\frac{\partial \lambda}{\partial a_i} + \tfrac{1}{2}(b_{vu0} + \gamma_{vu0})\frac{\partial \lambda}{\partial \psi_v} - \tfrac{1}{2}\frac{\partial (b_{iu0} + \gamma_{iu0})}{\partial a_i} \tag{25}$$

If compatible λ_0, λ_P and $\overline{\lambda}_P$ can be obtained from equation (24), the stationary solution of the averaged system governed by equation (19) is obtained by substituting $\lambda(a_1, \ldots, a_n, \psi_1, \ldots, \psi_m)$ into equation (22).

5. EXAMPLES

5.1. EXAMPLE 1

As the first example of application, consider a linear oscillator with harmonic excitation of the stiffness parameter and external excitation of the wide band stationary process. The equation of motion is

$$\ddot{X} + 2\zeta\bar{\omega}\dot{X} + \bar{\omega}^2[1 + f\sin(2vt)]X = \xi(t), \tag{26}$$

where $\xi(t)$ is a wide band stationary random process with spectral density $S(\omega)$. ζ, f are of order ε and $\xi(t)$ is of order $\varepsilon^{1/2}$. When $\bar{\omega} - v = o(\varepsilon)$, external resonance occurs in the system. Equation (26) can be rewritten as follows

$$\ddot{X} + v^2 X = -2\zeta\bar{\omega}\dot{X} + (v^2 - \bar{\omega}^2)X - f\bar{\omega}^2\sin(2vt)X + \xi(t). \tag{27}$$

Let $X = A\cos\Phi$, $\dot{X} = -vA\sin\Phi$, $\Phi = vt + \varphi$. Following the procedure in section 2, one obtains the following set of averaged Itô equations for A and ψ by using stochastic and deterministic averaging.

$$\mathrm{d}A = [-\zeta\bar{\omega}A + \zeta\bar{\omega}\rho A\cos(2\psi) + \pi K/A]\,\mathrm{d}t + \sqrt{2\pi K}\,\mathrm{d}B_1(t),$$
$$\mathrm{d}\psi = [-\sigma - \zeta\bar{\omega}\rho\sin(2\psi)]\,\mathrm{d}t + (\sqrt{2\pi K}/A)\,\mathrm{d}B_2(t), \tag{28}$$

where $\psi = \varphi$, $\sigma = (\upsilon^2 - \bar{\omega}^2)/2\upsilon$, $\rho = \bar{\omega}f/4\zeta\upsilon$, $K = S(v)/2\upsilon^2$ and $B_i(t)$ are independent unit Wiener processes. Then following the procedure in section 4, λ and γ_{il} are assumed to be of the form

$$\lambda(a, \psi) = \lambda_0(a) + \lambda_2(a)\cos(2\psi) + \overline{\lambda}_2(a)\sin(2\psi),$$
$$\lambda_{12}(a, \psi) = -\gamma_{21}(a, \psi) = d/a, \tag{29}$$

where d is an arbitrary constant. Substituting equation (29) into equation (24), one obtains $d = \pi K\sigma/\zeta\bar{\omega}$ and the joint probability density of the amplitude a and the phase ψ

$$p(a, \psi) = (\zeta\bar{\omega}\sqrt{1 - \mu^2}/2\pi^2 K)a\exp\{-(\zeta\bar{\omega}a^2/2\pi K)[1 - \mu\cos(2\psi - 2\psi_0)]\} \tag{30}$$

where

$$\cos(2\psi_0) = 1/\sqrt{1+\eta^2}, \qquad \sin(2\psi_0) = \eta/\sqrt{1+\eta^2}, \qquad \mu = \rho/\sqrt{1+\eta^2}, \qquad \eta = \sigma/\zeta\bar{\omega}.$$

Exact stationary solution (30) is the same as that given by Lin and Cai [13].

When the damping in equation (26) is a non-linear function of X, $\dot{X}$, the exact stationary solution of the stochastically averaged equation can be obtained only for very special case in which the frequency of the parametrically harmonic excitation is exactly tuned to the natural frequency of the system.

5.2. EXAMPLE 2

As the second example of application, consider a stochastically excited system of two coupled linear oscillators and two van der Pol oscillators. This system has been studied by Hall and Iwan [17] in the case when stochastic excitation is absent. The equations of motion are

$$\ddot{X}_1 + \bar{\omega}_1^2 X_1 = \alpha_1\dot{X}_3 + \beta_1 X_3 - \mu_1\dot{X}_1 + \xi_1(t), \qquad \ddot{X}_2 + \bar{\omega}_2^2 X_2 = \alpha_2\dot{X}_4 + \beta_2 X_4 - \mu_2\dot{X}_2 + \xi_2(t),$$
$$\ddot{X}_3 + \bar{\omega}_3^2 X_3 = \chi_1\dot{X}_1 + \delta_1 X_1 - (p_1\dot{X}_3^2 + q_1\dot{X}_4^2 - \eta_1)\dot{X}_3 + \xi_3(t),$$
$$\ddot{X}_4 + \bar{\omega}_4^2 X_4 = \chi_2\dot{X}_2 + \delta_2 X_2 - (p_2\dot{X}_4^2 + q_2\dot{X}_3^2 - \eta_2)\dot{X}_4 + \xi_4(t) \tag{31}$$

where the first two equations represent the response of two adjacent modes of structure and the last two equations represent the influence of the shed vortices coupled to the two modes. α_i, β_i, χ_i, δ_i, μ_i, η_i, p_i, q_i $(i = 1, 2)$ are small constants of the same order of ε and $\bar{\omega}_3 = 1$, $\bar{\omega}_4 = 1$. $\xi_i(t)$ are weakly independent stationary wide band random processes with zero mean value and spectral densities $S_i(\omega)$, which are of the same order of ε. The internal resonances do occur in the system when $\bar{\omega}_1 - 1 = o(\varepsilon)$, $\bar{\omega}_2 - 1 = o(\varepsilon)$. Following the procedure in section 2, the following set of averaged equations for A_i and ψ_i were obtained by using stochastic and deterministic averaging.

$$\mathrm{d}A_i = \tfrac{1}{2}(-\mu_i A_i + \alpha_i A_{2+i}\cos\psi_i - \beta_i A_{2+i}\sin\psi_i + \pi K_i/A_i)\,\mathrm{d}t + \sqrt{2\pi K_i}\,\mathrm{d}B_i(t),$$
$$\mathrm{d}A_{2+i} = \tfrac{1}{2}\{-\eta_i A_{2+i} + \chi_i A_i\cos\psi_i + \delta_i A_i\sin\psi_i - (3p_i^2/4)A_{2+i}^3$$
$$- (q_i/4)[2 + \cos(2\psi_3)]A_3^2A_4^2/A_{2+i} + \pi K_{2+i}/A_{2+i}\}\,\mathrm{d}t + \sqrt{2\pi K_{2+i}}\,\mathrm{d}B_{2+i}(t),$$
$$\mathrm{d}\psi_i = \tfrac{1}{2}[-\sigma_i - (\alpha_i A_{2+i}/A_i + \chi_i A_i/A_{2+i})\sin\psi_i - (\beta_i A_{2+i}/A_i - \delta_i A_i/A_{2+i})\sin\psi_i$$
$$+ (-1)^{i+1}(q_i/4)A_{5-i}^2\sin(2\psi_3)]\,\mathrm{d}t + (\sqrt{2\pi K_i}/A_i)\,\mathrm{d}B_{4+i}(t)$$
$$- \sqrt{2\pi K_{2+i}}/A_{2+i}\,\mathrm{d}B_{6+i}(t),$$
$$\mathrm{d}\psi_3 = \tfrac{1}{2}[(A_1/A_3)(-\chi_1\sin\psi_1 + \delta_1\cos\psi_1) + (A_2/A_4)(-\chi_2\sin\psi_2 + \delta_2\cos\psi_2)$$
$$+ \tfrac{1}{4}(q_1A_4^2 + q_2A_3^2)\cos(2\psi_3)] + (\sqrt{2\pi K_4}/A_4)\,\mathrm{d}B_8(t)$$
$$- (\sqrt{2\pi K_3}/A_3)\,\mathrm{d}B_7(t), \qquad i = 1, 2, \tag{32}$$

where

$$K_i = S_i(1)/2, \qquad (i = 1, 4); \qquad \sigma_i = 1 - \bar{\omega}_i^2\,(i = 1, 2);$$
$$\psi_i = \varphi_i - \varphi_{2+i}, \qquad (i = 1, 2); \qquad \psi_3 = \varphi_4 - \varphi_3.$$

Then following the procedure in section 4, one may assume that λ, γ_{il} are of the following form

$$\lambda = \lambda_0(a_1, \ldots, a_4) + \sum_{k=1}^{2} [\lambda_{1k}(a_1, \ldots, a_4) \cos \psi_k + \overline{\lambda}_{1k}(a_1, \ldots, a_4) \sin \psi_k]$$

$$+ \lambda_{23}(a_1, \ldots, a_4) \cos (2\psi_3) + \overline{\lambda}_{23}(a_1, \ldots, a_4) \sin (2\psi_3),$$

$$\gamma_{15} = -\gamma_{51} = d_1/a_1, \qquad \gamma_{26} = -\gamma_{62} = d_2/a_2, \tag{33}$$

where d_1 and d_2 are arbitrary constants. Substituting equation (33) into equation (24), one obtains $d_i = 2\pi K_i \sigma_i/\mu_i$ $(i = 1, 2)$. If the parameters satisfy the following conditions

$$\frac{q_1}{K_3} = \frac{q_2}{K_4}, \qquad \frac{\chi_i}{\pi K_{2+i}} = \frac{\pi K_i \alpha_i + d_i \beta_i}{\pi^2 k_i^2 + d_i^2}, \qquad \frac{-\delta_i}{\pi K_{2+i}} = \frac{\pi K_i \beta_i + d_i \alpha_i}{\pi^2 k_i^2 + d_i^2}, \qquad (i = 1, 2) \tag{34}$$

the joint stationary probability density of the action variables $I_i (I_i = \frac{1}{2} a_i^2)$ and ψ_u is

$$p(I_1, I_2, I_3, I_4, \psi_1, \psi_2, \psi_3) = C \exp\{-\zeta_1 I_1 - \zeta_2 I_2 + \zeta_3 I_3 + \zeta_4 I_4 - \zeta_7 I_3^2 - \zeta_8 I_4^2$$
$$+ \zeta_5 \sqrt{I_1 I_3} \cos (\psi_1 + \psi_{10}) + \zeta_6 \sqrt{I_2 I_4} \cos (\psi_2 + \psi_{20})$$
$$- 2\zeta_9 I_3 I_4 [2 + \cos (2\psi_3)]\}, \tag{35}$$

where C is a normalization constant, and

$$\zeta_i = \mu_i/2\pi K_i, \qquad \zeta_{2+i} = \eta_i/2\pi K_{2+i}, \qquad \zeta_{4+i} = \sqrt{\chi_i^2 + \delta_i^2}/2\pi K_{2+i},$$
$$\zeta_{6+i} = 3p_i/8\pi K_{2+i}, \qquad \zeta_9 = q_1/8\pi K_3,$$
$$\cos \psi_{i0} = \chi_i/\sqrt{\chi_i^2 + \delta_i^2}, \qquad \sin \psi_{i0} = -\delta_i/\sqrt{\chi_i^2 + \delta_i^2}, \qquad (i = 1, 2).$$

At the extreme points of the probability density (35), one has

$$4\zeta_1^2 I_1^0 = \zeta_5^2 I_3^0, \qquad 4\zeta_2^2 I_2^0 = \zeta_6^2 I_4^0, \qquad \psi_1^0 = -\psi_{10}, \qquad \psi_2^0 = -\psi_{20}, \qquad \psi_3^0 = \pi/2 \tag{36}$$

The extreme points of the probability density (35) coincide with that of the stable stationary solutions of the deterministic system [17]. It means that the probability density (35) describes the diffusion of the stable stationary solutions of the deterministic system.

The property of probability density (35) is mainly determined by the coefficients of the second order terms of I_i in the exponent function. The second invariant of the second order terms is of the form

$$J_2 = (1/64\pi^2 K_3 K_4)\{9p_1 p_2 - q_1 q_2 [2 + \cos (2\psi_3)]^2\} \tag{37}$$

$J_2 > 0$, $= 0$ and < 0 correspond to whether the second order terms are elliptic, parabolic and hyperbolic function, respectively. $J_2 = 0$ is associated with the bifurcation point in the sense of probability and the bifurcation point is the same as that of the deterministic bifurcation [17]. To demonstrate the difference in probability densities between the cases of $J_2 > 0$ and $J_2 < 0$, the marginal probability density of I_3 and I_4 are shown in Figure 1 ($J_2 > 0$) and Figure 2 ($J_2 < 0$) where Figure 1(a) and Figure 2(a) represent the analytical

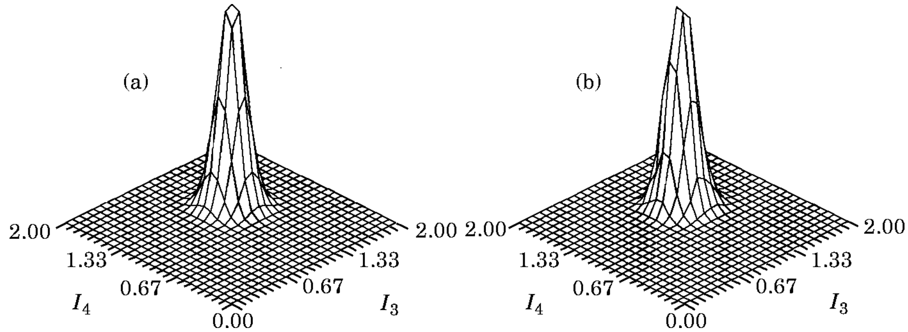

Figure 1. Probability density $p(I_3, I_4)$ in example 2. $\bar{\omega}_i = 1{\cdot}0$; $\pi k_i = 0{\cdot}01$, $(i = 1, 2, 3, 4)$; $\alpha_i = \beta_i = \chi_i = -\delta_i = 0{\cdot}04$; $\mu_i = 0{\cdot}06$; $\eta_i = p_i = 0{\cdot}05$; $q_i = 0{\cdot}02$, $(i = 1, 2)$. (a) Analytical solution; (b) digital simulation.

results while Figure 1(b) and Figure 2(b) are those from digital simulation. The analytical solution of the probability density $p(I_3, I_4)$ is obtained from equation (35) as follows

$$p(I_3, I_4) = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} p(I_1, I_2, I_3, I_4, \psi_1, \psi_2, \psi_3)\, \mathrm{d}\psi_1\, \mathrm{d}\psi_2\, \mathrm{d}\psi_3\, \mathrm{d}I_1\, \mathrm{d}I_2 \quad (38)$$

It is seen from Figures 1 and 2 that the analytical results agree well with those from digital simulation. For the case of $J_2 > 0$, the marginal probability density $p(I_3, I_4)$ has only one peak where the action variables I_3 and I_4 are not small. Taking account of the conditions (36) at the extreme point, one can imagine that the stationary probability density (35) also has one peak where the action variables I_1, I_2, I_3, I_4 are not small, it means that the response of the structure is the combination of the two modes. For the case of $J_2 < 0$, the marginal probability density $p(I_3, I_4)$ consists of two peaks where one of the action variables I_3 and I_4 is small. Taking account of the conditions (36) at the extreme point, one can imagine that the stationary probability density (35) also has two peaks where the action variables I_1 and I_3 or I_2 and I_4 are small. It means that the response of the structure is in one of the two modes.

6. CONCLUDING REMARKS

For stochastically and harmonically excited MDOF quasi-linear systems with internal and/or external resonances, the exact stationary solutions of the averaged equations have been obtained as functions of both n independent amplitudes and m combinations of phase

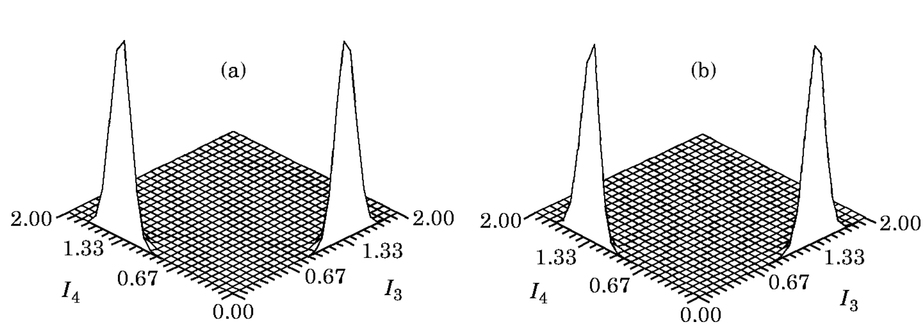

Figure 2. Probability density $p(I_3, I_4)$ in example 2. The parameters are the same as in Figure 1 except $q_i = 0{\cdot}3$, $(i = 1, 2)$. (a) Analytical solution; (b) digital simulation.

angles. The probability potentials of the exact stationary solutions are expanded into a m-fold harmonic series of m combinations of phase angles because of the periodic boundary conditions with respect to m combinations of phase angles. To make the solution more general, the equivalent stochastic systems of the averaged equations have been obtained by using the differential forms and exterior differentiation. For the special case in which the averaged equations belong to the class of stationary potential and the second moments of the averaged equations are functions of n independent amplitudes, the exact stationary solutions have been obtained for two examples with external and internal resonances, respectively, and the analytical results agree well with those from digital simulation. In the general case, although it is very difficult to obtain the exact stationary solutions of the averaged equations, the approximate stationary solutions may be obtained for a set of residual harmonic series.

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