Journal of Sound and Vibration (1997) **204**(2), 259–269

VIBRATION FREQUENCIES OF A CONSTRAINED FLEXIBLE ARM CARRYING AN END MASS

E. H. K. Fung and Z. X. Shi†

Department of Mechanical Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

(*Received* 2 *July* 1996, *and in final form* 24 *January* 1997)

The natural vibration frequencies of a constrained arm carrying an end mass are studied in this paper. A clamped–free Euler–Bernoulli beam is used to model the arm. An axial compressive force which is derived from the contact force between the tip of the arm and the constrained curve is applied at the free end. Hamilton's principle is used to derive the governing equation and the boundary conditions of the beam. By defining a new variable, the non-homogeneous boundary condition is transformed into a homogeneous one. An exact characteristic equation is derived giving the relationship between the non-dimensional frequency and the two non-dimensional parameters, i.e., the axially compressed force and the end mass. Frequencies are obtained for the first three modes by solving numerically the transcendental equation. Results are presented for the frequencies of this beam under different force and mass conditions. The natural frequency characteristics of the constrained clamped–free beam carrying an end mass are extremely important in predicting and understanding the dynamic behavior of the flexible arm.

1. INTRODUCTION

It is well known that light-weight elastic robot arms are of extreme importance for many industrial applications. The reduction of the component weight allows the actuators to move faster and carry heavier loads with longer links. The study of dynamics and control of this flexible robot generally begins with the modelling of the robot link by an elastic beam [1–7]. When this type of flexible robots perform contact tasks with the environment such as polishing and grinding, the constrained surface exists and the end of the last link of the robot experiences a force resulting from the contact with the surface. The problem of the position and force control of the multi-link flexible manipulator under constrained motions has received considerable attention in the past decade [8–12]. In the modelling of the elastic beam, approximations for its transverse vibrations were proposed. Hu and Ulsoy [10] used a weighted sum of the mode shapes for the clamped–hinged and the clamped–free cases, but ignored the end mass while Matsuno and Yamamoto [11] chose the B-spline function for the spatial variable. Matsuno *et al*. in another paper [12] studied the control of a constrained one-link flexible arm along a one-dimensional trajectory. An eigenfunction expansion method was used to approximate the vibration. The axial force was assumed to be zero in the calculation of mode shape functions. Although exact frequency equations for the lateral vibrations of beams with different boundary conditions

† On leave from Department of Electronic Engineering, Jilin University of Technology, Jilin, People's Republic of China.

0022–460X/97/270259 + 11 $25.00/0/sv970950

have been derived in many references [13–23], not many of them considered the vibrations under an axial load, which is of direct relevance not only to the above-mentioned constrained flexible robot, but also to other areas such as fuel pins in nuclear reactors, beam-like structural components in space crafts, cold water pipes of OTEC (Ocean Thermal Energy Conversion) plants and many others. Among them, Bokaian presented the frequencies and mode shapes for the Euler–Bernoulli beam with different boundary conditions subjected to constant axial compressive [19] and tensile [21] forces. Exact characteristic equations were also given for the relationship between the non-dimensional frequency and the non-dimensional axial force parameter. Liu and Ertekin in a recent paper [23] presented the results of a free–free beam under both tensile and compressive load, and found the fundamental frequency for the tensile case. However, the above analyses were performed on the axial compressive or tensile beams without the end mass. There is a need to consider, in addition to the constrained force effect, the end mass effect in the derivation of the frequency characteristic equation as the payload including the wrist or tool is usually present in many robotic applications.

The present study aims to determine the natural vibration frequencies of a constrained arm. A clamped–free Euler–Bernoulli beam with an end mass is used to model the arm. An axial compressive force resulting from the normal force between the tip of the arm and the constrained curve is applied to the beam. Hamilton's principle is used to derive the governing equation of the beam along with the boundary conditions. By defining a new variable, the non-homogeneous boundary condition is transformed into a homogeneous one. An exact characteristic equation is derived giving the relationship between the non-dimensional frequency and the non-dimensional force and mass parameters. Frequencies are obtained for the first three modes by solving numerically the transcendental equation.

2. THEORY

Consider a single-link flexible arm which is modelled by an Euler–Bernoulli beam as shown in Figure 1. A detailed derivation of the dynamic equation is given in Appendix A. The equation of vibration of the flexible beam is

$$\rho(r\ddot{\theta} - \ddot{w} + \dot{\theta}^2 w) - EIw'''' - Qw'' = 0. \tag{1}$$

The boundary conditions at the constrained end is

$$w''(t, L) = 0 \tag{2}$$

and

$$EI[(M/\rho)w_E'''' + w_E'''] + Qw_E' = -\lambda\partial\Phi/\partial w_E. \tag{3}$$

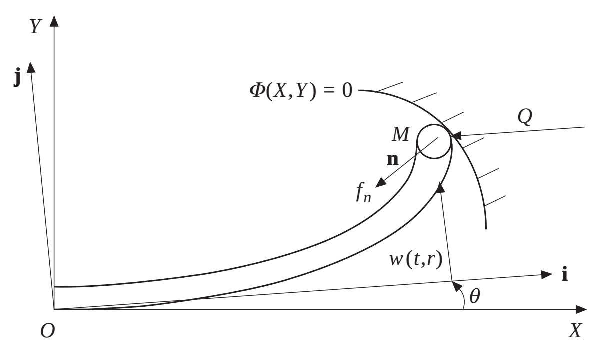

Figure 1. An axially compressed beam carrying an end mass.

TABLE 1

Calculated values of ε_{1L}, δ_{1L}, Ω_1 under different end masses and axial forces

$N = M/\rho L_2$	$U/U_{CR} = Q/Q_{CR}$	U	ε_{1L}	δ_{1L}	Ω_1
0	0	0	1·875	1·875	3·516
	0·25	0·617	1·668	1·843	3·074
	0·50	1·234	1·411	1·796	2·535
	0·75	1·851	1·053	1·720	1·811
	0·999	2·465	0·074	1·572	0·116
1	0	0	1·248	1·248	1·557
	0·25	0·617	1·039	1·302	1·352
	0·50	1·234	0·807	1·373	1·107
	0·75	1·851	0·477	1·482	0·707
	0·999	2·465	0·032	1·570	0·050
2	0	0	1·076	1·076	1·158
	0·25	0·617	0·862	1·166	1·005
	0·50	1·234	0·641	1·283	0·823
	0·75	1·851	0·410	1·421	0·583
	0·999	2·465	0·024	1·570	0·037
3	0	0	0·981	0·981	0·963
	0·25	0·617	0·763	1·095	0·836
	0·50	1·234	0·551	1·240	0·684
	0·75	1·851	0·345	1·404	0·485
	0·999	2·465	0·020	1·570	0·031
4	0	0	0·917	0·917	0·842
	0·25	0·617	0·696	1·049	0·730
	0·50	1·234	0·492	1·215	0·597
	0·75	1·851	0·304	1·394	0·423
	0·999	2·465	0·017	1·570	0·027

Equations (2) and (3) can also be obtained by the dynamic balance at the constrained end. At the clamped end of the flexible beam, the boundary conditions require the displacement and the slope to be zero, that is:

$$w(t, 0) = 0, \qquad w'(t, 0) = 0. \tag{4, 5}$$

Since the boundary condition given by equation (3) is non-homogeneous, it is difficult to treat it directly. This non-homogeneous boundary condition is therefore transformed into a homogeneous one by defining a new variable $v(t, r)$ [12] such that

$$v(t, r) = w(t, r) + g(t) f(r), \tag{6}$$

where

$$g(t) = \lambda \partial \Phi / \partial w_E, \qquad f(r) = (1/EI)[(L/2)r^2 - \tfrac{5}{6}r^3 + (1/2L)r^4 - (1/10L^2)r^5], \tag{7, 8}$$

$$w_E = w(t, L) \quad \text{and} \quad \lambda \text{ is a Lagrange multiplier.}$$

Substituting equation (6) into equations (2)–(5), the boundary conditions become

$$v(t, 0) = 0, \qquad v'(t, 0) = 0, \qquad v''(t, L) = 0, \qquad EI[(M/\rho)v_E'''' + v_E'''] + Qv_E' = 0, \tag{9–12}$$

where $v_E = v(t, L)$.

The terms containing derivatives of θ, i.e., $\dot{\theta}$, $\ddot{\theta}$ can be taken as zero in the determination of the natural frequencies of the beam. Based on these, and substituting equation (6) into equation (1), one obtains:

$$EIv'''' + \rho\ddot{v} + Qv'' = 0 \tag{13}$$

The method of assumed modes is applied, so the variable v is expressed as

$$v(t, r) = \sum_{i=1}^{\infty} Y_i(r)q_i(t), \tag{14}$$

hence

$$(EIY_i'''' + QY_i'')/\rho Y_i = -(\ddot{q}_i/q_i). \tag{15}$$

Assuming harmonic vibration, the equation for the generalized co-ordinate is

$$\ddot{q}_i = -\omega_i^2 q_i. \tag{16}$$

Using equations (15) and (16), the following equation for the mode shape function can be obtained:

$$(EIY_i'''' + QY_i'')/\rho Y_i = \omega_i^2 \quad \text{or} \quad EIY_i'''' + QY_i'' - \rho Y_i \omega_i^2 = 0. \tag{17}$$

The solution of equation (17) may be written as

$$Y_i(r) = C_{i1}\cosh \varepsilon_i r + C_{i2}\sinh \varepsilon_i r + C_{i3}\cosh \delta_i r + C_{i4}\sinh \delta_i r, \tag{18}$$

TABLE 2

Calculated values of ε_{2L}, δ_{2L}, Ω_2 under different end masses and axial forces

$N = M/\rho L_2$	$U/U_{CR} = Q/Q_{CR}$	U	ε_{2L}	δ_{2L}	Ω_2
0	0	0	4·694	4·694	22·035
	0·25	0·617	4·612	4·678	21·575
	0·50	1·234	4·527	4·662	21·105
	0·75	1·851	4·441	4·644	20·623
	0·999	2·465	4·352	4·626	20·131
1	0	0	4·031	4·031	16·250
	0·25	0·617	3·961	4·038	15·991
	0·50	1·234	3·889	4·044	15·727
	0·75	1·851	3·816	4·051	15·459
	0·999	2·465	3·742	4·058	15·187
2	0	0	3·983	3·983	15·861
	0·25	0·617	3·913	3·991	15·615
	0·50	1·234	3·842	3·999	15·364
	0·75	1·851	3·770	4·008	15·109
	0·999	2·465	3·697	4·017	14·851
3	0	0	3·965	3·965	15·720
	0·25	0·617	3·895	3·974	15·478
	0·50	1·234	3·825	3·983	15·232
	0·75	1·851	3·753	3·992	14·982
	0·999	2·465	3·681	4·002	14·729
4	0	0	3·956	3·956	15·647
	0·25	0·617	3·886	3·965	15·408
	0·50	1·234	3·816	3·974	15·164
	0·75	1·851	3·744	3·984	14·917
	0·999	2·465	3·672	3·994	14·666

TABLE 3

Calculated values of ϵ_{3L}, δ_{3L}, Ω_3 under different end masses and axial forces

$N = M/\rho L_2$	$U/U_{CR} = Q/Q_{CR}$	U	ϵ_{3L}	δ_{3L}	Ω_3
0	0	0	7·855	7·855	61·697
	0·25	0·617	7·810	7·850	61·310
	0·50	1·234	7·766	7·845	60·919
	0·75	1·851	7·721	7·840	60·527
	0·999	2·465	7·676	7·834	60·133
1	0	0	7·134	7·134	50·896
	0·25	0·617	7·093	7·137	50·623
	0·50	1·234	7·052	7·139	50·349
	0·75	1·851	7·012	7·142	50·074
	0·999	2·465	6·970	7·145	49·798
2	0	0	7·103	7·103	50·448
	0·25	0·617	7·062	7·106	50·179
	0·50	1·234	7·021	7·108	49·909
	0·75	1·851	6·980	7·111	49·636
	0·999	2·465	6·939	7·114	49·364
	0	0	7·092	7·092	50·291
	0·25	0·617	7·051	7·095	50·023
	0·50	1·234	7·010	7·098	49·754
	0·75	1·851	6·969	7·101	49·483
	0·999	2·465	6·928	7·104	49·212
4	0	0	7·086	7·086	50·211
	0·25	0·617	7·045	7·089	49·944
	0·50	1·234	7·004	7·092	49·675
	0·75	1·851	6·963	7·095	49·405
	0·999	2·465	6·922	7·098	49·134

where

$$\varepsilon_{iL} = \varepsilon_i L = \sqrt{-U/2 + \sqrt{U^2/4 + \Omega_i^2}}, \qquad \delta_{iL} = \delta_i L = \sqrt{U/2 + \sqrt{U^2/4 + \Omega_i^2}},$$
$$U = QL^2/EI, \qquad \Omega_i = \omega_i L^2/\sqrt{EI/\rho} = \varepsilon_{iL}\delta_{iL} \tag{19–22}$$

and C_{i1}, C_{i2}, C_{i3}, and C_{i4}, are four constants to be determined from the following boundary conditions which are derived from equations (9)–(12) and equation (14).

$$Y_i(0) = 0, \qquad Y_i'(0) = 0, \qquad Y_i''(0) = 0, \qquad EIY_i'''(L) + QY_i'(L) + (M/\rho)EIY_i''''(L) = 0. \tag{23–26}$$

From equation (23),

$$C_{i1} = -C_{i3} \tag{27}$$

From equation (24),

$$\varepsilon_i C_{i2} = -\delta_i C_{i4} \tag{28}$$

From equation (25) and introducing the mode shape coefficient σ_i,

$$\sigma_i = -C_{i2}/C_{i1} = (\varepsilon_i^2 \cosh \varepsilon_i L + \delta_i^2 \cos \delta_i L)/(\varepsilon_i^2 \sinh \varepsilon_i L + \varepsilon_i \delta_i \sin \delta_i L) \tag{29}$$

From equation (26),

$$C_{i2}/C_{i1} = -((M/\rho)(EI[\varepsilon_i^4 \cosh \varepsilon_i L - \delta_i^4 \cos \delta_i L] + EI[\varepsilon_i^3 \sinh \varepsilon_i L - \delta_i^3 \sin \delta_i L] + Q[\varepsilon_i \sinh \varepsilon_i L + \delta_i \sin \delta_i L])/((M/\rho)EI[\varepsilon_i^4 \sinh \varepsilon_i L - \varepsilon_i \delta_i^3 \sin \delta_i L] + \mathrm{EI}[\varepsilon_i^3 \cosh \varepsilon_i L + \varepsilon_i \delta_i^2 \cos \delta_i L] + Q[\varepsilon_i \cosh \varepsilon_i L - \varepsilon_i \cos \delta_i L]) \quad (30)$$

Combining equations (29) and (30) and using the following non-dimensional parameters, $N = M/\rho L$, $U = Q/(EI/L^2)$, $\varepsilon_{iL} = \varepsilon_i L$, $\delta_{iL} = \delta_i L$, the following equation is obtained.

$$(N[\varepsilon_{iL}^4 \cosh \varepsilon_{iL} - \delta_{iL}^4 \cos \delta_{iL}] + [\varepsilon_{iL}^3 \sinh \varepsilon_{iL} - \delta_{iL}^3 \sin \delta_{iL}] + U[\varepsilon_{iL} \sinh \varepsilon_{iL} + \delta_{iL} \sin \delta_{iL}])/(N[\varepsilon_{iL}^4 \sinh \varepsilon_{iL} - \varepsilon_{iL}\delta_{iL}^3 \sin \delta_{iL}] + [\varepsilon_{iL}^3 \cosh \varepsilon_{iL} + \varepsilon_{iL}\delta_{iL}^2 \cos \delta_{iL}] + U[\varepsilon_{iL} \cosh \varepsilon_{iL} - \varepsilon_{iL} \cos \delta_{iL}]) - (\varepsilon_{iL}^2 \cosh \varepsilon_{iL} + \delta_{iL}^2 \cos \delta_{iL})/(\varepsilon_{iL}^2 \sinh \varepsilon_{iL} + \varepsilon_{iL}\delta_{iL} \sin \delta_{iL}) = 0 \quad (31)$$

Simplification of equation (31) leads to the following generalized frequency equation.

$$\Omega_i^2 + (0{\cdot}5U^2 + \Omega_i^2) \cosh \varepsilon_{iL} \cos \delta_{iL} - 0{\cdot}5U\Omega_i \sinh \varepsilon_{iL} \sin \delta_{iL} + N\Omega_i \sqrt{0{\cdot}25U^2 + \Omega_i^2}(\varepsilon_{iL} \sinh \varepsilon_{iL} \cos \delta_{iL} - \delta_{iL} \cosh \varepsilon_{iL} \sin \delta_{iL}) = 0 \quad (32)$$

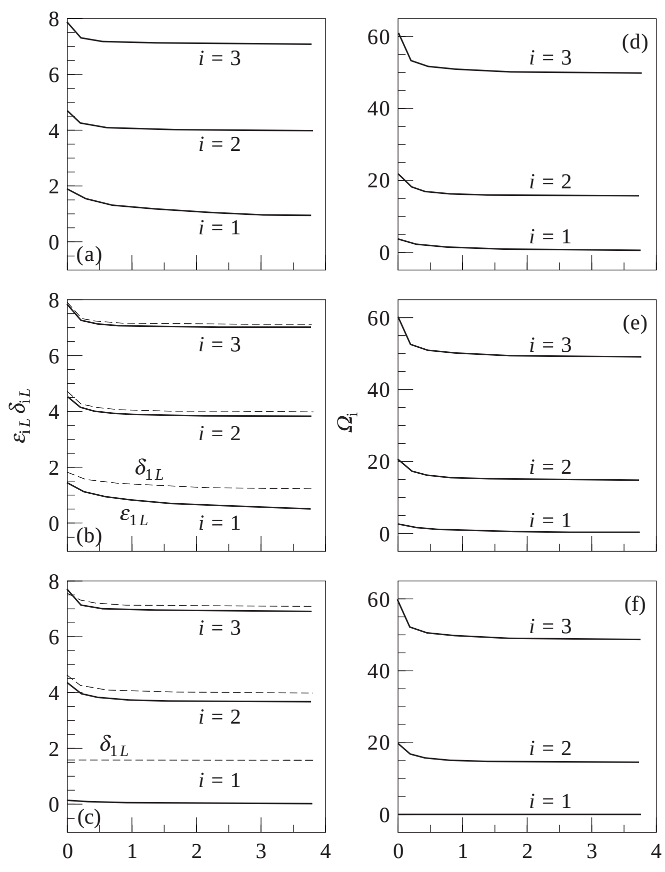

Figure 2. Non-dimensional mode shape parameters ε_{iL}, δ_{iL} and frequencies Ω_i as functions of end mass N. Values of Q: (a) 0; (b) 0·5 Q_{CR}; (c) 0·999 Q_{CR}; (d) 0; (e) 0·5 Q_{CR}; (f) 0·999 Q_{CR}.

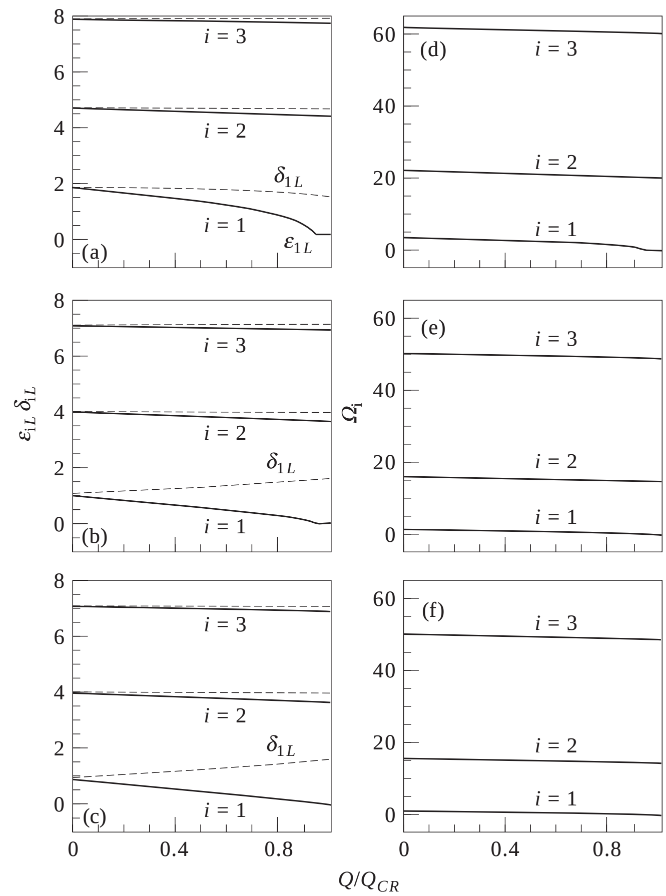

Figure 3. Non-dimensional mode shape parameters ε_{iL}, δ_{iL} and frequencies Ω_i as functions of force ratio Q/Q_{CR}. Values of N: (a) 0; (b) 2; (c) 4; (d) 0; (e) 2; (f) 4.

Putting $U = 0$ in equation (32), one has $\varepsilon_{iL} = \delta_{iL} = \sqrt{\Omega_i}$. The vibration frequency equation of the unconstrained clamped–free beam carrying an end mass is obtained and given by the following equation.

$$1 + \cosh\sqrt{\Omega_i}\cos\sqrt{\Omega_i} + N\sqrt{\Omega_i}(\sinh\sqrt{\Omega_i}\cos\sqrt{\Omega_i} - \cosh\sqrt{\Omega_i}\sin\sqrt{\Omega_i}) = 0 \quad (33)$$

which agrees with the equation given in [18].

Putting $N = 0$, the vibration frequency equation of the constrained clamped–free beam without the end mass is obtained and given by the following equation.

$$\Omega_i^2 + (0{\cdot}5U^2 + \Omega_i^2)\cosh\varepsilon_{iL}\cos\delta_{iL} - 0{\cdot}5U\Omega_i\sinh\varepsilon_{iL}\sin\delta_{iL} = 0 \quad (34)$$

which agrees with the characteristic equation of the clamped–free case as given in [19].

Analytical solutions for ε_{iL}, δ_{iL} in equation (31) cannot be found and numerical solutions are obtained by software packages MATHEMATICA and MATLAB. Values of ε_{iL}, δ_{iL} and $\Omega_i (i = 1, 2, 3)$ are given for $N = 0, 1, 2, 3, 4$ and $Q/Q_{CR} = 0, 0{\cdot}25, 0{\cdot}5, 0{\cdot}75$, and $0{\cdot}999$ in Tables 1–3. Table 1 gives the ε_{1L}, δ_{1L} and Ω_1 values at different values of N and Q/Q_{CR}. Tables 2 and 3 give their values for the second and third modes respectively. The buckling load of a clamped free beam Q_{CR} is $\pi^2 EI/4L^2$ and the corresponding non-dimensional value U_{CR} is $\pi^2/4$. Figures 2(a)–(c) show the variation of ε_{iL}, δ_{iL} with N for $Q = 0$ (Figure 2(a)),

$Q = 0{\cdot}5Q_{CR}$ (Figure 2(b)) and $Q = 0{\cdot}999Q_{CR}$ (Figure 2(c)) and the corresponding non-dimensional frequencies Ω_i are given in Figures 2(d)–(f) respectively. It can be seen in general the frequency decreases with an increase in N. The rate of decrease of the frequency is higher for smaller $N(0 \leqslant N \leqslant 0{\cdot}5)$ as compared with the large ones. Also, the higher modes yield the greater decrease rate for the same N. It can be seen from Figures 3(d)–(f) that the non-dimensional frequency gradually decreases with increasing Q/Q_{CR}. For $N \geqslant 2$, the first mode frequency is close to zero for all Q/Q_{CR}.

3. CONCLUSIONS

An exact frequency equation is derived for the natural vibrations of the flexible arm. The tip of the uniform beam used to model this flexible arm makes contact with the constrained surface at one end and is clamped at the other end. This axially compressed beam carries a concentrated end mass at the constrained end. Numerical solutions are found from the transcendental equation, which give the variation of frequencies with axial compressive force and end mass for such a beam. It was found that an increase in axial compressive force or end mass causes a decrease in the frequencies of all the vibration modes. For a variation in the axial force, the frequency decrease rate is more or less constant. For variation in end mass, the frequency shows a larger decrease rate at lower values of end mass. Results of this work can be applied to other mechanical systems that involve constrained clamped–free beam as one of the mechanical elements.

ACKNOWLEDGMENTS

This research work is supported by The Hong Kong Polytechnic University Research Grant Committee under project account code 0351.101.A3.430. Thanks are extended to Professor B. Wang of Jilin University of Technology for her invaluable encouragement and advice throughout the work. The authors wish to thank the reviewers for their useful comments.

REFERENCES

1. W. J. Book, O. Maizza-neto and D. E. Whitney 1975 *ASME Journal of Dynamic Systems, Measurement and Control* **97**, 424–431. Feedback control of two beam, two joint systems with distributed flexibility.
2. R. H. Cannon, Jr. and E. Schmitz 1984 *International Journal of Robotics Research* **3**, 62–75. Initial experiments on the end point control of a one link flexible experimental manipulator.
3. Y. Sakawa, F. Matsuno and S. Fukushima 1985 Journal of *Robotic Systems* **2**, 453–472. Modeling and feedback control of a flexible arm.
4. H. Kojima 1986 *Journal of Sound and Vibration* **107**, 149–154. Transient vibration of a beam/mass system fixed to a rotating body.
5. N. G. Chalhoub and A. G. Ulsoy 1987 *ASME Journal of Dynamic Systems, Measurement and Control* **109**, 299–309. Control of a flexible robot: experimental and theoretical results.
6. J. C. Simo and L. Vu-Quoc 1987 *Journal of Sound and Vibration* **119**, 487–508. The role of non-linear theories in transient dynamic analysis of flexible structure.
7. N. Popplewell and D. Chang 1996 *Journal of Sound and Vibration* **190**, 721–725. Influence of an off-set payload in flexible manipulator.
8. B. C. Chiou and M. Shahinpoor 1988 Journal of *Robotic Systems* **5**, 443–451. Dynamic stability analysis of a one-link force-controlled flexible manipulator.
9. F. Matsuno, Y. Sakawa and T. Asano 1991 *Proceedings of the IEEE International Conference on Robotics and Automation*, Sacramento, California **3**, 2838–2843. Quasi-static hybrid position/force control of a flexible manipulator.
10. F. L. Hu and A. G. Ulsoy 1994 *ASME Journal of Dynamic Systems, Measurement and Control* **116**, 56–65. Dynamic modeling of constrained flexible robot arms for controller design.

11. F. MATSUNO and K. YAMAMOTO 1994 Journal of *Robotic Systems* **11**, 355–366. Dynamic hybrid position/force control of a two-degree-of-freedom flexible manipulator.
12. F. MATSUNO, K. WAKASHIRO and M. IKEDA 1994 *Proceedings of the IEEE International Conference on Robotics and Automation* **3**, 2107–2112 Force control of a flexible arm.
13. A. H. CHURCH 1963 *Mechanical Vibrations*. New York: John Wiley; second edition.
14. L. MEIROVITCH 1967 *Analytical Methods in Vibrations*. New York: Macmillan.
15. F. S. TSE 1978 *Mechanical Vibrations*: *Theory and Applications*. Boston: Allyn and Bacon.
16. R. F. STEIDEL 1989 *An Introduction to Mechanical Vibrations*. New York: John Wiley.
17. C. L. AMBA-RAO 1967 *Journal of the Acoustical Society of America* **42**, 900–901. Effect of end conditions on the lateral frequencies of uniform straight columns.
18. W. G. BICKLEY and A. TALBOT 1961 *An Introduction to the Theory of Vibrating Systems*. Oxford: Clarendon Press.
19. A. BOKAIAN 1988 *Journal of Sound and Vibration* **126**, 49–65. Natural frequencies of beams under compressive axial loads.
20. N. G. STEPHEN 1989 *Journal of Sound and Vibration* **131**, 345–350. Beam vibration under compressive axial load—upper and lower bound approximation.
21. A. BOKAIAN 1990 *Journal of Sound and Vibration* **142**, 481–498. Natural frequencies of beams under tensile axial loads.
22. H. MATSUNAGA 1996 *Journal of Sound and Vibration* **191**, 917–933. Free vibration and stability of thin elastic beams subjected to axial force.
23. X. Q. LIU and R. C. ERTEKIN 1996 *Journal of Sound and Vibration* **190**, 273–282. Vibration of a free—free beam under tensile axial loads.
24. M. W. D. WHITE and G. R. HEPPLER 1995 *ASME Journal of Applied Mechanics* **62**, 193–199. Vibration modes and frequencies of Timoshenko beams with attached rigid bodies.
25. M. GÜRGÖZE 1996 *Journal of Sound and Vibration* **190**, 149–163. On the eigenfrequencies of a cantilever beam with attached tip mass and a spring mass system.

APPENDIX A: DERIVATION OF THE DYNAMIC EQUATION OF THE FLEXIBLE BEAM

Figure 1 shows a single-link flexible arm which is in contact with the constrained curve at one end and clamped at the other end. The meaning of the symbols are given in Appendix B. Let OXY denote the inertial Cartesian axes, and (X_p, Y_p) denote the co-ordinates of the end point of the beam. Hence,

$$X_p = L\cos\theta - w_E\sin\theta, \qquad Y_p = L\sin\theta + w_E\cos\theta, \tag{A1}$$

since the end is constrained to move along the curve $\Phi(X, Y) = 0$,

$$\Phi(X_p, Y_p) = 0. \tag{A2}$$

Substituting equation (A1) into equation (A2), one has

$$\Phi(\theta, w_E) = \Phi(L\cos\theta - w_E\sin\theta, \qquad L\sin\theta + w_E\cos\theta) = 0 \tag{A3}$$

In Figure 1, a pair of orthogonal unit vectors $(\mathbf{i}, \mathbf{j})$, which is fixed at the vertical shaft of the motor, is shown. They are given by

$$\mathbf{i} = [\cos\theta, \sin\theta]^{\mathrm{T}}, \qquad \mathbf{j} = [-\sin\theta, \cos\theta]^{\mathrm{T}} \tag{A4}$$

The flexible arm is modelled by an Euler–Bernoulli beam in which rotary inertia and shear deformation effects are ignored. Also, elastic deformation $w(t, r)$ is assumed to be small. Because the end point of the arm is constrained, a reaction force f_n is generated along the normal direction of the constrained curve. The relationship of the constrained force f_n, the axial force Q and λ are given by

$$f_n\mathbf{n} = \lambda\begin{bmatrix} b(\theta, w_E) \\ c(\theta, w_E) \end{bmatrix}, \qquad Q = -f_n\mathbf{i}^{\mathrm{T}}\mathbf{n} = -\lambda[b(\theta, w_E)\cos\theta + c(\theta, w_E)\sin\theta]$$

where

$$b(\theta, w_E) = \partial\Phi/\partial X, \qquad c(\theta, w_E) = \partial\Phi/\partial Y, \qquad \text{both at } X = X_p\ Y = Y_p \tag{A5}$$

Let **P** be the position vector of the end point of the flexible beam, and **r** be the position vector of the flexible beam at a general position r. The position vectors and their derivatives are given by

$$\mathbf{P} = L\mathbf{i} - w_E\mathbf{j}, \qquad \dot{\mathbf{P}} = L\dot{\theta}\mathbf{j} + w_E\dot{\theta}\mathbf{i} - \dot{w}_E\mathbf{j}, \qquad \mathbf{r} = r\mathbf{i} - w(t, r)\mathbf{j},$$

$$\dot{\mathbf{r}} = r\dot{\theta}\mathbf{j} + w(t, r)\dot{\theta}\mathbf{i} - \dot{w}(t, r)\mathbf{j}, \tag{A6, 7}$$

where a dot denotes the time derivative. Now, the total kinetic energy T of the arm and the potential energy V are

$$T = \frac{1}{2}\int_0^L \rho\dot{\mathbf{r}}^{\mathrm{T}}\dot{\mathbf{r}}\,\mathrm{d}r + \frac{1}{2}M\dot{\mathbf{P}}^{\mathrm{T}}\dot{\mathbf{P}} + \frac{1}{2}J\dot{\theta}^2, \qquad V = \frac{1}{2}\int_0^L EI(w'')^2\,\mathrm{d}r - \frac{1}{2}\int_0^L Q(w')^2\,\mathrm{d}r, \tag{A8, 9}$$

where a prime denotes the derivative with respect to the spatial variable. The virtual work is given by

$$\delta W = \tau\delta\theta \tag{A10}$$

The governing equation of the flexible arm is derived by Hamilton's principle, which can be written as

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W + \lambda\delta\Phi)\,\mathrm{d}t \equiv 0 \tag{A11}$$

According to Hamilton's principle, the coefficients of $\delta\theta_1$, $\delta\theta_2$, δw_E, $\delta w(t, r)$, $\delta w'_E$, are zero. For simplicity, only the results relevant to this paper are presented.
coefficient of $\delta\ w(t, r)$:

$$\rho(r\ddot{\theta} - \ddot{w} + \dot{\theta}^2 w) - EIw'''' - Qw'' = 0 \tag{A12}$$

With $r = L$, equation (A12) becomes

$$L\ddot{\theta} = \ddot{w}_E - \dot{\theta}^2 w_E + (EI/\rho)w_E'''' + Qw_E'' \tag{A13}$$

coefficient of $\delta w'_E$:

$$EIw_E'' = 0 \tag{A14}$$

coefficient of δw_E:

$$ML\ddot{\theta} - M\ddot{w}_E + M\dot{\theta}^2 w_E + EIw_E''' + \lambda\,\partial\Phi/\partial w_E = 0 \tag{A15}$$

From equations (A13), (A14), and (A15), one obtains the force equilibrium equation:

$$EIw_E''' + Qw'_E + (M/\rho)EIw_E'''' = -\lambda\partial\Phi/\partial w_E \tag{A16}$$

APPENDIX B: NOMENCLATURE

b, c	functions defined in equation (A5)
C_{i1}, C_{i2}, C_{i3}, C_{i4}	constants in equation (18)
EI	flexural rigidity of flexible beam
f	function defined in equation (8)
f_n	reaction force along **n**

g	function defined in equation (7)
J	moment of inertia of hub at clamped end
L	length of flexible beam
M	end mass
q_i	generalized co-ordinate
Q	axial force along **i**
r	position of a point on flexible beam
t	time
T	total kinetic energy of flexible arm
U	non-dimensional axial force defined in equation (21)
v	variable defined in equation (14)
V	total potential energy of flexible arm
w	transverse displacement of the flexible beam
w_E	transverse displacement at tip of flexible beam
X_p, Y_p	co-ordinates of end point of flexible beam
Y_i	mode shape function defined in equation (18)
θ	hub angle of flexible beam
Φ	constrained curve function
λ	Lagrange multiplier associated with the constrained curve
ρ	mass per unit length of flexible beam
δW	virtual work
τ	torque developed by motor
ω_i	vibration frequency of flexible beam
ε_i, δ_i	mode shape parameters in equation (18)
ε_{iL}, δ_{iL}	non-dimensional mode shape parameters defined in equations (19) and (20) respectively
Ω_i	non-dimensional vibration frequency of flexible beam defined in equation (22)
σ_i	mode shape coefficient defined in equation (29)
P	position vector of the end point of flexible beam
r	position vector of a point on flexible beam
(i, j)	a pair of orthogonal unit vectors for flexible beam
n	normal direction of constrained curve