Journal of Sound and Vibration (1997) **204**(2), 359–369

FREE VIBRATIONS OF UNIFORM TIMOSHENKO BEAMS WITH ATTACHMENTS

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(*Received* 8 *October* 1996, *and in final form* 4 *February* 1997)

The problem of free transverse vibrations of Timoshenko beams with attachments like translational and rotational springs, concentrated mass including the moment of inertia, linear undamped oscillators and additional supports is considered. The frequency equation for the combined system is derived by means of the Lagrange multiplier formalism. The exact solution of the free vibration problem of the beam without attachments is taken into account for the formulation of the free vibration problem of the combined system. Numerical examples show the separate or coupling influences of the additional elements on the combined system's frequencies. The comparison of results obtained by using the present approach with results of the exact solution indicates a good agreement.

1. INTRODUCTION

The problem of free vibrations of beams with attachments has been considered by many authors. Most of their works have presented the solution for the situations where the beams have been considered according to Bernoulli–Euler beam theory [1–9]. The exact solutions as well as the approximate ones have been obtained for systems with various additional elements (elastic supports, rigidly or elastically mounted masses, etc.) and for different combinations of the beam end conditions. Combined systems consisting of a uniform or non-uniform beam and different numbers of additional elements have been considered.

Several authors have studied the problem for situations where the beams have been treated according to the Timoshenko beam theory [10–17]. These recently published works concern some of the most frequently existing cases. White and Heppler [10] have reported the results of free vibration investigations of the beam with rigid bodies attached at its ends. They have included the effects of the body mass, first moment of mass and moment of inertia. Rossi *et al*. [11] have solved analytically the problem of free vibrations of beams carrying elastically mounted concentrated masses. Three combinations of boundary conditions: simply supported, simply supported–clamped and clamped at both ends have been considered. Maurizi and Belles [12] and Abramovich and Hamburger [13] have investigated cantilever beams with the attached masses. The cantilever beam with a tip mass and intermediate rotational and translational springs has been investigated by Abramovich and Hamburger [14].

In references [15–17] vibrations of non-uniform beams have been analyzed. Lee and Lin [15] have presented the exact solution for the free vibration of a symmetric beam with tip mass at one end and elastically restrained at the other. An approximate method has been developed by Matsuda *et al*. [16] to study the vibration of a tapered beam with constraint

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at any point and carrying a heavy tip body. Farghaly [17] has investigated the natural frequencies and the critical buckling load coefficients for multi-span beam systems.

In the present work the solution of the free vibration problem of a Timoshenko beam with additional elements attached is presented. The solution is obtained by using the Lagrange multiplier formalism. The frequency equation is derived for the combined system consisting of a uniform Timoshenko beam and additional elements. Some numerical examples are shown together with other solutions in order to show the accuracy of the results obtained. Other numerical results are presented to show the influence of the various parameters on the frequencies of the combined system.

2. FORMULATION

A dynamical system consisting of a uniform Timoshenko beam, rotational and translational springs, concentrated mass and element with rotary inertia, linear undamped oscillator and additional supports against the beam translation or rotation is considered (see Figure 1(a)). The beam without the additional elements is the base system that must satisfy any arbitrary chosen boundary conditions.

The beam kinetic energy is expressed as (see reference [18])

$$T_b(t) = \frac{1}{2}\int_0^L \left[\frac{\partial y(x,t)}{\partial t}\right]^2 \rho A(x)\,\mathrm{d}x + \frac{1}{2}\int_0^L \left[\frac{\partial \psi(x,t)}{\partial t}\right]^2 \rho I(x)\,\mathrm{d}x\,, \tag{1}$$

where $y(x, t)$ is the total deflection of the beam at a point x, $\psi(x, t)$ is the angle of rotation due to bending, $\rho A(x)$ is the mass per unit length, $\rho I(x)$ is the mass moment of inertia

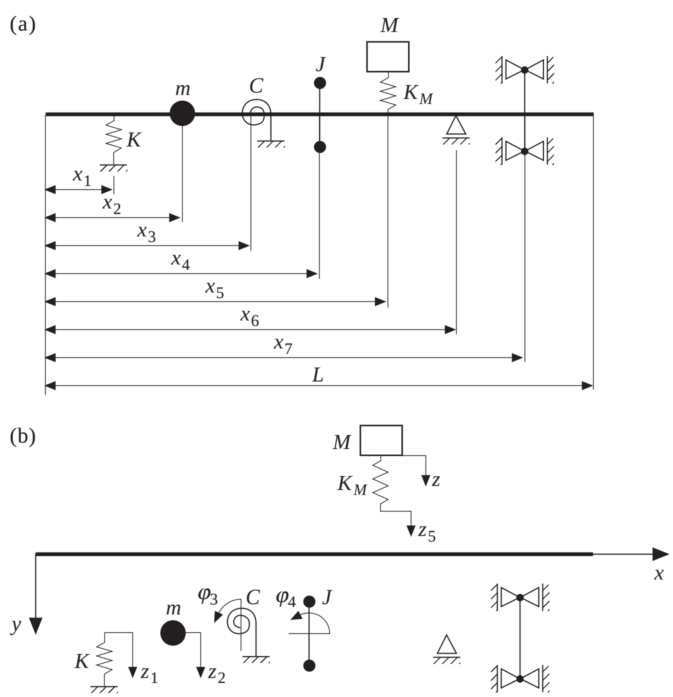

Figure 1. A model of the combined dynamical system.

per unit length about the neutral axis which passes through the center and ρ is the mass density.

The beam potential energy is expressed as

$$V_b(t) = \frac{1}{2}\int_0^L EI(x)\left[\frac{\partial\psi(x,t)}{\partial x}\right]^2 \mathrm{d}x + \frac{1}{2}\int_0^L k'GA(x)\left[\frac{\partial y(x,t)}{\partial x} - \psi(x,t)\right]^2 \mathrm{d}x\,, \qquad (2)$$

where E is the modulus of elasticity, $I(x)$ is the area moment of inertia about the neutral axis, G is the shear modulus, $A(x)$ is the cross-sectional area and k' is a numerical factor depending on the shape of the cross-section.

Based on the solutions obtained for the Timoshenko beam without any attachments one can express the total deflection y and rotation ψ as

$$y(x,t) = \sum_{i=1}^{n} Y_i(x)\xi_i(t)\,, \qquad \psi(x,t) = \sum_{i=1}^{n} \Psi_i(x)\xi_i(t)\,, \qquad (3, 4)$$

where $Y_i(x)$ denotes the ith transverse vibrational mode and $\Psi_i(x)$ the ith rotational vibrational mode.

Substituting equations (3) and (4) in equations (1) and (2), one obtains

$$T_b(t) = \frac{1}{2}\sum_{i=1}^{n} M_i\dot{\xi}_i^2, \qquad V_b(t) = \frac{1}{2}\sum_{i=1}^{n} K_i\,\xi_i^2, \qquad (5, 6)$$

where

$$M_i = \int_0^L Y_i^2(x)\rho A(x)\,\mathrm{d}x + \int_0^L \Psi_i^2(x)\rho I(x)\,\mathrm{d}x,$$

$$K_i = \int_0^L EI(x)\Psi_i'^2(x)\,\mathrm{d}x + \int_0^L k'GA(x)[Y_i'(x) - \Psi_i(x)]^2\,\mathrm{d}x. \qquad (7)$$

Initially, the beam and the additional elements of the system are considered to be unconnected (see Figure 1b)), so there is no influence of the additional elements on the total deflection y and rotation ψ of the beam. The additional elements are not influenced by the beam as well as they are not influenced by each other. From equations (5) and (6) the total kinetic energy of all components is

$$T = \frac{1}{2}\sum_{i=1}^{n} M_i\dot{\xi}_i^2 + \frac{1}{2}m\dot{z}_2^2 + \frac{1}{2}M\dot{z}^2 + \frac{1}{2}J\dot{\varphi}_4^2\,, \qquad (8)$$

and the total potential energy is

$$V = \frac{1}{2}\sum_{i=1}^{n} K_i\xi_i^2 + \frac{1}{2}Kz_1^2 + \frac{1}{2}C\varphi_3^2 + \frac{1}{2}K_M(z - z_5)^2, \qquad (9)$$

where K and K_M are the linear translational spring stiffnesses, m and M are the masses, C is the linear rotational spring stiffness, J is the rotary inertia and z, z_1, z_2, z_5, φ_3 and φ_4 are the co-ordinates of the additional elements as shown in Figure 1(b).

TABLE 1

Frequency coefficients $\Omega_i = \omega_i L^2 \sqrt{\rho A/EI}$ of the simply supported Timoshenko beam carrying elastically mounted concentrated mass ($x_5/L = 2/3$)

Model	α_{K_M}	α_M	Ω_1	Ω_2	Ω_3	Ω_4
Exact solution [11]	1	0·2	2·21494	9·42271	33·5706	101.390
Seven terms			2·21506	9·49272	33·5706	101·390
Fifteen terms			2·21500	9·49271	33·5706	101·390
Exact solution [11]		1	0·99093	9·48904	33·5706	101·390
Seven terms			0.99099	9.48905	33.5706	101.390
Fifteen terms			0·99096	9·48905	33·5706	101·390
Exact solution [11]		3	0·57215	5·48846	33·5705	101·390
Seven terms			0·57218	9·48847	33·5706	101·390
Fifteen terms			0·57217	9·48847	33·5705	101·390
Exact solution [11]	100	0·2	8·10814	23·0376	37·1183	102·088
Seven terms			8·10997	23·1258	37·1815	102·098
Fifteen terms			8·10904	23·0810	37·1491	102·093
Exact solution [11]		1	5·28137	16·3251	35·9766	102·061
Seven terms			5·28801	16·3836	36·0092	102·070
Fifteen terms			5·28465	16·3539	35·9925	102·065
Exact solution [11]		3	3·30383	15·1237	35·8436	102·057
Seven terms			3·30972	15·1712	35·8731	102·065
Fifteen terms			3·30674	15·1471	35·8580	102·061
Exact solution [11]	10^{15}	0·2	8·25154	30·3093	91·1007	128·787
Seven terms			8·25310	30·3564	92·3888	130·504
Fifteen terms			8·25231	30·3327	91·7431	129·624
Exact solution [11]		1	5·88427	26·6447	81·1391	123·906
Seven terms			5·89176	26·8021	84·4594	126·864
Fifteen terms			5·88797	26·7226	82·7824	125·329
Exact solution [11]		3	3·91991	25·1348	77·9003	122·799
Seven terms			3·92886	25·3364	81·8107	125·991
Fifteen terms			3·92432	25·2344	79·8179	124·322

The additional elements are connected to the beam at points x_k ($k = 1, 2, \ldots, 7$) as shown in Figure 1(a) by requiring that

$$f_1 \equiv y(x_1) - z_1 = 0, \qquad f_2 \equiv y(x_2) - z_2 = 0, \qquad f_3 \equiv \psi(x_3) - \varphi_3 = 0,$$
$$f_4 \equiv \psi(x_4) - \varphi_4 = 0, \qquad f_5 \equiv y(x_5) - z_5 = 0, \qquad f_6 \equiv y(x_6) = 0, \qquad f_7 \equiv \psi(x_7) = 0. \quad (10)$$

The Lagrangian for the combined system may be written as

$$L = T - V + \sum_{r=1}^{R} \lambda_r f_r, \quad (11)$$

TABLE 2

Frequency coefficients $\Omega_i = \omega_i L^2 \sqrt{\rho A/EI}$ of the cantilever Timoshenko beam with a tip mass

Model	α_m	α_J	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Exact solution [20]	1·0	0·125	1·40	5·73	23·64	58·41	106·54
Seven terms	–	–	1·40	6·27	26·92	66·08	118·73
Fifteen terms	–	–	1·40	5·83	24·52	60·48	108·83

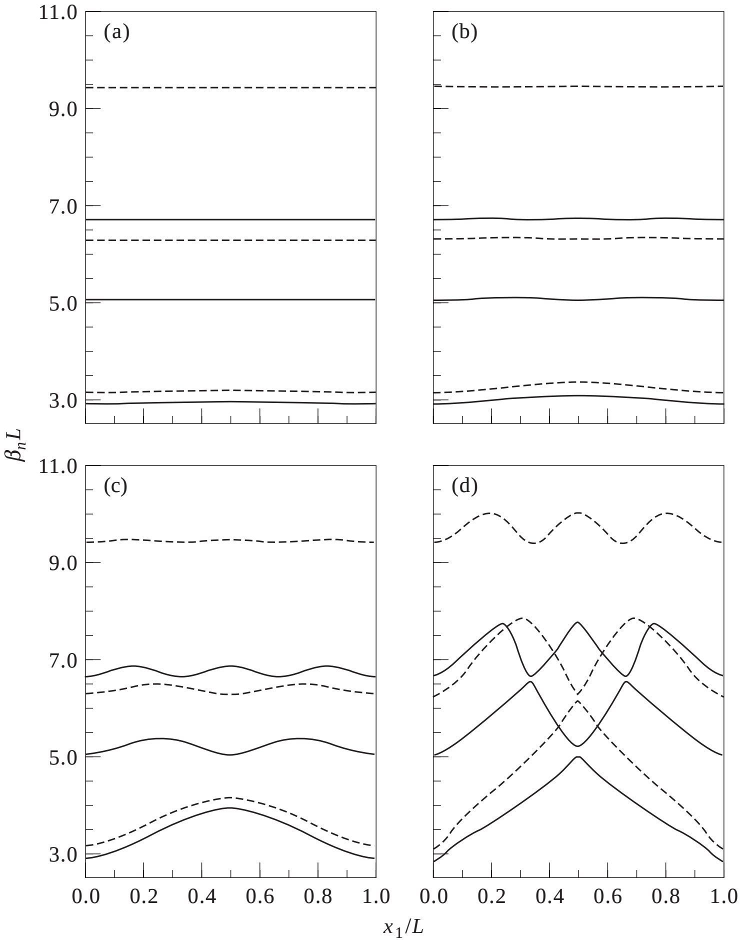

Figure 2. Frequency parameter values $\beta_n L$ versus co-ordinate x_1/L of the connection point between the beam and the translational spring· α_K values: (a) 1; (b) 10; (c) 100; (d) 1000. (See text for key.)

where λ_r is the Lagrange multiplier and R is the number of the attachments in the system. Using the Lagrange equations one obtains

$$M_i \ddot{\xi}_i + K_i \xi_i - \sum_{r=1}^{R} \lambda_r b_{ir} = 0 \qquad i = 1, 2, \ldots, n,$$

$$K z_1 + \lambda_1 = 0, \qquad m\ddot{z}_2 + \lambda_2 = 0, \qquad C\varphi_3 + \lambda_3 = 0,$$

$$J\ddot{\varphi}_4 + \lambda_4 = 0, \qquad -K_M(z - z_5) + \lambda_5 = 0, \qquad M\ddot{z} + K_M(z - z_5) = 0, \tag{12}$$

where

$$b_{ir} = \begin{Bmatrix} Y_i(x_r) & \text{for} & r = 1, 2, 5, 6 \\ \Psi_i(x_r) & \text{for} & r = 3, 4, 7 \end{Bmatrix} \tag{13}$$

Assuming simple harmonic motion,

$$\xi_i = A_i\, \mathrm{e}^{\mathrm{j}\omega t}, \qquad i = 1, 2, \ldots, n, \qquad z_k = Z_k\, \mathrm{e}^{\mathrm{j}\omega t}, \qquad k = 1, 2, 5,$$

$$\varphi_k = \Phi_k\, \mathrm{e}^{\mathrm{j}\omega t}, \qquad k = 3, 4, \qquad z = Z\, \mathrm{e}^{\mathrm{j}\omega t}, \qquad \lambda_r = \Lambda_r\, \mathrm{e}^{\mathrm{j}\omega t}, \qquad r = 1, 2, \ldots R,$$

one can solve the system of equations (12) for the A_i, Z_k, Φ_k and Z in terms of the Λ_r's:

$$A_i = \left(\sum_{r=1}^{R} \Lambda_r b_{ir}\right) \Big/ (K_i - \omega^2 M_i), \qquad Z_1 = -\Lambda_1/K,$$

$$Z_2 = \Lambda_2/(m\omega^2), \qquad \Phi_3 = -\Lambda_3/C, \qquad \Phi_4 = \Lambda_4/(J\omega^2).$$

$$Z_5 = \Lambda_5\left[-1/K_M + 1/(M\omega^2)\right], \qquad Z = \Lambda_5/(M\omega^2). \tag{15}$$

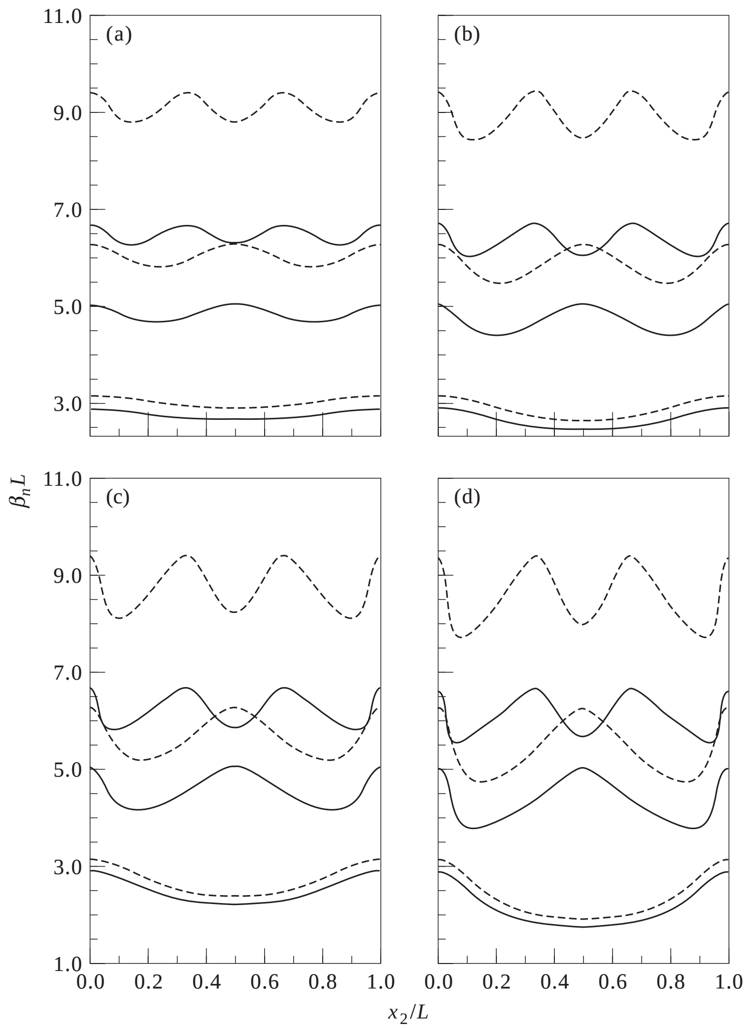

Figure 3. Frequency parameter values $\beta_n L$ versus co-ordinate x_2/L of the connection point between the beam and the concentrated mass· α_m values: (a) 0·2; (b) 0·5; (c) 1; (d) 3.

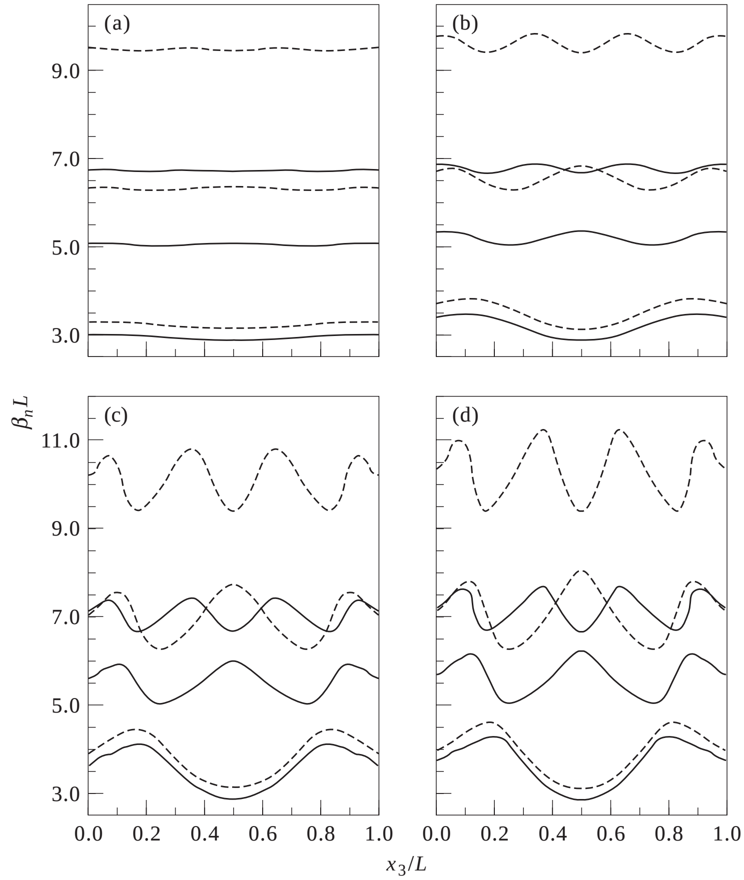

Figure 4. Frequency parameter values $\beta_n L$ versus co-ordinate x_3/L of the connection point between the beam and the rotational spring. α_C values: (a) 1; (b) 10; (c) 100; (d) 1000.

Substitution of equations (15) into equations (10) gives

$$\sum_{r=1}^{R} (C_{kr} + \delta_{kr}\varepsilon_k)\Lambda_r = 0, \qquad k = 1, 2, \ldots, R, \tag{16}$$

where δ_{kr} is the Kronecker delta, and

$$C_{kr} = \sum_{i=1}^{n} \frac{b_{ik}b_{ir}}{K_i - \omega^2 M_i}, \tag{17}$$

$$\varepsilon_1 = 1/K, \qquad \varepsilon_2 = -1/(m\omega^2), \qquad \varepsilon_3 = 1/C,$$

$$\varepsilon_4 = -1/(J\omega^2), \qquad \varepsilon_5 = 1/K_M - 1/(M\omega^2), \qquad \varepsilon_6 = 0, \qquad \varepsilon_7 = 0. \tag{18}$$

For non-trivial solutions the determinant of the coefficients of the Λ_r's in the system of equations (16) must be zero, e.g.,

$$|C_{kr} + \delta_{kr}\varepsilon_k| = 0, \tag{19}$$

which is an eigenvalue equation for ω^2. In this equation, similarly as in reference [8], the coefficients C_{kr} characterize the base system and the coefficients ε_k the additional elements attached to the base system. A number of attachments corresponds to an increase in the size of the matrix.

3. NUMERICAL RESULTS

In order to check the reliability and accuracy of the numerical solutions obtained by the present method, a uniform simply supported beam carrying an elastically mounted concentrated mass was taken as a first example. The same system has been investigated in reference [11]. For the system the numerical values of the frequency coefficients $\Omega_i = \omega_i L^2 \sqrt{\rho A/EI}$ $(i = 1 \cdots 4)$ are present in Table 1. The parameters presented are defined as $\alpha_M = M/\rho AL$, $\alpha_{K_M} = K_M L^3/EI$. For all situations considered $v = 0{\cdot}3$, $k' = 5/6$, $\sqrt{I/A}/L = 0{\cdot}05$ and the dimensionless distance to the left end of the beam $x_5/L = 2/3$. The Timoshenko beam without additional elements according to the theory taken into account in reference [11] was used as the base system for the present calculations. The second example, a comparison between the present results and those presented in reference [20], is shown in Table 2. The frequency coefficients $\Omega_i (i = 1 \cdots 5)$ have been obtained for the

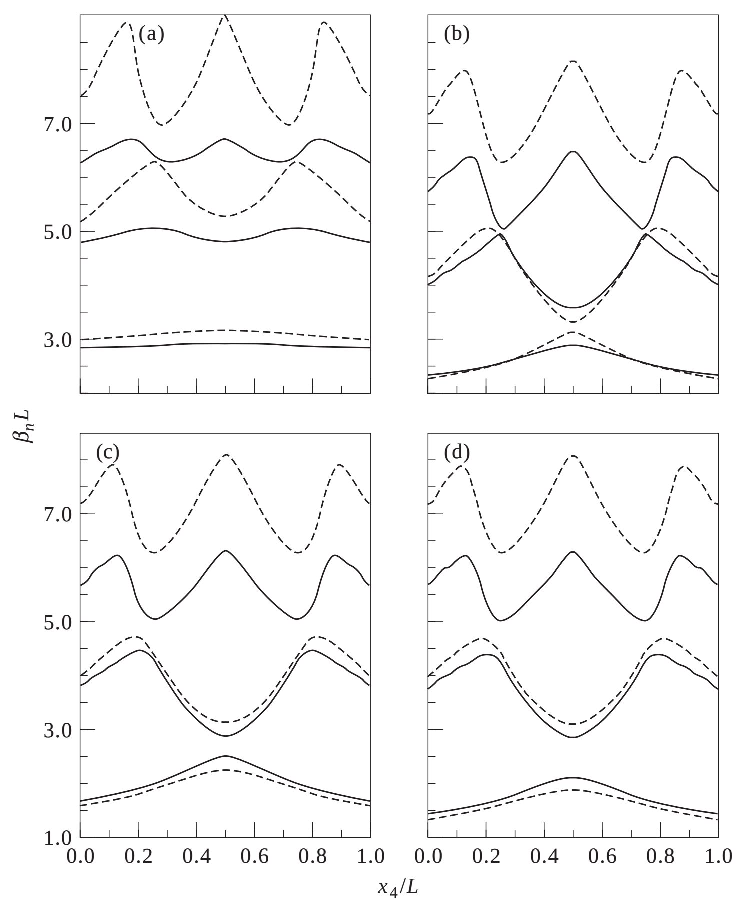

Figure 5. Frequency parameter values $\beta_n L$ versus co-ordinate x_4/L of the connection point between the beam and the element with rotary inertia. α_J values: (a) 0·01; (b) 0·1; (c) 0·5; (d) 1·0.

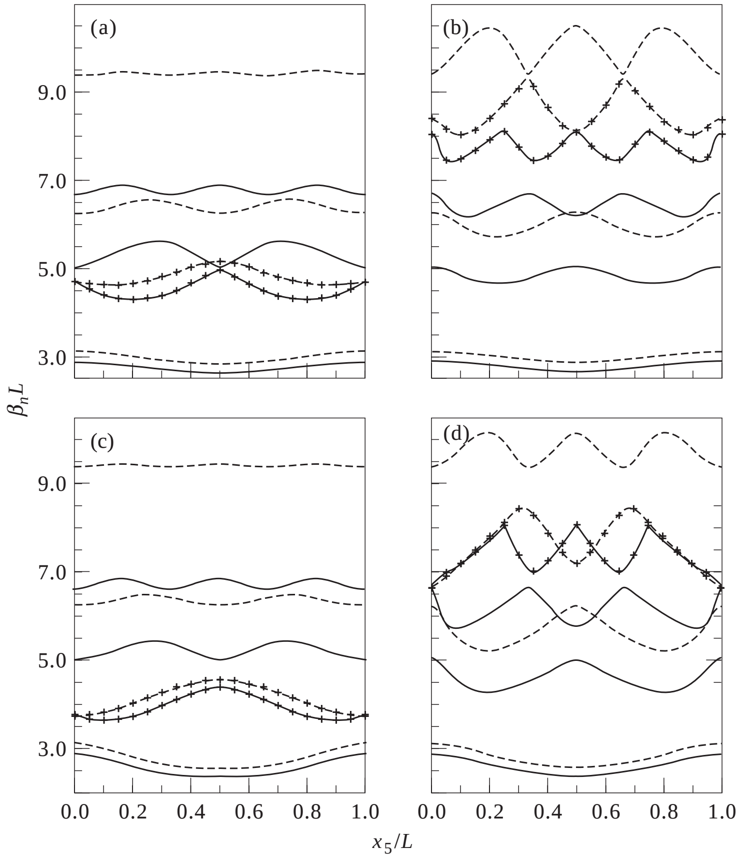

Figure 6. Frequency parameter values $\beta_n L$ versus co-ordinate x_5/L of the connection point between the beam and the linear undamped oscillator. α_M, α_{K_M} values: (a) 0·2, 100; (b) 0·2, 1000; (c) 0·5, 100; (d) 0·5, 1000.

cantilever Timoshenko beam with a tip mass. The system parameters are defined as $\alpha_m = m/\rho AL$, $\alpha_J = J/\rho AL^3$ and $\sqrt{I/A}/L = 0{\cdot}02$, $\sqrt{EI/k'AGL^2} = 0{\cdot}04$. The numerical results obtained by the present method show good accuracy when compared with the results in references [11] and [20].

For the calculations presented graphically, the uniform simply supported beam according to the theory presented by Abramovich and Elishakoff [19] has been taken as the base system. For all the situations considered $\nu = 0{\cdot}3$ and $k' = 5/6$. The results are obtained by taking into account fifteen terms in calculation of the coefficients C_{kr} for various non-dimensional values of $\alpha_K = KL^3/EI$, α_m, $\alpha_C = CL/EI$, α_J, α_M and α_{K_M}.

Values of the frequency parameters $\beta_n L(\beta_n^4 = \rho A\omega_n^2/EI)$ presented in Figures 2–6 show the separate influences of the additional elements on the lower frequencies ω_n ($n = 1 \cdots 3$ or $n = 1 \cdots 4$) of the combined system as functions of the element's locations x_i/L (i = 1, 2, 3, 4 or 5). Otherwise, the coupling influences of the additional elements for chosen values of $\alpha_K = 100$, $\alpha_m = 0{\cdot}5$, $\alpha_C = 100$, $\alpha_J = 0{\cdot}1$, $\alpha_M = 0{\cdot}2$ and $\alpha_{K_M} = 100$ on the combined system's frequencies are shown in Figure 7. On these figures the dashed lines represent the values obtained for $\sqrt{I/A}/L = 0{\cdot}001$ (this corresponds to the vibrating Bernoulli–Euler beam case) and the solid lines represent the values for $\sqrt{I/A}/L = 0{\cdot}1$. Additionally, the appearance of additional frequency for the system with an undamped oscillator is marked by dots on the proper lines.

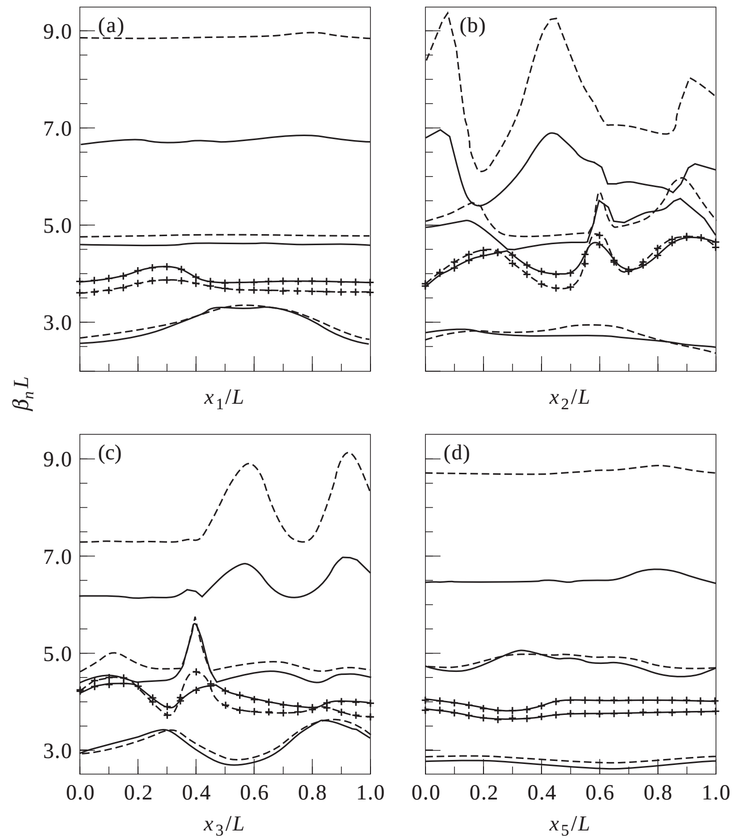

Figure 7. Frequency parameter values $\beta_n L$ versus co-ordinate x_i/L ($i = 1, 2, 3$ or 5) of the connection point for the combined system. (a) $x_2 = x_4 = 0{\cdot}4$, $x_3 = 0{\cdot}6$, $x_5 = 0{\cdot}8$; (b) $x_1 = 0{\cdot}2$, $x_3 = 0{\cdot}6$, $x_4 = x_2$, $x_5 = 0{\cdot}8$; (c) $x_1 = 0{\cdot}2$, $x_2 = x_4 = 0{\cdot}4$, $x_5 = 0{\cdot}8$; (d) $x_1 = 0{\cdot}2$, $x_2 = x_4 = 0{\cdot}4$, $x_3 = 0{\cdot}6$.

The figures presented show that the change in the natural frequency of the combined system depends not only on the characteristic parameter value of the additional element but also on its position along the base system. It could be interesting that the same elements placed along the beam in a different way may cause quite different changes in the natural frequencies of the combined system (compare Figure 7(a) or 7(d) with Figure 7(b) or 7(c)). Additionally, comparing the coupling influences of the additional elements on the natural frequencies of the combined system shown by Figure 7 with Figures 2(c), 3(b), 4(c), 5(b) and 6(a) showing the separate influences of the same elements on the natural frequencies, one can notice quite different behaviours of the system.

CONCLUSION

Equation (19) seems to be especially useful in cases of calculating the frequencies of combined systems that consist of many miscellaneous elements. There is a possibility to replace, in an easy way, the description of the base system by another one according to any arbitrary chosen beam theory. Only the form of the C_{kr} must be properly changed in the frequency equation. The additional elements can also be easily introduced into the description of the combined system. The proper ε_k must be used to form the frequency equation.

Using the exact solution of a free vibration problem for the description of the base system has a fundamental influence on the accuracy of the vibration analysis of the combined system. However, the next problem, which has to be properly solved, is the number of terms needed in the calculation of the C_{kr}. It depends on the convergence rate but also it is important to notice that according to the number of eigenvalues of equation (19) to be calculated, one has to choose an adequate number of terms (this means the number of vibrational modes of the base system). At least the number of terms must be such that the largest calculated eigenvalue must be smaller than the natural frequency of the base system for the last vibrational mode used for calculation of the C_{kr}.

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