Journal of Sound and Vibration (1997) **204**(2), 373–376

THE FUNDAMENTAL FREQUENCY OF NON-HOMOGENEOUS RECTANGULAR MEMBRANES

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(*Received* 16 *December* 1996)

1. INTRODUCTION

The vibrating membrane constitutes a dynamic system which can be found in a multitude of scientific and technological applications [1, 2]. When the mass per unit area of the membrane, ρ, is constant the problem is a classical one and exact and approximate solutions are available for a great variety of situations.

If the parameter ρ is not constant, e.g., due to desired service conditions or undesired homogeneities introduced by the manufacturing process, one must make use, in general, of approximate analytical and/or numerical procedures [3, 4].

The present study deals with the case in which the density of the membrane, ρ, is a linear function of the x-variable:

$$\rho = \rho_0(1 + \alpha(x/a)). \tag{1}$$

Two independent solutions are obtained using the optimized Galerkin–Kantorovich approach [5] and the differential quadrature (DQ) method [6].

2. APPLICATION OF THE OPTIMIZED GALERKIN–KANTOROVICH APPROACH

When the system executes transverse, normal modes of vibration, its behavior is described by

$$\nabla^2 W + \frac{\rho_0(1 + \alpha\bar{x}/a)}{S}\omega^2 W = 0, \qquad W[L(\bar{x}, \bar{y}) = 0] = 0, \tag{2a, b}$$

where W is the amplitude of vibration and $L(x, y) = 0$ is the functional relation which describes the boundary of the membrane.

Introducing the dimensionless variables $x = \bar{x}/a$ and $y = \bar{y}/b$, substituting in equation (2) and making

$$W \simeq W_\alpha = [A(x^\gamma - x) + B(x^{\gamma+1} - x)] \sin \pi y, \tag{3}$$

where γ is Rayleigh's optimization parameter, one is able to use the well established Galerkin–Kantorovich procedure. Once the frequency determinant has been generated, one determines its lowest root $\Omega_1 = (\sqrt{\rho_0/S})\omega_1 a$, which constitutes the fundamental frequency coefficient.

Since

$$\Omega_1 = \Omega_1(\gamma), \tag{4}$$

0022–460X/97/270373 + 04 $25.00/0/sv960931

by requiring

$$d\Omega_1/d\gamma = 0, \tag{5}$$

one obtains an optimized value of Ω_1.

3. SOLUTIONS BY MEANS OF THE DIFFERENTIAL QUADRATURE METHOD

The DQ method was originally proposed by Bellman and Casti [7], but its use has become quite popular in recent years, thanks to the efforts of Bert and coworkers [6].

Through application of the technique, and making use of Bert's notation [6], one obtains the following linear system of equations:

$$\sum_{k_1=2}^{N-1} B_{ik_1} W_{k_1 j} + \lambda^2 \sum_{k_2=2}^{N-1} B_{jk_2} W_{ik_2} + \Omega^2 g_i W_{ij} = 0, \qquad i, j = 2, 3, \ldots, N-1, \tag{6}$$

where $g_i = g(x_i) = 1 + \alpha x_i$, $\lambda = b/a$, $\Omega = (\sqrt{\rho_0/S})\omega a$, x_i are the co-ordinates of the partition station points in the x direction, and N is the number of points of the partition in each direction (all calculations were performed with $N = 9$).

The lowest root of the corresponding determinantal equation constitutes the fundamental frequency coefficient Ω_1.

4. NUMERICAL RESULTS

In Table 1 is depicted a comparison of values of Ω_1 obtained by means of the optimized Galerkin–Kantorovich method and the differential quadrature method for $\alpha = 0$ (constant density case), 0·1 and 1. For $\alpha = 0$ the results are in excellent agreement with the exact eigenvalues and for $\alpha = 0{\cdot}1$ and 1 the agreement is remarkably good between the results obtained by both techniques.

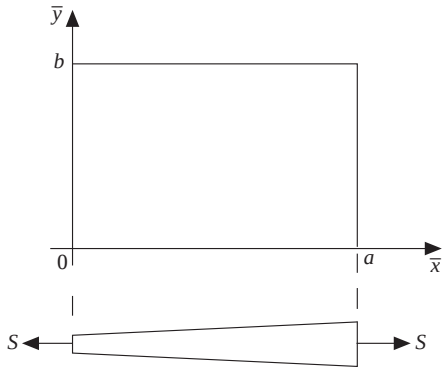
Apparently, the numerical results for Ω_1 previously published in the technical literature [4] are in error for $\alpha = 1$ and $b/a = 0{\cdot}6$, 0·4 and 0·2, since they are almost 30% higher than the results published in the present investigation.

Furthermore, for $b/a = 0{\cdot}6$ and 0·2 they turn out to be higher than the fundamental frequency coefficient corresponding to a homogeneous membrane ($\alpha = 0$). This is clearly unacceptable from a physical viewpoint since for the present problem, the non-homogeneous membrane possesses more mass.

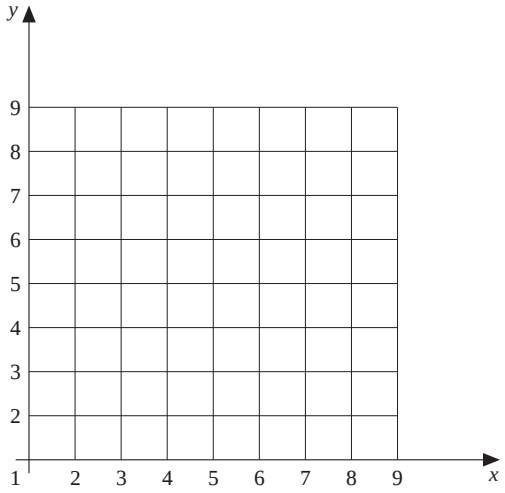
TABLE 1

Values of the fundamental frequency coefficient Ω_1 for $\alpha = 0$, 0·1 and 1·0, obtained by means of the optimized Galerkin–Kantorovich method, the DQ technique and those determined in reference [4] *for $\alpha = 1$, for several values of b/a*

	Optimized Galerkin–Kantorovich			Differential Quadrature			Reference [4]
b/a	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}1$	$\alpha = 1{\cdot}0$	$\alpha = 0{\cdot}0$	$\alpha = 0{\cdot}1$	$\alpha = 1{\cdot}0$	$\alpha = 1{\cdot}0$
1·0	4·44291	4·33539	3·61050	4·44289	4·33539	3·61043	3·61048
0·8	5·02902	4·90719	4·08151	5·02901	4·90719	4·08144	4·08151
0·6	6·10618	5·95791	4·94227	6·10617	5·95790	4·94214	6·70702
0·4	8·45901	8·25221	6·80011	8·45901	8·25221	6·79752	8·25861
0·2	16·01905	15·61337	12·56272	16·01906	15·61334	12·56414	16·5987



(a)



(b)

Figure 1. The mechanical system under study. (a) Membrane of non-uniform density, $\rho = \rho_0(1 + \alpha\bar{x}/a)$; (b) partition of the interval when applying the DQ method.

The approaches used in the present study are also applicable if the mass density function is a more complicated functional relation. If it possesses discontinuities, it will be more convenient to make use of the optimized Rayleigh–Ritz method.

ACKNOWLEDGMENTS

The present study has been sponsored by CONICET PIA 1996–1997 and by Secretaría General de Ciencia y Tecnología of Universidad Nacional del Sur.

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